A simple model for estimating the sensitivity of runoff to long-term changes in precipitation without a change in vegetation

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Abstract

Forecasts of changes in precipitation \( (P) \) and potential evaporation \( (PE) \) can be applied to hydrologic models calibrated on existing conditions to obtain predictions of changes in runoff. This study describes an alternative approach, which uses a simple soil-moisture accounting model with a small number of independent and physically based parameters to explore the sensitivity of runoff to climate change for three simplified climates. The climate types chosen initially are those for which a piecewise analytical solution can be obtained so that computer programmes involving numerical solutions can be verified before being applied to field data. Sensitivity factors are calculated for the various cases and their relationships with climatic conditions and soil conditions are explored. Breakpoint values were determined for each type of climate studied. These correspond to situations in which the soil becomes momentarily saturated once during the seasonal cycle but does not remain saturated for any finite duration. For humidity ratios greater than the breakpoint, the sensitivity of runoff to precipitation increases abruptly. For the climates studied, the sensitivity factor approaches the value of the soil parameter \( c \) as the humidity index approaches zero. The other climates studied exhibit the same sensitivity at this limit. A particular feature of the model is that analytical solutions can be determined in many cases to check and confirm the results of the numerical simulations. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

There are many layers of uncertainty in the problem area of modelling the sensitivity of runoff to climate changes. These include the uncertainty in the climate models, the uncertainty in the hydrologic models, the uncertainties in relation to the predictions of the economic development and population growth with consequent uncertainties in future water resources use and the complexities arising from the interactions between these diverse elements. Analysis of present data alone, no matter how sophisticated, cannot provide a complete basis for prediction of climate change impacts.

There is a deficiency of knowledge concerning hydrologic behaviour at the catchment and the regional scales. Upscaling of the various approaches applied at the plot and field scales cannot easily be applied to the entire catchment or on a regional scale. An exception is the equation of continuity which can be formulated at any scale provided the boundary fluxes can be determined or accurately estimated. This is possible because the continuity equation is intrinsically linear and does not contain parameters that impede the upscaling process.

The uncertainties in relation to fluxes of water, energy and momentum at the land surface constitute a major obstacle to the linking of atmospheric components of Global Climate Models with the land phase of the hydrological cycle. An outline of the complex feedback processes involved is shown in the form of a conventional Forrester diagram in Fig. 1 due to O’Kane [2,9]. In addition to the division of available water at the soil surface into infiltration and runoff, there are further key partitionings dividing percolating water into transpiration and groundwater recharge at the root zone and dividing evaporated water into re-precipitation and outflow of precipitable water in the local atmosphere. These are shown in Fig. 2 due to [12].

The present paper aims to contribute to the understanding of the effect of the first two of those partitionings on climate modelling. A model is built up from simplest possible analysis in order to gain basic insight into the key properties of the system as an aid to directing a later fuller analysis of the problem. The results therefore are indicative only but do represent a reliable if small step on the long road to a final solution.

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The approach used in this study has certain elements in common with previous studies in the US \cite{23,22}, in Poland \cite{15,14} and Australia \cite{7}. In particular the approach is similar to that of Schaake \cite{22} except for the treatment of the parameterisation of subsurface storage. The application of this approach to the classical climatological formulae for actual evaporation \cite{24,19,1,26,21} has been already described \cite{10}. In the latter paper, it has been shown that the sensitivity of runoff ($Q$) to changes in precipitation ($P$) are similar for the formulae due to Schreiber, Ol’dekop and Turc–Pike in the case of semi-humid and humid biomes but that for the arid case the limiting value of the sensitivity factor for complete aridity is infinite for the Schreiber formula compared with a value of 3.0 for the formulae of Ol’dekop and Turc–Pike. In all cases, the sensitivity of runoff decreases monotonically from the limiting value for a zero value of the humidity index ($P/PE$) and approaches the limiting value of 1.0 asymptotically as the humidity index becomes very large. One purpose of the present study was to investigate the discrepancy for arid conditions and the form of the decrease for higher humidity ratios on the basis of a highly simplified model of catchment behaviour.

2. Evaporation ratio and humidity index

The long-term water balance at a catchment or regional scale can be written as

$$P = \overline{AE} + Q,$$

(1)

where $P$ is the long-term average of precipitation, $\overline{AE}$ is the long-term average of evapotranspiration, and $Q$ is the long-term average of runoff of all types (Hortonian overland flow, saturation excess runoff, interflow and baseflow). This equation of continuity must be supplemented by further relationships or assumptions to obtain a solution of the water balance. Global scale hydrology takes into account the interaction of hydrologic processes at large scales with changes in vegetation and with climate variation and change. One plausible starting hypothesis for a simplified analysis is that, on the space scale and time scale of interest, there is an equilibrium between the climate factors, the pedological factors and vegetative factors. This concept of geographical zonality put forward \cite{17} over a hundred years ago suggests that undisturbed catchments reflect an equilibrium between climatic and biological factors that results in a limited number of biomes linked to climatic regions.

The key problem in large-scale water balances is the reduction of the potential evapotranspiration ($PE$), which depends on atmospheric and land surface factors, to the actual evapotranspiration ($AE$) used in the water balance of Eq. (1). Budyko \cite{5} applied the concept of geographical zonality by assuming (a) that the evaporation ratio ($\overline{AE}/PE$) was a function only of what he termed the aridity ratio ($PE/P$) and (b) that the biome type is also determined by the value of the aridity ratio. The use of a humidity ratio ($P/PE$), which is the reciprocal of Budyko’s aridity ratio, has the advantage of reducing some important limiting cases to straight line relationships and is adopted in this presentation. The two basic assumptions in the present analysis thus are the long-term water balance as described by Eq. (1)
above and the equilibrium concept represented by the relationship

\[
\frac{AE}{PE} = \phi \left( \frac{P}{PE} \right), \quad \text{catchment parameters}
\]

(2a)

which for the commonly used empirical formulae may be simplified to

\[
\frac{AE}{PE} = \phi \left( \frac{P}{PE} \right)
\]

(2b)

which may be termed the Budyko hypothesis. These two equations are sufficient for the first analysis of the sensitivity of runoff to changes in precipitation \((P)\) and evapotranspiration \((PE)\) as predicted by climate models which all have the form of Eq. (2b).

The sensitivity of long-term catchment runoff \(Q\) to changes in \(P\) and \(PE\) can be characterised by

\[
\frac{\Delta Q}{Q} = \psi_\tau \frac{\Delta P}{P} + \psi_{\text{PE}} \frac{\Delta PE}{PE},
\]

(3)

where \(\psi_\tau\) and \(\psi_{\text{PE}}\) are the sensitivity factors for changes in \(P\) and \(PE\), respectively.

These two sensitivity factors are identical to the elasticities \(\phi_\tau\) and \(\phi_{\text{PE}}\), respectively, defined by [23].

It is shown in Appendix A, for the special case defined by Eq. (2b), that

\[
\psi_\tau = \frac{P/PE \left[ 1 - \phi \left( P/PE \right) \right]}{P/PE - \phi \left( P/PE \right)}
\]

(4)

and that

\[
\psi_\tau + \psi_{\text{PE}} = 1
\]

(5)

so that we can write

\[
\frac{\Delta Q}{Q} = \psi_\tau \frac{\Delta P}{P} + (1 - \psi_\tau) \frac{\Delta PE}{PE}.
\]

(6)

It is clear that once the function \(\phi \left( P/PE \right)\) is known the sensitivity can be determined for any given value of the humidity ratio. The task thus reduces to the need to determine the function \(\phi \left( P/PE \right)\) for typical climate types and typical parameter values.

The equilibrium hypothesis is not directly applicable to the study of vegetated surfaces under climate changes because the parameters for evapotranspiration are sensitive to CO2 enrichment of the atmosphere. Two possible mechanisms for such an effect are (a) the effect of CO2 enrichment on stomatal density and hence of leaf evaporation [13,28] and (b) the effect of changes in water table levels and soil moisture content on the leaf area index [18]. In this case, the complementarity of Eqs. (5) and (6) will not apply. In the present paper the analysis is restricted to the evaluation of \(\psi_\tau\) for reasons of space but the approach is similar for the case of \(\psi_{\text{PE}}\).

3. Use of a simple soil moisture accounting

When the soil moisture at the land surface is at or close to its maximum, evaporation is atmosphere-controlled and takes place at the potential rate which depends on atmospheric conditions and especially on the energy available at the land surface. When the soil moisture is significantly below its maximum, the actual evaporation will be soil-controlled i.e. will depend on the ability of the soil to provide water at the root level or at the surface. The key problem, therefore, is to provide some method of soil moisture accounting which will give a reasonable simulation of the variation of soil moisture throughout the year and to link the actual evaporation to the state of the soil moisture.

The catchment water balance of Eq. (1) can be divided into a surface water balance and a subsurface water balance. Assuming that the surface is vegetated so that the evaporation can come from the entire active depth of the soil column, the former is given by

\[
P(t) = I(t) + Q_s(t)
\]

(8)

without serious error. The variable \(t\) may be expressed as days since the beginning of the year (either calendar year or water year) or as a fraction of the year.

The long-term subsurface water balance is given by

\[
I = \Delta AE + Q_b,
\]

(9)

where \(\Delta AE\) is the long-term average of the actual evapotranspiration (including evaporation from any bare soil surface) and \(Q_b\) is the long-term average of surface runoff (both interflow and groundwater runoff). In order to determine the seasonal variation of \(\Delta AE(t)\) throughout the year, the relatively slow variation in soil water content \(W(t)\) must be taken into account thus giving rise to the soil moisture accounting equation

\[
\frac{d}{dt} [W(t)] = I(t) - \Delta AE(t) - Q_b(t),
\]

(10)

where \(Q_b(t)\) is the rate of water percolating downward through the base of the soil column whose long-term average will be equal to \(Q_b\). In order to solve Eq. (10) it is necessary to know the given inputs \(P(t)\) and \(PE(t)\) and to derive or assume the necessary relationships between the terms in Eq. (10) and the moisture content \(W(t)\). These relationships depend critically on whether the surface fluxes at a given time are atmosphere-controlled or soil-controlled.
When precipitation follows a dry spell, there will be an initial period during which the rate of infiltration \( I(t) \) at the surface is equal to the rate of precipitation \( P(t) \) if the latter is less than the rate of potential infiltration \( P^*(t) \) (commonly referred to as infiltration capacity). During this period of atmosphere-controlled infiltration, Eq. (10) governing the soil moisture accounting will become

\[
\frac{d}{dt}[W(t)] = P(t) - AE(t) - Q_b(t). \tag{11}
\]

As soon as the surface of the ground becomes ponded, the amount of infiltration is limited by the soil conditions and any precipitation above this limit appears as Hortonian surface runoff \( Q_H(t) \).

Accordingly, the first step is to compare the rate of precipitation \( P(t) \) with the rate of potential infiltration \( P^*(t) \) (see Appendix B). If we have \( P(t) < P^*(t) \) then the rate of infiltration is atmosphere-controlled and

\[
I(t) = P(t), \quad (12a)
\]

\[
Q_H(t) = 0. \quad (12b)
\]

If, however, \( P(t) > P^*(t) \) then the surface flux is soil-controlled and we have

\[
I(t) = P^*(t), \quad (13a)
\]

\[
Q_H(t) = P(t) - P^*(t). \quad (13b)
\]

For many combinations of climate types and soil types the conditions for Eqs. (13a) and (13b) will seldom or ever occur and the step may be omitted in a first approach to the problem and Eqs. (11), (12a) and (12b) used throughout.

The second major step is to estimate the reduction from potential to actual evapotranspiration when soil water content is depleted. Here the assumption used is that of a Budyko bucket [25,6] which was used in a number of early global climate models (e.g. [16]). This simple model specifies that

\[
\begin{align*}
0 & \leq W(t) \leq W_B & AE(t) = PE(t) \frac{W(t)}{W_B} & \quad (14a) \\
W_B & \leq W(t) \leq W_{sat} & AE(t) = PE(t) & \quad (14b)
\end{align*}
\]

where \( W_B \) is the soil moisture content below which the evaporation rate is less than potential, is sometimes referred to as ‘field capacity’.

The third step is to seek a simple representation of the recharge from the base of soil column to the groundwater store. Many of the land surface schemes used in global climate models assume that outflow condition at the base of the soil column is that of free drainage i.e. zero variation in water content and consequently a downward flow at a rate equal to the hydraulic conductivity corresponding to that constant soil water content. Since unsaturated conductivity varies sharply with local water content [4] it is plausible to use a relationship of the form

\[
Q_b(t) = K_{sat} \left( \frac{W(t)}{W_{sat}} \right)^c, \tag{15}
\]

where the dimensionless parameter \( c \) depends only on the soil type.

Although this simple model does not produce Hortonian runoff, it can produce runoff if precipitation excess occurs when the soil is saturated. In this case \( W = W_{sat} \) and then \( Q_b(t) \) of an amount will be

\[
Q_b(t) = P(t) - PE(t) - K_{sat} \tag{16}
\]

but for \( W(t) < W_{sat} \), \( Q_b(t) = 0 \).

Thus for a first analysis the water balance of Eq. (10) can be written

\[
\frac{d}{dt}[W(t)] = \text{Min}[P(t), P^*(t)] - PE(t) \cdot \text{Min} \left[ 1, \left( \frac{W(t)}{W_B} \right) \right] - K_{sat} \left( \frac{W(t)}{W_{sat}} \right)^c \tag{17a}
\]

and if, for a first analysis, soil-control rarely occurs and can be neglected, this reduces to

\[
\frac{d}{dt}[W(t)] = P(t) - PE(t) \cdot \text{Min} \left[ 1, \left( \frac{W(t)}{W_B} \right) \right] - K_{sat} \left( \frac{W(t)}{W_{sat}} \right)^c. \tag{17b}
\]

This equation can be solved for any specified climate characterised by specified \( P(t) \) and \( PE(t) \) and a soil characterised by the soil column parameter \( W_{sat} \) and the soil property parameters \( K_{sat} \), \( c \) and \( W_B/W_{sat} \).

4. Evaporation function for typical climates

The shape of the evaporation function implied by the above model can readily be derived by the insertion of appropriate expressions for \( P(t) \) and \( PE(t) \) in Eqs. (17a) and (17b) above, solving for \( W(t) \) and hence \( AE(t) \), and summing to obtain the annual value of \( AE \). In the present section, this is done for (a) a non-seasonal climate, (b) a simplified tropical climate and (c) a simplified temperate climate.

5. Non-seasonal climate

For the case of a simple non-seasonal climate, we have no long-term variation i.e.

\[
P(t) = \bar{P}, \tag{18a}
\]

\[
PE(t) = \bar{PE}. \tag{18b}
\]

For such constant climatic conditions the soil water content \( W(t) \) will for any initial condition approach an equilibrium value \( \bar{W} \) and remain at this value. If this
value of $\bar{W}$ is less than $W_B$ then the long-term actual evaporation $AE$ will be less than the potential. In this case, the steady-state solution satisfies

$$\frac{\mathcal{P}}{\mathcal{P}_E} = \left( \frac{\bar{W}}{W_B} \right) + K_{sat} \left( \frac{W_B}{W_{sat}} \right)^c \left( \frac{\bar{W}}{W_B} \right)^c. \tag{19}$$

The humidity ratio for which $\bar{W} = W_B$ is given by

$$\left( \frac{\mathcal{P}}{\mathcal{P}_E} \right)_B = 1 + R_0, \tag{20a}$$

where

$$R_0 = \left( \frac{K_{sat}}{\mathcal{P}_E} \right) \left( \frac{W_B}{W_{sat}} \right)^c \tag{20b}$$

which is an important parameter for this and other climate types reflecting the relative contributions of evaporation and groundwater recharge to the water balance. From Eqs. (19), (20a) and (20b) it is clear that the form of the equation for the evaporation ratio becomes

$$\frac{AE}{\mathcal{P}_E} = \Phi \left( \frac{\mathcal{P}}{\mathcal{P}_E}, c, R_0 \right) \tag{21}$$

which replaces the simpler form of Eq. (2b) applicable to the commonly used empirical formulae.

For the lower range $0 < \bar{W} < W_B$, Eq. (19) readily gives the evaporation function $\Phi(\mathcal{P}/\mathcal{P}_E)$ defined by Eqs. (2a) and (2b) in inverse form. Since we have from Eq. (14b)

$$\frac{AE}{\mathcal{P}_E} = \frac{\bar{W}}{W_B}, \tag{22}$$

then we can write Eq. (19) as

$$\frac{\mathcal{P}}{\mathcal{P}_E} = \left( \frac{AE}{\mathcal{P}_E} \right) + R_0 \left( \frac{AE}{\mathcal{P}_E} \right)^c \tag{23}$$

which is the inverse solution of the basic problem. For the upper range $W_B < \bar{W} < W_{sat}$ we will have $AE = \mathcal{P}_E$ and this water balance of Eq. (19) becomes

$$\frac{\mathcal{P}}{\mathcal{P}_E} = 1 + R_0 \left( \frac{\bar{W}}{W_B} \right)^c \tag{24a}$$

and the additional precipitation appears as groundwater recharge

$$\frac{Q_s}{\mathcal{P}_E} = R_0 \left( \frac{\bar{W}}{W_B} \right)^c = \frac{\mathcal{P}}{\mathcal{P}_E} - 1. \tag{24b}$$

For $\bar{W} = W_{sat}$, the soil will be saturated throughout the year and we will have

$$AE(t) = \mathcal{P}_E, \tag{25a}$$

$$R(t) = K_{sat}, \tag{25b}$$

$$Q_s(t) = \mathcal{P} - \mathcal{P}_E - K_{sat}. \tag{25c}$$

Thus the evaporation function will consist of a curved segment in accordance with Eq. (23) up to a values of $\mathcal{P}/\mathcal{P}_E = 1 + R_0$ and a constant value of 1 beyond this point, as shown in Fig. 3. Fig. 3 shows the relationship between the evaporation ratio $(AE/\mathcal{P}_E)$ and the humidity ratio for the three type climates for $c = 4$ (typical for loam soils) and for $R_0 = 1$ which is a reasonable lower limit for this parameter (see Table 2 later). The latter parameter values were chosen to facilitate the verification of the computer simulation by an analytical solution but are reasonably close to typical values for a clay-loam.

6. Tropical climate

The second typical climate is a simple tropical climate defined by

$$P(t) = 0 \quad 0 < t < T, \tag{26a}$$

$$P(t) = \frac{\mathcal{P}}{1 - T} \quad T < t < 1 \tag{26b}$$

and

![Fig. 3. Evaporation function for $R_0 = 1$ and $c = 4$.](image-url)
derive analytical expressions for the breakpoint humidity ratio
which the elapsed time \(t\) and the length of the dry period are expressed as fractions of a year taken from the start of the dry period.

For this type of climate, the value of \(W(t)\) will vary throughout the year, a period \(T\) of soil water depletion being followed by a period of recharge during the remainder of the year. For this and other cases of seasonal variation in \(P(t)\) and \(PE(t)\), an additional parameter relating to time scaling enters the computation. The evaporation ratio is now

\[
\frac{\Delta E}{PE} = \Phi[\frac{\bar{P}}{PE}, c, R_0, PE, L/W_b, W_b/W_{max}]
\]

where \(L\) is the length of the cycle (i.e. 1 yr). Numerical simulations for the two simplified seasonal climates studied reveal that the sensitivity ratio \(\Psi_{2I}\) is more strongly influenced by changes in value of \(c\) and \(R\) rather than the other parameter \(PE, L/W_b\). For low values of the humidity ratio \(\frac{\bar{P}}{PE}\) the soil water will not reach the value \(W_b\) at any time during the year and hence the evaporation will never reach the potential rate. The values of the humidity index \(\frac{\bar{P}}{PE}\) for which the soil water content reaches \(W_b\) only instantaneously is defined as the breakpoint humidity ratio \(\frac{\bar{P}}{PE}\)\(_b\). For higher values than this breakpoint humidity ratio, the recharge to \(W_b\) will take place before the end of the year and these will be a period of evaporation at the potential rate for the remainder of the year.

For rational values of the soil parameter (and more easily for integral values 1, 2, 3 or 4), it is possible to derive analytical expressions for the breakpoint humidity ratio \(\frac{\bar{P}}{PE}\)\(_b\), the corresponding evaporation ratio \(\frac{\Delta E}{PE}\)\(_b\), the maximum evaporation ratio \(\frac{\Delta E}{PE}\)\(_{sat}\), the period required for recharge and the average evaporation ratio \(\frac{\Delta E}{PE}\). However, it is more convenient to use a procedure of numerical simulation which has first been verified on the basis of the simpler analytical solution for the non-seasonal case described above. A typical result is shown on Fig. 3 using \(T = 0.5\) yr (i.e. equal length of dry and wet seasons), \(\frac{PE \cdot L}{W_{max}} = 10\), \(W_b/W_{max} = 0.75\) and the same parameters as in the non-seasonal case i.e. \(R_0 = 1\) and \(c = 4\). The main difference is seen to be that the maximum rate of evaporation is less than the potential rate, the ratio being approximately \((1 - T)\). This reflects the fact that potential evaporation not utilised during the dry season cannot be recovered in the wet season, no matter how intense the rainfall. The effect of the length of the dry season on the actual evaporation is shown in Fig. 4. As might be expected from a comparison of Eqs. (18a), (18b), (26a)–(26c), the non-seasonal case represents a limiting form of the tropical climate with \(T\) approaching zero.

7. Temperate climate

The third type of climate examined was the temperate climate defined by

\[
P(t) = \bar{P} \cdot [1 - r \cos(2\pi t)],
\]

where the parameter \(r\), which lies between 0 and 1, is related to the minimum scale of the potential evaporation during the year by

\[
\frac{PE_{min}}{PE} = 1 - r.
\]

For values of \(\frac{\bar{P}}{PE}\) less than \([R_0 + (1 - r)]\) the soil water would never reach the breakpoint value and thus the evaporation would not reach the potential rate at any time during the year. For values of \(\frac{\bar{P}}{PE}\) greater than \([R_0 + (1 + r)]\) the soil would remain saturated and the evaporation would be at potential rate throughout the year. At each of these two points there would be a discontinuity and between them the value of \(\frac{\Delta E}{PE}\) can be derived either analytically involving quadrature or by numerical simulation. The result obtained by

![Fig. 4. Effect of length of dry season on evaporation.](image-url)
numerical simulation for the standard value of the parameters $R_0 = 1$ and $c = 4$ is shown in Fig. 3 for the case of $r = 0.7$ which was the value chosen by Schaake and Liu on the basis of US data [23,22]. This figure shows that the temperate climate resembles the non-seasonal case more closely than it does the tropical case. This case of the temperate climate reduces to the non-seasonal case as $r$ approaches zero.

8. Sensitivity to long-term precipitation

We are now in a position to derive the sensitivity ratio of catchment runoff as defined by Eq. (3). For cases where an analytical solution is available for the evaporation function defined by Eqs. (2a) and (2b), the value of $\Psi_T(P/PE)$ is derived in Appendix A as Eq. (A.5)

$$\Psi_T(P/PE) = \frac{P/PE (1 - \phi'(P/PE))}{(P/PE - \phi(P/PE))}. \quad (A.5)$$

Where it is more convenient to use a computer simulation based on the soil moisture Eqs. (17a) and (17b), the sensitivity can be determined for any values of $P/PE$ by increasing $P$ to $P + \Delta P$ and calculating the value of $\Delta PE$ for the climate involved. The sensitivity factor is then given by

$$\Psi_T(P/PE) = 1 - \frac{\Delta AE/\Delta P}{1 - AE/P} \quad (28)$$

for each value of $(P/PE)$. For the climates examined two areas of particular sensitivity appear. One is in the neighbourhood of $(P/PE)$ equal to zero and the second one in the neighbourhood of the breakpoint humidity ratio.

For the case of the non-seasonal climate, the values of $\Psi_T$ at lower humidity ratios can readily be derived from the implicit evaporation function given by Eq. (23) as

$$\Psi_T(P/PE) = \frac{c}{1 + R_0 \left(\frac{AE/PE}{c - 1}\right)} \quad (29a)$$

with the limiting values

$$\Psi_T(0) = c \quad (29b)$$

at the extreme arid limit and

$$\Psi_T(P/PE)_{\text{b1}} = \frac{(1 + R_0)c}{1 + R_0c} \quad (29c)$$

at the breakpoint which is defined by Eq. (20a):

$$(P/PE)_{\text{b1}} = 1 + R_0.$$  

For higher humidity ratios, there is a discontinuity in the value of $\Psi$ which rises from the value given by Eq. (29c) to

$$\Psi_T(P/PE)_{\text{b2}} = 1 + \frac{1}{R_0} \quad (30a)$$

and then declines thereafter in accordance with

$$\Psi_T(P/PE) = \frac{P/PE}{P/PE - 1} \quad (30b)$$

The relative magnitudes of the parameters $c$ and $R_0$ will determine whether the breakpoint maximum, given by Eq. (30a), or the maximum at the zero humidity ratio, given by Eq. (29b), is the greater.

For the case of the tropical climate the relation of the sensitivity parameter $\Psi$ to the humidity ratio $(P/PE)$ is quite similar to that for the non-seasonal climate. The result can be derived analytically but the algebra is cumbersome. Fig. 5 shows the comparison for the two climates for the case of $c = 4$ and $R_0 = 1$. The differences are seen to be small. For very low values of the humidity ratio $(P/PE)$ the values of the sensitivity factor $\Psi_T$ for the tropical and temperate climates approach the value for the non-seasonal climate as given by Eqs. (29a)–(29c). For values of the humidity ratio higher than the breakpoint the sensitivity factor for the tropical climate is somewhat lower than that given for the non-seasonal climate by Eqs. (30a) and (30b).

Fig. 6 shows the effect of variation in the length of the dry season $T$ on the sensitivity ratio $(\Psi_T)$ for the tropical climate. The figure shows that the sensitivity factor $(\Psi_T)$ decreases as the relative duration of the dry season $(T)$ increases for a given value of the humidity ratio $(P/PE)$. The main differences between the three climate types are seen to occur for the case of semi-arid regions. The effect of $T$ on the breakpoint value of the humidity ratio is to be expected since, over a wide range of values of $R_0$ and $\Psi_T$, the breakpoint value of $(P/PE)$ is equal to $2(1 - T)$. The effect of the parameter $R_0$ on the sensitivity at the breakpoint maximum for a tropical climate might be expected to be similar to that for the non-seasonal climate given by Eq. (30a), which could be considerable particularly for $R_0$ less than 1.

For the temperate climate, the procedure is similar. In this case, the concentrated breakpoint discontinuity of the non-seasonal case is smoothed out into the intermediate range from $[R_0 + (1 - r)]$ to $[R_0 + (1 + r)]$ with a smaller discontinuity at each end. At the breakpoint maximum, the value of the sensitivity factor for the temperate climate is less than the value for the tropical climate which in turn is approximately equal to the value for a non-seasonal climate. The comparative results for the variation of the sensitivity factor $(\Psi_T)$ with the humidity ratio $(P/PE)$ are shown in Fig. 5.

For the non-seasonal climate case, it can readily be shown that for a humidity ratio $(P/PE)$ greater than the breakpoint value given by Eq. (20a), the relationship of complementarity between $\Psi_T$ and $\Psi_{PE}$ given by Eq. (5) will still hold. For humidity ratios below the breakpoint, the relationship is no longer valid because of the occurrence in Eq. (23) of the parameter $R_0$ which, as defined in Eq. (21), is dependent on the value of $PE$. Similarly for the other two cases of tropical climate and
temperate climate it can be shown that for humidity ratios greater than the appropriate breakpoint values, the complementarity property of Eq. (5) will still hold. However, this does not apply for humidity ratios below the breakpoint where the evaporation ratio \( AE/PE \) is not a constant.

These results are compatible with the semi-empirical results of Schaake & Liu who derived elasticities (i.e. sensitivity factors) for the Central and Eastern US [23,22]. Representative values of \( WP \) based on the results of their analysis as plotted on maps in [22] are shown for comparative purposes in Fig. 5. These values show the same functional relationship between sensitivity and aridity as the model over the range of Schaake’s data.

9. Typical values of the key parameters

The simple model represented by Eqs. (17a) and (17b) contains two critical parameters: (1) the parameter \( c \) in the assumed power relationship between hydraulic conductivity and moisture content and (2) the parameter \( R_0 \) defined by which represents the ratio of groundwater recharge to evapotranspiration at the “field capacity” \( W_B \). The value of \( c \) may be based on the tabulations on the literature for such models of soil moisture properties as those due to [4] and to [27]. The recharge/evaporation ratio is given by Eq. (20b):

\[
R_0 = \left( \frac{K_{sat}}{PE} \right) \left( \frac{W_B}{W_{sat}} \right)^c.
\]

Here, the first factor on the right-hand side \( (K_{sat}/PE) \) represents the ratio of groundwater recharge to surface evaporation when the soil is saturated, the second factor represents the adjustment between field capacity and saturation and the exponent \( c \) is the same as that used in the evaporation term of Eqs. (2a) and (2b). The values of \( W_{sat} \) will be given by the product of the soil porosity and the depth of the soil column. The value of \( W_B \) above
which the evaporation is at potential rate is usually specified as a given fraction of $W_{\text{sat}}$. The traditional formulae relating permeability to soil moisture content based on the Brooks–Corey approach [4] are in the form of a power relationship

$$\frac{K}{K_{\text{sat}}} = \left( \frac{W}{W_{\text{sat}}} \right)^c, \quad (31)$$

where the parameters $K_{\text{sat}}$ and $c$ depend only on the soil type.

Typical values of $K_{\text{sat}}$ and $c$ have been tabulated by a number of researchers and the values in Table 1 are those recommended by Entekhabi [3].

The intermediate value of $c = 4$ is taken in the examples on Figs. 3–6.

For the case of the non-seasonal climate, we have from Eqs. (29a)–(29c) above the limiting value of the sensitivity factor ($\Psi_\pi$) for a vanishingly small value of the humidity ratio ($\overline{\text{P}}/\overline{\text{PE}}$),

$$\Psi_\pi(0) = c$$

so that this arid maximum is on the basis of the simplified model completely determined by the type of soil. The numerical results suggest that the same is true for the simplified temperate climate defined by Eqs. (27a)–(27c) and for the simplified tropical climate defined by Eqs. (26a)–(26c). For values of the humidity ratio greater than 0.1, the values of the other two simple climates are smaller than those for the non-seasonal climates.

The values of the sensitivity factor ($\Psi_\pi$) for the range of humidity ratio ($\overline{\text{P}}/\overline{\text{PE}}$) normally encountered depend critically on the value of $R_0$ as defined by Eq. (21) above which is heavily dependent on the value of the break-point parameter ($W_{\text{b}}/W_{\text{sat}}$) used in Eqs. (14a) and (14b) and later equations dependent on it. The variation of $R_0$ with this ratio and with soil is shown in Table 2 below on the basis of a value of $\overline{\text{PE}} = 1000$ mm/yr and values of $K_{\text{sat}}$ and $c$ taken from Table 1.

### Table 1

<table>
<thead>
<tr>
<th>Soil</th>
<th>$K_{\text{sat}}$ (mm/year)</th>
<th>$c$ (dimensionless)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clay</td>
<td>$11 \times 10^3$</td>
<td>7.5</td>
</tr>
<tr>
<td>Silty loam</td>
<td>$107 \times 10^3$</td>
<td>4.7</td>
</tr>
<tr>
<td>Sandy loam</td>
<td>$1072 \times 10^3$</td>
<td>3.6</td>
</tr>
<tr>
<td>Sand</td>
<td>$2712 \times 10^3$</td>
<td>3.4</td>
</tr>
</tbody>
</table>

This shows that the value of the parameter $R_0$ is sensitive to both the type of soil and the value assumed for $\beta = W_{\text{b}}/W_{\text{sat}}$.

For present purposes the value of $W_{\text{b}}/W_{\text{sat}} = 0.75$ used by Manabe in his original simple land coupling of a GCM is adopted [16,11]. For this case and for the values of the sensitivity factor ($\Psi_\pi$) for the usual range of the values of the humidity ratio,

$0 < \left( \frac{\overline{\text{P}}}{\overline{\text{PE}}} \right) < 3$,

are given for the non-seasonal climate in Table 3.

The only type of soil for which a discontinuity in $\Psi_\pi$ due to the occurrence of a breakpoint occurs within the range of Table 3 is the clay soil where there is a breakpoint at $\left( \frac{\overline{\text{P}}}{\overline{\text{PE}}} \right) = 1 + R_0 = 2.27$ where it rises sharply from 1.62 to 1.73. It is interesting to note that this is close to the value of $\left( \frac{\overline{\text{P}}}{\overline{\text{PE}}} \right) = 2.25$ generally recognised as the separation value between a forest biome and a tundra biome.

### 10. Conclusions

1. For the case where actual long-term evaporation ($\overline{AE}$) is a function only of the long-term precipitation ($\overline{P}$) and the long-term potential evaporation ($\overline{PE}$), the sum of the sensitivity factor due to changes in $\overline{P}$ ($\Psi_\pi$) and the sensitivity factor due to changes in $\overline{PE}$ ($\Psi_{\overline{PE}}$) is always equal to unity. This complementarity of Eq. (5) holds for the empirical evaporation formulae commonly used and for the non-seasonal case of soil moisture accounting. For the other two simple climates studied this complementarity existed only for humidity ratios higher than the breakpoint ratio.

2. For a non-seasonal climate, the relation of ($\overline{AE}/\overline{PE}$) to $\left( \frac{\overline{P}}{\overline{PE}} \right)$ was a monotonically rising curve concave downwards followed by an almost horizontal line with $\left( \frac{\overline{AE}}{\overline{PE}} \right) = 1$ after the breakpoint (Fig. 3).

3. For a simplified tropical climate, the relation of ($\overline{AE}/\overline{PE}$) to $\left( \frac{\overline{P}}{\overline{PE}} \right)$ was a monotonically rising curve concave downwards followed by an almost horizontal line with $\left( \frac{\overline{AE}}{\overline{PE}} \right) < 1$ after the breakpoint (Fig. 3).

4. For a simplified temperate climate the relation between ($\overline{AE}/\overline{PE}$) and $\left( \frac{\overline{P}}{\overline{PE}} \right)$ was a monotonically rising curve concave downwards with a discontinuity at the breakpoint followed by a horizontal line with $\left( \frac{\overline{AE}}{\overline{PE}} \right) = 1$ (Fig. 3).
For all three of the simplified climates studied, two areas of special interest emerge: (1) the neighborhood of zero humidity index where the sensitivity factor \((\Psi_P)\) approaches asymptotically a constant value and (2) the neighborhood of the breakpoint humidity ratio where the sensitivity factor shows an abrupt rise in value (Fig. 5).

The critical values of the sensitivity factor \((\Psi_P)\) for these two key values of the humidity ratio can be determined from Eqs. (17a) and (17b) either numerically or analytically for all three simple climates.

(6) The critical values of the sensitivity factor \((\Psi_P)\) for the value of parameter \(c\) which is the exponent relating unsaturated hydraulic conductivity to the soil moisture content and depends only on soil type. For the same case, the peak sensitivity value corresponding to the neighbourhood of the breakpoint humidity ratio can be evaluated if \(\Phi[P/PE]\) is known.

For a perturbation in \(PE\), we have for the special case of Eq. (2b) where the catchment parameters do not involve \(PE\),

\[
\frac{\Delta Q}{\Delta PE} = \frac{P}{PE} \Phi' \left( \frac{P}{PE} \right) \Phi \left( \frac{P}{PE} \right) \frac{\Delta PE}{PE},
\]

The relative perturbation is given by

\[
\Psi_P = \frac{P}{PE} \Phi' \left( \frac{P}{PE} \right) \Phi \left( \frac{P}{PE} \right) \left( \frac{P}{PE} - \Phi \left( \frac{P}{PE} \right) \right).
\]

Addition of Eqs. (A.5) and (A.8) gives the result \(\Psi_P + \Psi_{PE} = 1\) so that, for cases where the catchment parameters do not involve \(P\) or \(PE\), only a single sensitivity factor is involved. If any of the parameters involves either \(P\) or \(PE\), then the complementarity relationship is no longer valid.

Appendix A

The long-term catchment water balance defined by Eq. (1) in the main text can be written as

\[
\bar{Q} = \bar{P} - \bar{AE}
\]

so that we can write, for the case of Eq. (2a),

\[
\bar{Q} = \bar{P} - \bar{PE} \Phi \left[ \frac{P}{PE}, \Phi \left( \frac{P}{PE} \right), \text{catchment parameters} \right].
\]

For a perturbation in \(P\) we have

\[
\frac{\Delta \bar{Q}}{\Delta \bar{P}} = 1 - \Phi' \left( \frac{P}{PE} \right)
\]

where \(\Phi'(x)\) denotes the derivative of the function \(\Phi(x)\) with respect to \(P\). Eq. (A.3a) can also be written as

\[
\Delta \bar{Q} = \bar{P} \left[ 1 - \Phi' \left( \frac{P}{PE} \right) \right] \frac{\Delta P}{P}.
\]

The relative perturbation in \(\bar{Q}\) is therefore given by

\[
\frac{\Delta \bar{Q}}{\bar{Q}} = \frac{P}{\left( P - PE \Phi \left( \frac{P}{PE} \right) \right)} \frac{\Delta P}{P}
\]

so that the sensitivity to \(P\) in Eq. (3) of the main text is given by

\[
\Psi_P = \frac{P}{PE} \left[ 1 - \Phi' \left( \frac{P}{PE} \right) \right] \left( \frac{P}{PE} - \Phi \left( \frac{P}{PE} \right) \right).
\]
and $P(t)$ can be estimated by Eq. (A.10) or otherwise. By analogy with the solution of Richards equation for constant hydraulic diffusivity $D$, this can be written

$$P(t) = \left( \frac{W_{sat} - W_T}{W} \right) \cdot \sqrt[3]{\frac{D}{\pi}}, \quad (A.12a)$$

where $W_T$ is the soil moisture content at end of the dry period $T$ and

$$D = -K \frac{\partial h}{\partial W}; \quad (A.12b)$$

which can be estimated for any given model of soil moisture behaviour. In the present paper, this case is neglected and Eq. (17a) simplified to Eq. (17b). This means that the model does not allow for Hortonian overland flow.

References


