Analytical and semi-analytical solutions of horizontal well capture times under no-flow and constant-head boundaries

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Abstract

The closed-form analytical solutions and semi-analytical solutions of capture times to horizontal wells are derived for different recovery scenarios. The capture time is the time a fluid particle takes to flow to the well. The first scenario is recovery from a confined aquifer in which the influence of regional groundwater flow upon the capture time is included. The second scenario is recovery from underneath a water reservoir in which the top boundary of the aquifer is constant-head. The third scenario is recovery from a low-permeability layer bounded above and below by much higher permeability media. Closed-form solutions are provided for the cases with: (1) a center or a bottom well for the first scenario; (2) a bottom well for the second scenario; and (3) a center well for the third scenario. Semi-analytical solutions are provided for general well locations for those scenarios. Solutions for both isotropic and anisotropic media are studied. These solutions can be used as quick references to calculate the capture times, and as benchmarks to validate numerical solutions. The limitations of the analytical solutions are analyzed. Our results show that the top and bottom no-flow boundaries of an aquifer constrain the vertical flow, but enhance the horizontal flow, resulting in elongated iso-capture time curves. When constant-head boundaries are presented, water can infiltrate vertically across those boundaries to replenish the aquifers, resulting in less elongated iso-capture time curves. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Horizontal well; Capture time; Regional flow; Potential theory; Analytical solution.

1. Introduction

Horizontal wells have been used extensively in the petroleum industry for oil and gas production in the past decade [15,21]. Extensive studies on pressure analysis of a horizontal well have been done in petroleum engineering, and these studies are useful for interpreting horizontal well pumping tests [5,7,13,18].

Historically, the cost of installing a horizontal well is significantly higher than that of a vertical well, but this cost has decreased with the advancement of new drilling technologies [11,12,16]. Horizontal wells have advantages in certain situations of groundwater contamination recovery. For instance, they can provide: (1) improved access to subsurface contaminant at sites with surface restrictions (e.g., landfills, lagoons, buildings, wetlands, lakes, utility lines, tanks), (2) minimal surface disturbance, and (3) increased surface-area contact with contaminants.

The reduced operational cost, combined with the advantages in recovery situations has led to increased utilization of horizontal well technology in the hydrological and engineering applications in recent years [4,14,23,24,27,28]. Horizontal wells are also commonly utilized in air sparging and soil vapor extraction to recover volatile organic contaminants.

Fluid and gas flows to a horizontal well or trench have been studied before. An early study of fluid flow into collector wells was done by Hantush and Papadopoulos [10] who provided analytical solutions for the drawdown distribution around collector wells. Tarshish [27] constructed a mathematical model of flow in an aquifer with a horizontal well located beneath a water reservoir. Falta [6] has developed analytical solutions of the transient and steady-state gas pressure and the steady-state stream function resulting from gas injection and extraction from a pair of parallel horizontal wells. Murdoch [17] has discussed the three-dimensional transient analyses of an interceptor trench; his results can be readily extended to horizontal wells. Parmentier and Klemovich [20] have pointed out that the contact area of a single horizontal well may equal that of ten
vertical wells. Rushing [23] has developed a semianalytical model for horizontal well slug testing in confined aquifers.

Despite previous research of horizontal well hydraulics in petroleum engineering, agricultural engineering, and hydrological engineering, several important problems related to groundwater hydrology still exist. One of them is the analytical determination of the time a fluid particle takes to flow to a horizontal well; that is, the capture time. These analytical solutions are very useful for quick assessments of recovery efficiency and can be used to validate the numerical solutions. Analytical solutions also provide better physical insights into the problem than the numerical solutions. Zhan [29] has derived a closed-form analytical solution of capture time to a horizontal well in semi-analytical studies [d]

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Definition</th>
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<tr>
<td>a</td>
<td>$a = 2\pi/d$, a parameter defined in Eq. (8) [m$^{-1}$]</td>
</tr>
<tr>
<td>$a_e$</td>
<td>$a_e = 2\pi/d_e$, effective “a” defined in Eq. (46) [m$^{-1}$]</td>
</tr>
<tr>
<td>A</td>
<td>integration defined in Eq. (A.5) [m]</td>
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<tr>
<td>b</td>
<td>$b = 2\psi_0/m$, a parameter defined in Eq. (8) [dimensionless]</td>
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<tr>
<td>$b_e$</td>
<td>$b_e = 2\psi_{b0}/m$, effective “b” defined in Eq. (46) [dimensionless]</td>
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<tr>
<td>B</td>
<td>integration defined in Eq. (A.5) [m]</td>
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<tr>
<td>c</td>
<td>$c = 2q_0/m$, a parameter defined in Eq. (8) [m$^{-1}$]</td>
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<tr>
<td>$c_e$</td>
<td>$c_e = 2q_{e0}/m$, a parameter defined in Eq. (46) [m$^{-1}$]</td>
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<tr>
<td>d</td>
<td>$d = d/2^{1/4}$, effective aquifer thickness defined in Eq. (44) [m]</td>
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<tr>
<td>$d_{wD}$</td>
<td>dimensionless $d_w$</td>
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<tr>
<td>$d_w$</td>
<td>distance from a horizontal well to an aquifer’s bottom boundary [m]</td>
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<tr>
<td>$d_{wD}$</td>
<td>dimensionless $d_w$</td>
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<tr>
<td>$e^{i}$</td>
<td>$i = \sqrt{-1}$</td>
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<td>j</td>
<td>integral numbers, $-2$, $-1$, $0$, $1$, $2$</td>
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<td>$K_e$</td>
<td>$K_e = \sqrt{K/K_y}$, effective hydraulic conductivity [m/d]</td>
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<tr>
<td>$K_x$</td>
<td>principal hydraulic conductivity in $x$-direction [m/d]</td>
</tr>
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<td>$K_y$</td>
<td>principal hydraulic conductivity in $y$-direction [m/d]</td>
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<td>L</td>
<td>horizontal well screen length [m]</td>
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<tr>
<td>$L_{wD}$</td>
<td>dimensionless well screen length defined in Eq. (2)</td>
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<tr>
<td>$m$</td>
<td>$m = Q/2\pi$</td>
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<td>$M$</td>
<td>truncation number used in the semi-analytical solutions</td>
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<td>n</td>
<td>effective porosity</td>
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<td>$q_0$</td>
<td>regional specific discharge along $-y$-direction [m$^3$/d]</td>
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<td>$q_{e0}$</td>
<td>$q_{e0} = x^{1/4}q_0$, effective regional specific discharge along $-y$-direction [m$^3$/d]</td>
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<tr>
<td>$q_{0x}$</td>
<td>regional specific discharge along the $+x$-direction [m$^3$/d]</td>
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<tr>
<td>Q</td>
<td>pumping rate per unit screen length of well ($Q &gt; 0$ for pumping) [m$^3$/d]</td>
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<tr>
<td>$S_s$</td>
<td>specific storativity [m$^{-1}$]</td>
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<td>t</td>
<td>time [d]</td>
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<tr>
<td>$t_D$</td>
<td>dimensionless time defined in Eq. (2)</td>
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<td>T</td>
<td>capture time to a horizontal well [d]</td>
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<td>$T_m$</td>
<td>approximation of capture time to a horizontal well in semi-analytical studies [d]</td>
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<tr>
<td>u</td>
<td>$u = x + iy$</td>
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<tr>
<td>$V_x$</td>
<td>$x$-component of groundwater flow velocity [m/d]</td>
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<td>$V_y$</td>
<td>$y$-component of groundwater flow velocity [m/d]</td>
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<tr>
<td>x</td>
<td>$x$-coordinate of the starting point [m]</td>
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<tr>
<td>$x_0$</td>
<td>$x_0 = (K_e/K_y)^{1/2}x$, effective $x$ defined in Eq. (44) [m]</td>
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<tr>
<td>$x_{0e}$</td>
<td>$(K_e/K_y)^{1/2}x_0$, effective $x_0$ [m]</td>
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<tr>
<td>$x_{1+}$</td>
<td>discretization of $x$ along the streamline [m]</td>
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<td>$x_{p1}$</td>
<td>$x_{p1}$, $x_{p2}$, dummy parameters used in Section 2.2</td>
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<tr>
<td>$y_0$</td>
<td>$y$-coordinate of the starting point [m]</td>
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<tr>
<td>$y_{0e}$</td>
<td>$(K_e/K_y)^{1/2}y_0$, effective $y_0$ [m]</td>
</tr>
<tr>
<td>$y_{1+}$</td>
<td>discretization of $y$ along the streamline [m]</td>
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<tr>
<td>$y_{1+}$</td>
<td>$y_{1+}$, $y_{1+}$, dummy parameters used in Eq. (A.6)</td>
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<tr>
<td>z</td>
<td>$z = z_{x}$, dimensionless $z$-coordinate</td>
</tr>
<tr>
<td>$z_D$</td>
<td>$z_D = x$, horizontal coordinate along the well axis [m]</td>
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<tr>
<td>$\Delta S$</td>
<td>$\Delta S$, drawdown [m]</td>
</tr>
<tr>
<td>$\beta_1$, $\beta_2$</td>
<td>dummy parameters used in Section 2.2</td>
</tr>
<tr>
<td>$\gamma_1$, $\gamma_2$</td>
<td>dummy parameters used in Eq. (A.6)</td>
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<tr>
<td>$\zeta$</td>
<td>$\zeta = \phi + i\psi$, complex function used to describe the steady-state flow [m$^2$/d]</td>
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<tr>
<td>$\phi$</td>
<td>real part of $\zeta$ [m$^2$/d]</td>
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<tr>
<td>$\psi$</td>
<td>imaginary part of $\zeta$ [m$^2$/d]</td>
</tr>
<tr>
<td>$\psi_0$</td>
<td>the streamline function defined by point $(x_0, y_0)$ [m$^2$/d]</td>
</tr>
<tr>
<td>$\psi_{0e}$</td>
<td>the effective streamline function defined by $(x_{0e}, y_{0e})$ [m$^2$/d]</td>
</tr>
<tr>
<td>$\psi_M$</td>
<td>approximation of the streamline function in the semi-analytical studies [m$^2$/d]</td>
</tr>
<tr>
<td>$\psi_M^{0e}$</td>
<td>approximation of $\psi_0$ in the semi-analytical studies [m$^2$/d]</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\omega = \cos(ax/4)$, a dummy parameter used in Appendix C</td>
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</table>
regional flow. This paper, however, will study the more general case including the regional flow. It will also study the cases with constant-head boundaries, which are useful when a horizontal well is in an aquifer underneath a water reservoir or wetland, or the horizontal well is in an aquitard bounded by two high-permeability layers.

2. Capture time to a horizontal well

2.1. Problem description

Fig. 1 is the schematic diagram of a horizontal well pumping in a confined aquifer. The lateral boundaries are sufficiently far and do not influence the hydraulic head. The center of the well is the origin of the coordinates. \( d_w \) is the distance to the bottom boundary, \( L \) the well length, and \( d \) is the aquifer thickness. For simplicity, we assume a homogeneous, isotropic medium with a hydraulic conductivity \( K \) and a specific storativity \( S_s \).

Flow to a finite horizontal well includes three stages: an early, nearly radial flow, an intermediate flow, and a late horizontal pseudoradial flow (Fig. 1). At the early time, the influences of the top and bottom boundaries are not perceptible, thus the flow is radial in a plane perpendicular to the well axis. Daviau et al. [5] (p. 717) showed that the top/bottom boundaries were perceptible roughly when dimensionless time \( t_D \) was

\[
t_D \approx \frac{0.32(d_w/L_D)^2}{1}, \quad \text{if } d_w/L_D \leq 0.5;
\]

\[
t_D \approx \frac{0.32((1 - d_w)/(L_D))^2}{1}, \quad \text{if } d_w/L_D > 0.5,
\]

(1)

where \( t_D, d_w, \) and \( L_D \) were

\[
t_D = \frac{K \cdot t}{S_s(L/2)} \frac{d}{d}, \quad d_w = \frac{d_w}{d}, \quad L_D = \frac{L}{2},
\]

(2)

and \( t \) was the time. The end effect was noticeable when \[5, \text{p. 717}\]

\[
t_D \approx (1 - z_D)^2/6,
\]

(3)

where the dimensionless \( z_D = z/(L/2) \). We should emphasize that Eqs. (1) and (3) only give orders of magnitude estimations. Other authors may use different formulae to estimate these times [17, p. 3027].

After the end of the early period, the flow enters the intermediate stage which has a complicated pressure behavior as described by Goode and Thambynayagam [7], Daviau et al. [5], and Ozkan et al. [18]. The ending time of the intermediate stage is approximately \( 0.8 < t_D < 3 \) [5, p. 718]. Once again, this result only gives order of magnitude estimation, and different authors may use slightly different formulae [17, p. 3027].

After the intermediate period, the flow enters the pseudoradial flow stage during which the equipotential surfaces are similar to vertical cylinders in the far field (Fig. 1). This situation is similar to a fully penetrated, large diameter vertical well [7,17].

2.2. Analytical solution of capture time to a horizontal well in a confined aquifer

Flow around a finite-length horizontal well is generally a three-dimensional problem owing to the influence of well ends. General study of capture time to a horizontal well can only be calculated numerically. The advantage of a numerical method is the feasibility of handling complicated boundary geometries, transient flow, and finite length of well. The disadvantage is the inevitable truncation error and the requirement of very small grid sizes to simulate the region near the wellbore. Analytical solutions, however, can provide benefits, which cannot be gained through other means. For instance, they provide quick references for calculating the
capture times, they are useful for validating the numerical codes, and they provide better physical insights, which may be useful in further investigations. However, analytical solutions also have their limitations. Some assumptions are necessary to have closed-form solutions. In the following analytical studies, we assume steady-state flow and an infinitely long horizontal well. It is important to justify these assumptions before the use of the analytical solutions.

1. First of all, the assumption of an infinitely long horizontal well is correct only when the aquifer is confined by two impermeable vertical walls which are perpendicular to the axis of well. This is possible in a field application in which two barriers are built to confine a contaminated plume. For a finite length well, the ends of the well will influence the pressure distribution and flow becomes three-dimensional. However, if the well is sufficiently long, the analytical solutions are expected to be approximately correct in the regions near the well center where the influences of the ends are the least. Considering the fact that many environmental horizontal wells use long screen lengths to intercept large areas of groundwater, the infinitely long well approximation is reasonable for estimating the near field capture time.

Daviau et al. [5] have analyzed the transient pressure change along the horizontal wellbore and their result showed that (Fig. 2 in their paper) with a well length \( L = 2d \), when \( |z|/L \leq 1/4 \), the well length has a negligible influence on the pressure, where \( z \) is the distance to the well center (see Fig. 1). Thus, we can approximately use \( L/4 \) as the scale of the near field, i.e., when the distance of the point to the well center is less than a quarter of the well length, that point is called a near field point and we can use the infinite well assumption.

2. When a horizontal well is installed in a confined aquifer, Murdoch [17] has pointed out that the flow around a horizontal well is better described as a transient process and the duration of the transient period is proportional to the square of well length (see \( t_D \) in Eq. (2)). This is indeed true if we check the transient pressure along the wellbore in Fig. 2 of Daviau et al. [5], which showed that drawdowns continuously increased when pumping continued.

However, as flow enters the pseudoradial flow, Ozkan et al. [18] (Eq. 11) showed that the increase of drawdown with time was uniform across the domain and had a simple form \( \Delta S = \beta_1 \times (\ln t_D + \beta_2) \), where \( \Delta S \) is the drawdown, \( \beta_1 \) a constant determined by the permeability, viscosity, and pumping rate, and \( \beta_2 \) is a constant related to \( d_{ed} \), \( L_D \), and the coordinates of the point where \( \Delta S \) was measured. Therefore, the groundwater flow velocity, which depends on the hydraulic head gradient, not on hydraulic head itself, is independent of time in the pseudoradial flow. Thus the steady-state assumption can be used after flow enters the pseudoradial flow.

3. If constant-head exists at the top and/or bottom boundaries, steady-state drawdowns can soon be established around 0.8 < \( t_D \) < 3 after the early radial flow [5,13]. That is, because water can flow across the constant-head boundaries to supplement the aquifers soon after pumping starts, therefore, the demand of water from lateral directions is reduced and the drawdowns will approach steady state. Thus the steady-state assumption is reasonable when constant-head boundaries exist.

Based on the above justification, we will make use of the steady-state and infinitely long well assumptions to calculate the capture times for near field points when flow enters the pseudoradial stage in a confined aquifer, or steady-state stage when constant-head boundaries are presented.

The solution for a confined, isotropic medium is first studied. Fig. 2 is the cross-section of an infinitely long horizontal well located at the center of a homogeneous, isotropic, confined aquifer. The \( x \)- and \( y \)-axes are perpendicular to and parallel with the confining bed, respectively. The well is located at \((x, y) = (0, 0)\). Groundwater regional flow direction can be arbitrary in
the \(yz\)-plane. However, the \(z\)-component of the regional flow (along the well axis) will only cause a shift of a fluid particle along the \(z\)-direction. As long as such a shift does not affect the assumption of an infinitely long well, i.e., the fluid particle is still far from the well ends, the \(z\)-component of the regional flow does not influence the fluid particle’s movement in the \(xy\)-plane and the capture time. Only the \(y\)-component of the regional flow affects the capture time. In the following discussion, we assume that the \(y\)-component of the regional flow is along the \(-y\)-axis with a specific discharge \(q_0\) (see Fig. 2).

To simulate the steady-state flow in this region, an infinite number of image wells with the identical pumping rate as the original well are assigned along the \(x\)-axis with locations at \(\pm jd\) (\(j = 1, 2, 3, \ldots\)) The effect of image wells is equivalent to that of the top and bottom no-flow boundaries. The image well method has been broadly employed in studying groundwater flow \([1,4,27]\) and heat transfer \([3]\) problems in bounded domains. Using the potential theory \([1]\), a complex function \(\zeta(u) = \phi + i\psi\) is assigned to describe the flow

\[
\zeta(u) = m \sum_{j=-\infty}^{\infty} \ln[u - jd] - iq_0u, \tag{4}
\]

where the real part, \(\phi\) and the imaginary part, \(\psi\) of \(\zeta(u)\) are the equipotential and streamline functions, respectively; \(m = Q/2\pi, Q\) is the pumping rate per unit screen length of the well \((L/2T)\) \((Q > 0\) means a pumping well and \(Q < 0\) means an injecting well); \(u = x + iy\); and \(i = \sqrt{-1}\). Bear \([1]\) (Eqs. (7.8.31)) gave a simplified expression to the first term of Eq. (4) without providing the details of derivation, and his (Eqs. (7.8.31)) had an inaccurate negative sign. Zhan \([29]\) has calculated the summation term of Eq. (4) and the result is

\[
\zeta(u) = m \ln \left[ \sin \left( \frac{\pi u}{d} \right) \right] - iq_0u, \tag{5}
\]

\(\phi\) and \(\psi\) are

\[
\phi = \frac{m}{2} \ln \left( \frac{1}{2} \left[ \cosh \frac{2\pi y}{d} - \cos \frac{2\pi x}{d} \right] + q_0y; \tag{6}
\]

\[
\psi = m \tan \left[ \frac{1}{\tan (\pi y/d)} \right] - q_0x.
\]

The \(x\)-component of flow velocity is \(V_x = -(1/n)(\partial \psi / \partial x)\), where \(n\) is the effective porosity. The capture time \(T\) for a water particle to flow from \((x_0, y_0)\) into the horizontal well is

\[
T = \left| \int_{x_0}^{x_0 + d/V_x} dx \right| = n \int_{0}^{e_0} \frac{dx}{\partial \phi / \partial x}, \tag{7}
\]

where \(|\cdot|\) is the sign of absolute value. Given the starting point \((x_0, y_0)\), the stream function \(\psi_0 = \psi(x_0, y_0)\) describes the flow path. Assign constants

\[
a = 2\pi/d, \quad b = 2\psi_0/m, \quad c = 2q_0/m. \tag{8}
\]

For any point \((x, y)\) in the flow path started from \((x_0, y_0)\), \(\psi(x, y) = \psi(x_0, y_0)\), thus the stream function in Eq. (6) becomes

\[
\tanh(ay/2) \tan(ax/2) = \tan \left( \frac{b + cx}{2} \right). \tag{9}
\]

The presumption of using Eq. (7) is that \(x \neq 0\), otherwise \(V_x = 0\). Thus special care should be taken when \(x \rightarrow 0\). If \(x \rightarrow 0\), keeping the first-order approximation of Eq. (9), we have

\[
\tanh(ay/2) ax/2 = \tanh(b/2) + cx/2 \quad 1 - \tanh(b/2)cx/2. \tag{10}
\]

Eq. (10) is simplified into

\[
\lim_{x \rightarrow 0} \frac{\tan(b/2)}{a + c \cdot \tanh(ay/2)} = 1. \tag{11}
\]

Eq. (11) is employed to calculate the capture time when \(x_0 \rightarrow 0\) (see the following case 2).

Using Eqs. (6), (7) and (9), we are able to derive closed-form analytical solutions of capture time. The mathematical details are shown in Appendix A. The result is

\[
T = \frac{2n}{ma} \left\{ \frac{1}{a + c} \ln \left[ \frac{\cos \left( \frac{a + c}{{2}} \right) + b}{\cos \left( \frac{b}{{2}} \right)} \right] \right\}.
\]

Special cases are those with \(a + c \rightarrow 0\) or \(a - c \rightarrow 0\) in using Eq. (12). For those cases, Eq. (12) can be simplified by invoking Eq. (A.8) or (A.10) (see Appendix A). Now we give an example to use these formulae. The following parameters are given: the aquifer thickness \(d = 20\) m, hydraulic conductivity \(K = 10\) m/d, effective porosity \(n = 0.25\), pumping rate per unit screen length \(Q = 15\) m/d, the regional groundwater discharge \(q_0 = 0.1\) m/d along the \(-y\)-axis, the starting point \((x_0, y_0) = (5, 10)\) m. Thus \(m = Q/2\pi = 2.387\) m/d. Substituting \(m, x_0, y_0, d,\), and \(q_0\) into Eq. (6) to obtain \(\psi_0 = 1.2716\) m/d. Substituting \(d, m, \psi_0,\) and \(q_0\) into Eq. (8) to obtain \(a = 0.3142\) m\(^{-1}\), \(b = 1.0654\), and \(c = 0.08379\) m\(^{-1}\). Substituting \(n, m, a, b, c,\) and \(x_0\) into Eq. (12) results in the capture time from \((x_0, y_0)\) to the well \(T = 4.858\) d.

A few special cases of Eq. (12) deserve discussion.

Case 1: No regional flow. When there is no regional flow, \(q_0 = 0\) (i.e., \(c = 0\)), as derived in Appendix B, the capture time becomes

\[
T = \frac{2n}{ma} \left\{ \frac{1 + \cosh(ay_0)}{1 + \cosh(ax_0)} \right\}. \tag{13}
\]

When \(x_0 = d/2, ax_0 = \pm \pi\) and Eq. (13) gives \(T \rightarrow \infty\). This simply reflects the fact that the particles at the top
or bottom boundaries will not be captured by the well. Eq. (13) is the result derived in Zhan [29] excluding the regional flow.

Case 2: Starting point is close to x-axis. \((x_0 \rightarrow 0)\).
When \(x_0 \rightarrow 0\), using Eq. (11), it is easy to show that Eq. (12) becomes

\[
T = \frac{2 n}{ma} \frac{1}{(a + c)} \ln \left| \frac{a - a \cdot \tanh(ay_0/2)}{a + c \cdot \tanh(ay_0/2)} \right| + \frac{1}{(a - c)} \ln \left| \frac{a + a \cdot \tanh(ay_0/2)}{a + c \cdot \tanh(ay_0/2)} \right|.
\]

(14)

For two special situations \((a + c) \rightarrow 0\) or \((a - c) \rightarrow 0\), using Eqs. (A.8) and (A.10), Eq. (14) becomes

\[
T = \frac{n}{ma^2} |1 + ay_0 - e^{ay_0}|, \text{ if } X_0 \rightarrow 0 \text{ and } a + c \rightarrow 0,
\]

(15)

\[
T = \frac{n}{ma^2} |1 - ay_0 - e^{-ay_0}|, \text{ if } X_0 \rightarrow 0 \text{ and } a - c \rightarrow 0.
\]

(16)

Case 3: The horizontal well is located at the bottom of the aquifer. In some applications, the well may be installed near the bottom of the aquifer. For instance, horizontal wells are commonly employed to recover dense non-aqueous liquids (DNAPLs) such as chlorinated solvents, creosote, and coal tar. DNAPLs are driven by gravitational forces deeper into the subsurface, where they are trapped by capillary forces and as pools of liquid suspended on low-permeability strata. Horizontal wells can provide much better recoveries because they can be installed at the bottoms of the aquifers to have much larger contact areas with the DNAPL plumes. The capture time for a bottom horizontal well is addressed below.

Fig. 2(B) shows the original and image wells for a bottom well scenario. The image wells are \(2d\) apart with a pumping rate \(2Q\), the original well is assigned an equivalent “\(2Q\)” pumping rate. Such an assignment can be explained in this way: in Fig. 2(A), if the original well starts to move downwards, then the image well D1 moves upward, the pair wells U2/U3 or D2/D3 tend to move closer to each other. Finally, the original well moves to the bottom and joins together with D1 to form one “equivalent” well with a pumping rate \(2Q\). Meanwhile, wells U2/U3 come to form one “equivalent” well, and wells D2/D3 form another “equivalent” well, etc. We find that Fig. 2(B) is similar to Fig. 2(A) except that \(d\) is replaced by \(2d\) and \(Q\) is replaced by \(2Q\). Thus a complex function describing the flow in Fig. 2(B) is obtained from a modification of Eq. (5) (notice that the origin of the coordinate system is at the bottom with the real well).

\[
\zeta(u) = 2m \ln \left[ \sin \left( \frac{\pi u}{2d} \right) \right] - i\eta u h.
\]

(17)

The capture time becomes

\[
T = \frac{4n}{ma} \frac{1}{(a + c)} \ln \left| \frac{\cos \left( \frac{ay_0 + b}{4} \right)}{\cos \left( \frac{b}{4} \right)} \right| + \frac{1}{(a - c)} \ln \left| \frac{\cos \left( \frac{ay_0 - b}{4} \right)}{\cos \left( \frac{b}{4} \right)} \right|,
\]

(18)

where parameters \(a, b,\) and \(c\) are the same as those defined in Eq. (8).

The solutions for special cases such as \(x_0 \rightarrow 0\) and/or \((a + c) \rightarrow 0\) or \((a - c) \rightarrow 0\) in case 3 can be easily obtained from the corresponding results in cases 1 and 2 of this section by simply replacing “\(d\)” and “\(Q\)” by “\(2d\)” and “\(2Q\)” there, respectively.

2.3. Analytical solutions of capture time to a horizontal well in an aquifer with constant-head boundaries

As mentioned in the introduction, horizontal wells are commonly used to recover contaminated water from under a water reservoir, or a wetland, or a lake. Under these circumstances, the top boundary of the aquifer is better described as constant-head if the aquifer is hydrologically connected with the bottom of the reservoir. Tarshish [27] has discussed the horizontal pumping under a water reservoir.

Another common application of horizontal well is to recover contaminants from a low-permeability layer bounded from above and below by highly permeable media. Vertical wells are inefficient in such cases due to the limited influence screen length. On the contrary, horizontal wells can be drilled into the low-permeability layer to substantially increase the interceptive area with the contaminated water there. The highly permeable media above and below can be described as constant-head boundaries if the permeability contrasts between the low-permeability layer and the highly permeable layers are very large. The reason that the highly permeable media are treated as constant-head boundaries is because any possible hydraulic head differences within the media will dissipate rapidly, resulting in the equal hydraulic head within the media.

In the following, we provide the analytical solutions for these two cases.

Case 1: Horizontal well located underneath a water reservoir. For this case, the top flow boundary is constant-head whereas the bottom boundary is still no-flow, thus water is supplemented into the aquifer from the reservoir when pumping starts. Therefore, the well is more likely to catch water from top region than from bottom region. To achieve the maximal efficiency of catchment, the well is better aligned near the bottom no-flow boundary. Similar situations occur in petroleum engineering. As pointed out by Kuchuk et al. [13, p. 86],
if a constant pressure such as a gas cap existed in an oil reservoir, the well was usually drilled close to the other no-flow boundary. Fig. 3 shows a schematic diagram of the real and image well system for the bottom well scenario. Both injecting and pumping image wells are used. The bottom no-flow boundary prohibits the vertical regional flow. The top constant-head boundary prohibits the horizontal regional flow components. Thus no regional flow can exist in this case. The complex function describing steady-state flow is

\[ \zeta(u) = 2m \ln \left[ \sin \left( \frac{\pi u}{4d} \right) \right] - 2m \ln \left[ \sin \left( \frac{\pi |u| + 2d}{4d} \right) \right], \tag{19} \]

and the real and imaginary parts of Eq. (19) are

\[ \phi = m \ln \left( \frac{1}{2} \left( \cosh \frac{\pi y}{2d} - \cos \frac{\pi x}{2d} \right) \right) - m \ln \left( \frac{1}{2} \left( \cosh \frac{\pi y}{2d} + \cos \frac{\pi x}{2d} \right) \right), \tag{20} \]

\[ \psi = 2m \tan^{-1} \left[ \frac{\sinh (\pi y/2d)}{\sin (\pi x/2d)} \right]. \tag{21} \]

Appendix C gives capture time, \( T \), as

\[ T = \frac{16\pi}{ma^2 \sin \left( \frac{\psi_0}{4} \right)} \left[ b - \sin^{-1} \left( \frac{b}{4} \cos \left( \frac{a\psi_0}{4} \right) \right) \right], \tag{22} \]

where \( a \) and \( b \) are defined in Eq. (8), and \( \psi_0 = \psi(x_0, y_0) \) is calculated from Eq. (21). When \( x_0 \to 0 \) or \( y_0 \to 0 \) Eq. (22) is simplified to

\[ \lim_{x_0 \to 0} T = \frac{8\pi}{ma^2} \left[ \cosh \left( \frac{a\psi_0}{4} \right) - 1 \right], \tag{23} \]

\[ \lim_{y_0 \to 0} T = \frac{8\pi}{ma^2} \left[ 1 - \cos \left( \frac{a\psi_0}{4} \right) \right]. \tag{24} \]

**Case 2:** Horizontal well located in a low-permeability stratum bounded by highly permeable aquifers above and below. Horizontal wells are more effective than vertical wells to extract the contaminants from a low-permeability stratum bounded above and below by highly permeable media. This phenomenon was also observed by Seines [25] who used a case study of the Troll field to demonstrate that one horizontal well operated as the equivalent of four vertical wells in recovering oil from a thin oil reservoir. Owing to the large permeability contrast, the top and bottom boundaries of the stratum are better characterized as constant-head boundaries. In this case, the horizontal well is better installed at the center of the stratum to achieve maximal catchment efficiency. Fig. 3(B) shows a schematic diagram of the real and image wells for this scenario. We assume that the top and bottom boundaries have an equal constant head, thus there is no vertical flow before pumping. In fact, the well configuration in Fig. 3(B) is almost identical to that of Fig. 3(A) except that the distance between neighboring wells becomes \( 2d \) rather than \( 4d \) and the pumping rate is \( Q \) rather than \( 2Q \). Therefore, if we put the origin of the coordinate system at the real well, the solution of capture time excluding regional flow can be directly modified from Eq. (22) and becomes

\[ T = \frac{8\pi}{ma^2 \sin (b)} \left[ \frac{b}{2} - \sin^{-1} \left( \frac{b}{4} \cos \left( \frac{a\psi_0}{2} \right) \right) \right], \tag{25} \]

where \( a \) and \( b \) are still defined by Eq. (8), but \( \psi_0 \) becomes

\[ \psi_0 = m \tan^{-1} \left[ \frac{\sinh (\pi y_0/2d)}{\sin (\pi x_0/2d)} \right]. \tag{26} \]

When \( x_0 \to 0 \) or \( y_0 \to 0 \), Eq. (25) is replaced by simplified forms

---

Fig. 3. The vertical cross-section view of an aquifer and the real and image wells. (A) A bottom well scenario with a constant-head top boundary and a no-flow bottom boundary. (B) A center well scenario with constant-head top and bottom boundaries.
lim \( T_{x=0} = \frac{4n}{ma^2} \left[ \cosh \left( \frac{a x_0}{2} \right) - 1 \right] \), \hspace{1cm} (27)

lim \( T_{y=0} = \frac{4n}{ma^2} \left[ 1 - \cos \left( \frac{a y_0}{2} \right) \right] \). \hspace{1cm} (28)

### 3. Semi-analytical solutions of capture times to horizontal wells with general well locations and regional flows

In previous sections, horizontal wells are either at the centers or at the bottoms of the aquifers and the regional flows are not included when the constant-head boundaries are presented. These special requirements are necessary in order to achieve the closed-form solutions. In this section, we will study three cases with arbitrary well locations and with possible regional flows. Closed-form analytical solutions are unlikely for these general situations, however, semi-analytical solutions are possible.

**Case 1: An arbitrary horizontal well location in a confined aquifer.** Fig. 4(A) shows the real and image wells for this case. The distance from the horizontal well to the bottom boundary is \( d_w \). As discussed in Section 2, only the \( y \)-component of the regional flow influences the capture time, the \( z \)-component of the regional flow causes the shift of a fluid particle along the well axis, but does not affect the capture time. We assumed that the \( y \)-component of the regional flow is along the \(-y\)-axis with a specific discharge \( q_0 \).

With a coordinate system shown in Fig. 4(A) where the origin is at the bottom boundary, the complex function describing the steady-state flow in the \( xy \)-plane is

\[
\zeta(u) = m \sum_{j=-\infty}^{\infty} \ln \left[ u - (2jd + d_w) \right] - iq_0 u, \hspace{1cm} (29)
\]

where the summation reflects the contributions from the original well and the image wells. Notice that the influence from an image well upon a fluid particle’s movement will decrease when the distance between that image well and the fluid particle increases, thus the summation of the infinite series in Eq. (29) is replaced by a summation of a finite series from \( j = -M \) to \( M \). The velocities along the \( y \)- and \( x \)-directions are \( V_y = (1/n)(\partial \phi / \partial y) \) and \( V_x = (1/n)(\partial \phi / \partial x) \), respectively, where \( \phi \) is the real part of \( \zeta(u) \). The approximation of the capture time is

\[
T_M = n \int_0^{y_0} dy \left[ m \sum_{j=-M}^{M} \frac{y}{\sqrt{(x - (2jd + d_w))^2 + y^2 + q_0}} \right],
\]

if \( y_0 \neq 0 \). \hspace{1cm} (30)

---

**Fig. 4.** The vertical cross-section view of an aquifer and the real and image wells for arbitrary well locations and possible regional flows. (A) A confined aquifer. (B) An aquifer with a constant-head top boundary and a no-flow bottom boundary. (C) An aquifer with constant-head top and bottom boundaries, the top and bottom heads could be different.
The corresponding streamline function is
\[
\psi_M(x, y) = m \sum_{j=-M}^{M} \tan^{-1} \left( \frac{y}{x - (2jd \pm d_w)} \right) - q_0 x. \tag{32}
\]

If a fluid particle starts from \((x_0, y_0)\), the streamline for that particle is determined by
\[
\psi_M(x, y) = \psi^0_M(x_0, y_0). \tag{33}
\]

The choice of \(M\) depends on a prior determined approximation criterion. For instance, we can choose \(M\) to be the smallest positive integer satisfying
\[
\left| \frac{\psi^0_{M+1} - \psi^0_M}{\psi^0_M} \right| \leq 10^{-6}. \tag{34}
\]

When \(M\) is found, Eqs. (32) and (33) determine the streamline. For a given \(y\)-coordinate at the streamline, the \(x\)-coordinate is numerically determined. This is carried out either using the particle tracking method or the Newton–Raphson root searching method [22]. We use the following particle tracking method to trace the streamline:
\[
x_{i+1} = x_i - \left( \frac{\partial \psi_M/\partial y}{\partial \psi_M/\partial x} \right)(y_{i+1} - y_i),
\]
where \((x_i, y_i)\) is the particle location at the point \(i\) along the streamline, \(i = 0, 1, 2, \ldots\). If \((y_{i+1} - y_i)\) is small enough, \(x_{i+1}\) can be accurately determined. After this Eq. (30) is numerically integrated using the Simpson’s rule [22, p. 113]. The numerical methods are straightforward to implement. A FORTRAN program including the particle tracking to find the streamline and the numerical integration to calculate the capture time is written by us and is available from the website http://geoweb.hydrog.tamu.edu/Faculty/Zhan/Zhan.html. Examples 1 and 2 in Table 1 show the results from these semi-analytical calculations in confined aquifers. In the example 1 of Table 1, \(d_w = d/2\), i.e., the well is at the aquifer center, the closed-form analytical solution Eq. (12) can be used. Using the parameters listed in example 1, the capture time calculated from Eq. (12) is \(T = 4.58\) d, which agrees with the semi-analytical solution \(T = 4.58\) d.

**Case 2: An arbitrary horizontal well location in an aquifer with a constant-head top boundary and a no-flow bottom boundary.** Fig. 4(B) shows the real and image wells for this case. As pointed out in the case 1 of Section 2.3, no regional flow can exist in this case. However, the well can be put in an arbitrary location. In Fig. 4(B), the real and image pumping wells are at \(x^+ = 4jd \pm d_w\) and the injecting image wells are at \(x^- = 4jd + 2d \pm d_w\) where \(j = 0, \pm 1, \pm 2, \ldots\). The complex function describing the steady-state flow, the capture time, and the streamline function are approximated by Eqs. (35)–(38).

\[
\zeta(u) = m \sum_{j=-M}^{M} \ln \left| u - (4jd \pm d_w) \right| - m \sum_{j=-M}^{M} \ln \left| u - (4jd + 2d \pm d_w) \right|,
\tag{35}
\]

### Table 1

**Analytical and semi-analytical solutions of capture times**

<table>
<thead>
<tr>
<th>Example</th>
<th>Case 1: Capture times in confined aquifers</th>
<th>Case 2: Capture times in aquifers with constant-head top boundaries and no-flow bottom boundaries</th>
<th>Case 3: Capture times in aquifers with constant-head top and bottom boundaries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input parameters:</td>
<td>(Q = 15) m(^3)/d, (q_0 = 0.1) m/d, (n = 0.25), (d = 20) m, (d_w = 10) m, (x_0 = 15) m, (y_0 = 10) m</td>
<td>(Q = 15) m(^3)/d, (q_0 = 0.1) m/d, (n = 0.25), (d = 20) m, (d_w = 5) m, (x_0 = 15) m, (y_0 = 10) m</td>
<td>(Q = 15) m(^3)/d, (q_0 = 0.01) m/d, (n = 0.25), (d = 20) m, (d_w = 5) m, (x_0 = 10) m, (y_0 = 10) m</td>
</tr>
<tr>
<td>Analytical solution [Eq. (12)]</td>
<td>(T = 4.58) d</td>
<td>Semi-analytical solution [Eq. (30)]</td>
<td>(T = 4.58) d</td>
</tr>
<tr>
<td>Semi-analytical solution [Eq. (30)]</td>
<td>(T = 7.41) d</td>
<td></td>
<td>(T = 4.39) d</td>
</tr>
<tr>
<td>Analytical solution [Eq. (22)]</td>
<td>(T = 3.41) d</td>
<td>Semi-analytical solution [Eq. (36)]</td>
<td>(T = 3.40) d</td>
</tr>
<tr>
<td>Semi-analytical solution [Eq. (36)]</td>
<td></td>
<td></td>
<td>(T = 4.39) d</td>
</tr>
<tr>
<td>Analytical solution [Eq. (25)]</td>
<td>(T = 8.02) d</td>
<td>Semi-analytical solution [Eq. (40)]</td>
<td>(T = 8.04) d</td>
</tr>
<tr>
<td>Semi-analytical solution [Eq. (40)]</td>
<td></td>
<td></td>
<td>(T = 10.49) d</td>
</tr>
</tbody>
</table>

*The origin of the coordinate system is at the bottom boundary. Notice that \(x_0\) used in this table is the vertical distance to the bottom boundary, not the vertical distance to the well.*
The capture time, and the streamline function are approximated by Eqs. (39)–(42).

\[
\zeta(u) = m \sum_{j=-M}^{M} \ln |u - (2jd + d_w)| \quad - m \sum_{j=-M}^{M} \ln |u - (2jd - d_w)| - q_{0y}u, \quad (39)
\]

\[
T_M = n \left[ \int_{-a}^{a} dy \right] \left[ \sum_{j=-M}^{M} \frac{y}{|x - (2jd + d_w)|^2 + y^2} \right] - m \sum_{j=-M}^{M} \frac{1}{|x - (2jd + d_w)|^2} + y^2 \], \quad \text{if } y_0 \neq 0 \quad (40)
\]

\[
\psi_M(x, y) = m \sum_{j=-M}^{M} \tan^{-1} \left[ \frac{y}{x - (2jd + d_w)} \right] - m \sum_{j=-M}^{M} \tan^{-1} \left[ \frac{y}{x - (2jd + 2d + d_w)} \right]. \quad (41)
\]

\[
\psi_M(x, y) = m \sum_{j=-M}^{M} \tan^{-1} \left[ \frac{y}{x - (2jd + d_w)} \right] - m \sum_{j=-M}^{M} \tan^{-1} \left[ \frac{y}{x - (2jd - d_w)} \right] - q_{0y}y. \quad (42)
\]
4. Capture time in anisotropic media

If the horizontal and vertical conductivities are $K_x$ and $K_y$, respectively ($x$ is the vertical direction), let

$$\alpha = \frac{K_x}{K_y}, \quad K_e = \sqrt{K_x K_y},$$

(43)

where $\alpha$ is the anisotropic ratio and $K_e$ is the effective conductivity. Bear [2, p. 169–175] showed that an anisotropic medium could be transformed into an “equivalent” isotropic medium. The detailed transformation procedure was given there. For our purpose, we need to transfer information procedure was given there. For our purpose, we need to transfer $x$, $y$, $q_0$, and $d$ into the “equivalent”, $x_e$, $y_e$, $q_{0e}$, and $d_e$ as follows:

$$x_e = \left( \frac{K_y}{K_x} \right)^{1/2} \frac{x}{\alpha^{1/4}}, \quad y_e = \left( \frac{K_x}{K_y} \right)^{1/2} \frac{y}{\alpha^{1/4}},$$

(44)

$$y = \alpha^{1/4} y, \quad q_{0e} = \alpha^{1/4} q_0, \quad d_e = \frac{d}{\alpha^{1/4}}.$$

The pumping rate $Q$ and porosity $n$ are unchanged. Therefore, the capture time for a horizontal well in the middle of a confined aquifer is

$$T = \frac{2n}{ma_e} \left[ \frac{1}{(a_e + c_e)} \ln \left| \frac{\cos \left( \frac{a_e + c_e}{2} \right)}{\cos \left( \frac{b_e}{2} \right)} \right| \right]$$

$$+ \frac{1}{(a_e - c_e)} \ln \left| \frac{\cos \left( \frac{a_e - c_e}{2} \right)}{\cos \left( \frac{b_e}{2} \right)} \right|,$$

(45)

where

$$a_e = 2\pi/d_e, \quad b_e = 2\psi_{0e}/m_e, \quad c_e = 2q_{0e}/m.$$

(46)

$$\psi_{0e} = m \tan^{-1} \left[ \frac{\tanh \left( \frac{\pi q_{0e}}{d_e} \right)}{\tan \left( \frac{\pi q_{0e}}{d_e} \right)} \right] - q_{0e} x_{0e}.$$

(47)

Using Eq. (44) to do the parameter transformation, we can obtain the analytical and semi-analytical solutions of capture times for all the previously described scenarios in anisotropic aquifers. The details are trivial and will not be repeated here. The FORTRAN programs for all the semi-analytical solutions are capable of handling anisotropic media.

5. Analysis of the solutions

On the basis of the above analytical solutions, we can delineate the horizontal well capture zones. For example Fig. 5(A) shows iso-capture time contours for a middle well in confined, isotropic and anisotropic aquifers on the basis of Eqs. (12) and (45). The parameters used for that figure are: $n = 0.25$, $d = 21$ m, $q_0 = 0.1$ m/d, $Q = 15$ m$^2$/d, and $K = 10$ m/d for the isotropic aquifer, and $K_y = 10$ m/d, $K_x = 1$ m/d for the anisotropic aquifer. The capture time in Fig. 5(A) is 10 d. Fig. 5(B) shows the capture zone of a bottom well scenario whose input parameters are the same as those used in Fig. 5(A). Fig. 5(A) and (B) indicates that the top and bottom no-flow boundaries constrain the vertical flows and enhance the horizontal flows, resulting in horizontally elongated capture zones. The presence of vertical anisotropy with a smaller vertical conductivity further exaggerates this tendency and results in an even more elongated capture zone. For the bottom well scenario, the bottom boundary has a much stronger impact on the flow than the top boundary and the capture zones extend further in the horizontal direction compared with their counterparts of Fig. 5(A). Regional flow causes asymmetric shapes of capture zones in the $y$-direction.

Fig. 6(A) and (B) shows the iso-capture times for the cases 1 and 2 of Section 2.3. When constant-head boundaries are presented, water can infiltrate vertically across those boundaries to replenish the aquifers when pumping starts, thus the demand of water from the lateral directions is reduced, resulting in smaller lateral flow velocities. On the contrary, a confined aquifer prevents any water replenishment from the vertical direction across the no-flow boundaries, thus the pumped water entirely comes from the lateral flow, resulting in enhanced lateral flow velocities and the laterally elongated iso-capture time curves.

The derived closed-form analytical solutions have the following limitations:

- Because of the use of steady-state flow and infinitely long well assumptions, the analytical solutions can only be used for near field and for time after start of the pseudoradial flow in a confined aquifer or for the steady-state flow if constant-head boundaries are presented (see the justification in Section 2.2).
We provide analytical solutions for central and bottom well scenarios. For a case with a general well location, we can only provide a semi-analytical solution, which needs a numerical integration to fully calculate the capture time based on Eq. (7).

We derive the closed-form solutions for confined aquifers including the regional flow. When top and bottom boundaries are constant-head, we can only provide semi-analytical solutions if top and bottom boundaries have different heads, but we can have the closed-form solutions if top and bottom boundaries have equal constant-head.

Semi-analytical solutions are very flexible and can deal with cases with arbitrary well locations and possible regional flows.

Analytical and semi-analytical solutions obtained above can be compared with capture times calculated using numerical software such as Visual Modflow (VM) [9]. Zhan [29] has provided the numerical simulation results excluding the regional flow and he found that the numerical solutions were almost identical to the analytical solutions, especially for the isotropic cases. The numerical scheme is quite standard as described in Zhan [29], thus the details are not repeated here.

6. Summary and conclusions

We have obtained closed-form analytical solutions and semi-analytical solutions of capture times on the conditions of a steady-state flow and an infinitely long well. The suitability of these assumptions is justified and the applications and limitations of the analytical solutions are discussed.

We have derived the analytical solutions of capture times for center or bottom wells in confined aquifers including the horizontal regional groundwater flows. We have provided the analytical solutions of capture times to horizontal wells in aquifers under water reservoirs that serve as constant-head boundaries. Bottom wells can achieve the maximal capture efficiency in these aquifers and are investigated. We have obtained the analytical solutions of capture times in aquifers when top and bottom boundaries have equal constant heads. Center wells can achieve the maximal recovery efficiency in these aquifers and are investigated.

We have conducted the semi-analytical studies with general well locations and with possible regional flows in either confined aquifers, or aquifers under water reservoirs, or aquifers with top and bottom constant-head boundaries whose constant heads could be different. We first give the analytical expressions of capture times and streamline functions in the $xy$-plane, then use the particle tracking method to determine the streamlines in the $xy$-plane, and use the Simpson’s rule [22] to numerically integrate the capture times. FORTRAN programs are written to conduct the particle tracking and the numerical integrations. The semi-analytical solutions agree with the analytical solutions for three special examples where analytical solutions are possible.

Capture times in anisotropic media are derivable for all previously mentioned cases following simple modifications from the solutions of isotropic media. FORTRAN programs associated with the semi-analytical solutions are capable of handling anisotropic media.

The aquifer’s top and bottom no-flow boundaries play important roles in influencing flows to a horizontal well. These boundaries constrain vertical flow, but enhance lateral flow, resulting in elongated iso-capture time curves. If the aquifer is anisotropic with a smaller vertical hydraulic conductivity, constraint of vertical flow and enhancement of lateral flow are further exaggerated. When constant-head boundaries are presented, water can infiltrate across those boundaries to supplement the aquifer, thus demand of water flow from lateral direction is reduced.

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Appendix A. Derivation of Eq. (12)

Taking the derivative $\partial \phi / \partial x$ from Eq. (6) and substituting it into Eq. (7) results in

$$T = 2n \left| \int_0^{x_0} \frac{\cosh(ay) - \cos(ax)}{\sin(ax)} \, dx \right|. \quad (A.1)$$

Squaring Eq. (9) and using the following identities:

$$\tanh^2 \left( \frac{x}{2} \right) = \frac{\cosh(x) - 1}{\cosh(x) + 1},$$

$$\tan^2 \left( \frac{x}{2} \right) = \frac{1 - \cos(x)}{1 + \cos(x)},$$

we obtain

$$\cosh(ay) = \frac{1 + \cos(b + cx) \cos(ax)}{\cos(b + cx) + \cos(ax)}. \quad (A.3)$$

Substituting Eq. (A.3) into Eq. (A.1) leads to

$$T = 2n \left| \int_0^{x_0} \frac{\sin(ax)}{\cos(ax) + \cos(b + cx)} \, dx \right|. \quad (A.4)$$

To integrate Eq. (A.4), we use the following two integrations $A$ and $B$:

$$A = \int_0^{x_0} \frac{\sin(ax)}{\cos(ax) + \cos(b + cx)} \, dx;$$

$$B = \int_0^{x_0} \frac{\sin(ax)}{\cos(ax) - \cos(b + cx)} \, dx. \quad (A.5)$$

Recognizing the following identity:

$$\sin(\gamma_1) \pm \sin(\gamma_2) = 2\sin[(\gamma_1 \pm \gamma_2)/2] \cdot \cos[(\gamma_1 \pm \gamma_2)/2],$$

and the similar identities regarding "$\cos(\gamma_1) + \cos(\gamma_2)$", $A$ becomes

$$A = \int_0^{x_0} \frac{\sin \left[ \frac{(a+c)x+b}{2} \right]}{\cos \left[ \frac{(a+c)x+b}{2} \right]} \, dx$$

$$= -\frac{2}{a+c} \ln \left| \frac{\cos \left[ \frac{(a+c)x+b}{2} \right]}{\cos \left( \frac{b}{2} \right)} \right|, \quad \text{if} \quad a + c \neq 0, \quad (A.7)$$

$$A = \tan \left( \frac{b}{2} \right)x_0, \quad \text{if} \quad a + c = 0. \quad (A.8)$$

Similarly, we have

$$B = -\frac{2}{(a-c)} \ln \left| \frac{\cos \left[ \frac{(a-c)x-b}{2} \right]}{\cos \left( \frac{b}{2} \right)} \right|, \quad \text{if} \quad a - c \neq 0, \quad (A.9)$$

$$B = -\tan \left( \frac{b}{2} \right)x_0, \quad \text{if} \quad a - c = 0. \quad (A.10)$$

$(A + B)/2$ will give the integration need in formula Eq. (A.4). The final result is Eq. (12).

Appendix B. Derivation of Eq. (13) when $q_0 = 0$

When $q_0 = 0$, $c = 0$, from Eq. (12), we have

$$T = \frac{2n}{ma} \left\| \ln \left| \frac{\cos \left( \frac{ax_0+b}{2} \right)}{\cos \left( \frac{b}{2} \right)} \right| \right\|. \quad (B.1)$$

Considering $\ln(\gamma_1) + \ln(\gamma_2) = \ln(\gamma_1 \gamma_2)$ Eq. (B.1) then becomes

$$T = \frac{2n}{ma} \left\| \ln \left| \frac{\cos(ax_0) + \cos(b)}{1 + \cos(b)} \right| \right\|. \quad (B.2)$$

From Eq. (9) we obtain $\cos(b)$ and substitute it into Eq. (B.2) to have Eq. (13).

Appendix C. Derivation of Eq. (22)

For any point $(x,y)$ in the flow path started from $(x_0,y_0)$, $\psi(x,y) = \psi(x_0,y_0)$, Eq. (21) gives

$$\sinh(ay/4) = \sin(ay/4) \tan(\theta/4), \quad (C.1)$$

where $\alpha$ and $\beta$ are defined in Eq. (8). Deriving $\partial \phi / \partial x$ from Eq. (20) and substituting it into Eq. (7) results in

$$T = \frac{2n}{ma} \left\| \int_0^{x_0} \frac{\cosh \left[ \frac{ax}{4} \right] - \cos \left[ \frac{ax}{4} \right]}{\sin \left( \frac{ax}{4} \right)} \, dx \right\|. \quad (C.2)$$

From Eq. (C.1) we can find $\cosh(ax/4)$ and substitute it into Eq. (C.2) to have

$$T = \frac{2n}{ma} \left\| \int_0^{x_0} \frac{\sin \left( \frac{ax}{4} \right)}{\cos \left( \frac{ax}{4} \right) \left[ \cos^2 \left( \frac{x}{4} \right) + \sin^2 \left( \frac{x}{4} \right) \right]^{1/2}} \, dx \right\|. \quad (C.3)$$

Assigning $\omega = \cos(ax/4)$ then Eq. (C.3) becomes

$$T = \frac{8n}{ma^2 \cos \left( \frac{x}{4} \right)} \left\| \int_0^1 \frac{d\omega}{\sqrt{1 - \sin^2 \left( \frac{x}{4} \right) \omega^2}} \right\|. \quad (C.4)$$

Using the formulae 2.271.4 of Gradshteyn and Ryzhik [8, p. 105], the integration of Eq. (C.4) can be carried out and the final result is

$$T = \left\| \frac{16n}{ma^2 \sin \left( \frac{x}{4} \right)} \left[ 4 - \sin^2 \left( \frac{\beta}{4} \cos \left( \frac{ax_0}{4} \right) \right) \right] \right\|. \quad (C.5)$$
References