A non-linear filter for one- and two-dimensional open channel flows with shocks

Scott A. Yost *, Prasada Rao 1

Department of Civil Engineering, University of Kentucky, 161 Raymond Building, Lexington, KY 40506-0281, USA

Received 10 December 1999; received in revised form 14 March 2000; accepted 3 April 2000

Abstract

Modeling flows with shocks creates numerical challenges due to the oscillations in the vicinity of the discontinuity. This paper presents the advantage of using a non-linear filter for obtaining oscillatory free solution for one- and two-dimensional open channel flows. Numerical results are compared with those of other conventional high-resolution schemes to demonstrate the accuracy of the solution. As discussed in the course of this document, the present formulation is computationally faster, and simpler to code while maintaining consistent results with the standard schemes. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Surges; Shocks; Oscillations; Monotone; High-resolution

1. Introduction

Flows in open channels are often accompanied with moving discontinuities. Modeling of flows with shocks requires careful choice of the numerical scheme. In the field of computational aeronautics and hydraulics, it has been well documented that second and higher-order finite difference schemes, albeit easier to code, generate dispersive errors in the vicinity of a shock front [1–3]. Since the flow equations lack viscous (dissipative) terms, an additional numerical smoothing mechanism is used to dampen these oscillations. However, many numerical smoothing techniques result in a smeared shock front as well as a permanent loss of information due to the added artificial viscosity.

To overcome this limitation, in the early 1980s high-resolution schemes were introduced in the field of aeronautics. The most notable feature of these schemes was the ability to simulate sharp shocks without numerical oscillations. The early family of high-resolution schemes includes the total variation diminishing (TVD [4]) schemes, essentially non-oscillatory (ENO [5]) schemes and flux corrected transport (FCT [6]) algorithms. Although these methods are derived from different concepts, all of them introduce a finite amount of dissipation at selected locations in the flow field. This necessitated the use of limiters in the above formulations. While these schemes (FCT, TVD) eliminate oscillations at the shock, they reduce to first- or zeroth-order accuracy in the vicinity of the shock, even though the governing numerical scheme is second-order accurate or higher. By allowing a small amount of oscillation, the ENO schemes can achieve \((N-1)\)th order accuracy near the shocks for \(N\)th order schemes. For a thorough discussion of these techniques see the work of Delis and Skeels [7] and Alcrudo and Navarro [8].

Motivation for this work necessitates a brief outline of the limitations associated with the current family of high-resolution schemes. First TVD schemes result in additional CPU time (typically a factor of 2) in comparison with finite difference schemes employing simple artificial viscosity [9]. The ENO formulations require advanced mathematical understanding compared to TVD, owing to the necessary characteristic decomposition with the associated Riemann solver. Both the TVD and ENO formulations require the computation of eigen values at every grid node to determine the characteristic direction of wave propagation. This upwind bias forces the discrete approximation to directly simulate the signal propagation properties of hyperbolic equations. A major challenge arises when source terms are present or if the governing equations are expressed...
in a non-conservative form (particularly valid for higher-dimensional flows). Computing the true eigen values in either cases is a tedious task. The effect of using an appropriate limiter in all these techniques, which controls the second-order terms and hence the oscillations, has been investigated by others (for example see [9,10]). There appears to be no unanimity among researchers as to the choice of limiter. Another limitation is the difficulty involved in converting an existing finite difference code to a versatile TVD/ENO one. The FCT formulations, although less harsh in terms of portability, require a careful blending of first- and higher-order schemes so that the discontinuous solution are not smeared.

The present work differs from all the mentioned ones, in that we try to arrive at a monotone solution by way of using a non-linear filter. Originally proposed by Engquist et al. [11], a filter is a computational technique used after every time step to remove any numerical oscillations. For Euler equations, Engquist et al. [11] demonstrated that by tracking the location of the shock front a judicious amount of dissipation could be introduced at a few selected grid nodes on either side of it to overcome the oscillations. At the end of every time step, the filter identifies the nodes where the solution tends to oscillate, and then evaluates the corrections that need to be conservatively applied to the local flow variables. The associated increase in run times is marginal due to the small number of nodes where a correction is applied (see Figs. 5, 10 and Table 1). This filtering algorithm can be used in conjunction with any higher-order finite difference scheme as a post-processor step, and no modification need be done to the existing code.

In this work, we demonstrate the effectiveness and advantages of using a non-linear filter for simulating open channel flows with shocks. The present analysis has been conducted for both one- and two-dimensional flows. To demonstrate the capability of this filtering approach in arriving at monotone solutions, the results are compared with those of the other high-resolution schemes (i.e., FCT, TVD and ENO). The results indicate that the present solution is consistently equal or superior to those of other high-resolution techniques. The associated simplicity and CPU time savings underline the advantages of the present approach.

### 2. One-dimensional analysis

Since analytical solutions exist for one-dimensional flows, they act as benchmarks for numerical solutions. A large part of the discussion is devoted to the filtering approach, and the other material has been presented in abbreviated form. The reader can obtain the details in the many references cited.

For a rectangular channel, the basic governing equations based on the continuity and momentum principles can be written in conservation form as [3]

\[
\begin{align*}
 h_t + q_x &= 0, \\
 q_t + \left( \frac{q_x^2}{h} + \frac{gh^2}{2} \right)_x &= gh(S_0 - S_f),
\end{align*}
\]

where \( h \) is the flow depth, \( q_x \) the unit discharge, \( g \) the acceleration due to gravity, \( S_0 \) the bed slope of the channel and \( S_f \) is the frictional slope, computed using Manning equation, written as

\[
S_f = \frac{m^2 u |u|}{R^{4/3}}
\]

here \( m \) is the Manning roughness coefficient, \( u \) the flow velocity and \( R \) is the hydraulics radius (ratio of cross-sectional area to the wetted perimeter).

Eqs. (1) and (2) have been solved using the explicit finite difference MacCormack scheme [3,12]. The choice of the time step is governed by the CFL stability criteria, which can be written as

\[
\Delta t = C_n \frac{\Delta x}{\max(|u| + \sqrt{gh})},
\]

where \( C_n \) is the Courant number \((\leq 1)\), and \( \Delta x \) is the grid spacing.

#### 2.1. Filtering algorithm

The numerical solution contains oscillations near the vicinity of the shock, and needs to be smoothed/filtered to prevent malfunctioning of the code. The characteristic features that an ideal filter should possess include:

(i) Identifying the nodes where the solution tends to oscillate.

(ii) Determining whether a particular node is an undershoot or overshoot, the flow variables need to be correspondingly lifted or reduced.

(iii) Conserving mass by balancing a correction (addition/subtraction) at node with an equal and opposite correction (subtraction/addition) from an adjacent node.

(iv) Ensuring portability where the formulation is independent of the basic numerical scheme.

The above properties are illustrated in Fig. 1, which represents the solution at time level \( n + 1 \) obtained by a second-order accurate finite difference scheme. Here \( i \) indicates the grid node and \( \Delta_i, \Delta_{i+1} \) the forward and

<table>
<thead>
<tr>
<th>Scheme</th>
<th>( L_\infty ) norm</th>
<th>Factor of CPU increase</th>
</tr>
</thead>
<tbody>
<tr>
<td>MacCormack</td>
<td>0.0451</td>
<td>1.00</td>
</tr>
<tr>
<td>(without filter)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MacCormack + filter</td>
<td>0.0102</td>
<td>1.21</td>
</tr>
<tr>
<td>FCT [6]</td>
<td>0.0194</td>
<td>1.63</td>
</tr>
<tr>
<td>ENO [5]</td>
<td>0.0185</td>
<td>1.68</td>
</tr>
<tr>
<td>TVD [4]</td>
<td>0.0159</td>
<td>2.13</td>
</tr>
</tbody>
</table>

---

backward differences of the flow variable at this node, respectively \((\Delta h_i = h_{i+1} - h_i, \, \Delta h = h_i - h_{i-1})\). The first criteria (i), is achieved by checking if \(\Delta h > 0\). Having identified that node \(i\) needs a correction, being an overshoot in this illustration, the flow depth is reduced. The values at the adjacent node (either \(i-1\) or \(i+1\)) should be increased by an equal amount to preserve continuity. For this purpose, the filter selects node \(i+1\), since \(\Delta h_{i+1} > \Delta h_i\). Having filtered the flow depth, the analysis is repeated on the other variable, discharge.

To compute the filtered solution at the new time level (i.e., \(n+1\)), the algorithm for one-dimensional flows can be written as:

1. Use the basic MacCormack code to calculate the flow variables \((h, q)\) at all the interior nodes for the new time level.
2. Apply appropriate boundary conditions at the end nodes.
3. Identify the nodes where the solution oscillates do \(i = 2, \text{last} - 1\)
   if \(\Delta h_i > 0\) then
   4. Identify the grid node that needs to be corrected \((ic)\) along with node \(i\)
      if \(\Delta h_i > |\Delta h_{i-1}|\) then
      \[\delta_i = |\Delta h_i|, \quad \delta_{i-1} = |\Delta h_{i-1}|, \quad ic = i + 1\]
      else
      \[\delta_i = |\Delta h_{i-1}|, \quad \delta_{i-1} = |\Delta h_i|, \quad ic = i - 1\]
      end if
   end if
4. Determine the magnitude of correction and apply it at the associated nodes
   \[\delta = \min(\delta_{i-1}, \delta_i/2)\]
   \[s = \text{sign}(|\Delta h_{i-1}|)\]
   \[h_i = h_i + s\delta\]
   \[h_{ic} = h_{ic} - s\delta\]
end do

In this formulation, filtering is carried out only in selected regions, and hence differs from the conventional schemes. It is important to note that in all the previous investigations involving the application of high resolution schemes [7–10], including the work of Engquist et al. [11], the investigators had first decomposed the system of equations (Eqs. (1) and (2)) into Riemann equivalents and then filtered the variables. This can be achieved either with field-by-field decomposition or componentwise decomposition. In the field-by-field decomposition, the Jacobians associated with the flux components are diagonalized. Flux limiting is done in each eigen space, and the numerical fluxes are reconstructed from these components [10]. This decomposition is an expensive part in the formulation. Though the component-wise decomposition does offer some advantages (it is simpler to code and is faster than field-by-field), since it is based on cell averages extending it to multi-dimensions is much more complicated [5]. The results below show that one can avoid decomposing the equations and still solve the given scalar system (Eqs. (1) and (2)) coupled to this non-linear filter. In the process high-resolution at contact discontinuities is maintained.

Additional comparison with decomposition techniques like TVD and ENO indicate the reliability of this approach.

### 2.2. Application

The definition sketch of the one-dimensional test problem is shown in Fig. 2. The initial conditions in the 4000 m long horizontal, friction-less channel are a depth of 5 m and a flow velocity of 4 m/s \((q_x = 20 \, \text{m}^2/\text{s})\). The surge is created by an instantaneous closure of gate at the downstream end. A uniform grid spacing \((\Delta x)\) of 4 m and a Courant number of 0.8 were used in the simulation. The magnitude of the time step, as obtained using Eq. (4) was 0.1817 s. Investigations on similar flow cases have been previously studied in [13–15]. Along with the specified initial conditions at all the nodes, a zero velocity for all time levels is specified at the downstream end. The other flow variable (i.e., flow depth) was computed using the \(C^+\) form of Eqs. (1) and (2) [3]. With

![Fig. 2. Definition sketch for a sudden closure of gate (one-dimensional flow).](image-url)
or parameters (compared to all the others: formulation, coupled with the absence of any pseudo-
techniques are similar, the simplicity in the present for-
formulations (Fig. 4). Though the results of all the
flow profile with those of TVD, ENO and FCT [4–6]

solving the shock front over 2 grid nodes.

The peak oscillation is about 13.3% of the
terms of the oscillations in the vicinity of the shock
front. The peak oscillation is about 13.3% of the
characteristic feature of second-order schemes is manifest in
without the filtering mechanism are shown. The char-
acteristic is

\[
\hat{h}_{i+1} = \hat{h}_i^{n-1} + \left( \frac{q_s}{-u + \sqrt{gh}} \right)^n_{i+1} + gh \Delta t (S_\eta - S_\gamma)^n_{i-1}.
\]

Fig. 3 is the transient plot at \( t \approx 250 \) s, when the surge is
located at \( x = 2432 \) m. The depth profiles with and
without the filtering mechanism are shown. The char-
acteristic feature of second-order schemes is manifest in
via the shock front over 2 grid nodes.

To facilitate a comparison with the results of other
high-resolution schemes, we have compared the present
flow profile with those of TVD, ENO and FCT [4–6]
formulations (Fig. 4). Though the results of all the
techniques are similar, the simplicity in the present for-
mulation, coupled with the absence of any pseudo-
parameters (compared to all the others: \( \delta \) in [4], \( \lambda \) in [5]
or \( \eta \) in [6]) gives the present filter a distinct advantage.
Moreover, this algorithm can be implemented in a
straightforward way by coupling it with the existing
codes, an aspect absent in other formulations. Table 1
presents the error in the \( L_\infty \) norm and the required CPU
for the five numerical techniques. The \( L_\infty \) norm, an
indication of the peak conservation of solution, was
computed as

\[
|\text{error}|_{L_\infty} = \frac{1}{N} \sum_{i=1}^{N} \sqrt{(h_i - h_{\text{anal}})^2}
\]

with \( h_{\text{anal}} \) representing the analytical depth [15].

\[ h_{\text{anal}} = \begin{cases} 5 \text{ m}, & x < 2432 \text{ m}, \\ 8.18 \text{ m}, & x \geq 2432 \text{ m}. \end{cases} \]

The timing results are presented in Table 1. The factor
of increase in CPU for the different formulations has
been compared relative to the time required for the
MacCormack scheme (without filter). Consistent with
other findings, the TVD schemes are twice as costly as
other standard finite difference techniques [9]. Coupling
this non-linear filter to the MacCormack scheme mar-
ginally increases the CPU, but it is well beneath the
Corresponding TVD computational time. Fig. 5 shows
the number of nodes where the filtering mechanism is
triggered as a function of time. The total number of
nodes where the variables are smoothed, about 35 per
Time step is

\[ \Delta t = C_R \frac{\Delta x \Delta y}{(|R| + \sqrt{gh})(\sqrt{\Delta x^2 + \Delta y^2})} \]

with \( R \) indicating the resultant velocity at a grid node.

3. Two-dimensional analysis – governing equations

Many open channel flow situations require the solu-
tion of the two-dimensional equations. With \( x \) and \( y \)
indicating the two spatial directions, the governing flow
equations can be written as [3]:

\[
(h)_i + (q_s)_x + (q_v)_y = 0,
\]

\[
(q_s)_x + \left( \frac{q_s^2}{h} + \frac{gh^2}{2} \right)_x + \frac{q_s q_v}{h}_y = gh (S_{0x} - S_{1x}),
\]

\[
(q_s)_y + \left( \frac{q_s q_v}{h} \right)_x + \left( \frac{q_v^2}{h} + \frac{gh^2}{2} \right)_y = gh (S_{0y} - S_{1y}),
\]

where \( h \) indicates the flow depth, \( q_s \) and \( q_v \) are the unit
flow discharges in \( x \) and \( y \) directions, \( S_{0x} \) and \( S_{0y} \) are the
bottom slopes of the channel along the two axis, \( S_{1x} \) and
\( S_{1y} \) represent the friction slopes along the longitudinal
and normal directions. The frictional slopes were cal-
culated as

\[
S_{1x} = \frac{m^2 u \sqrt{u^2 + v^2}}{R^{1/3}},
\]

\[
S_{1y} = \frac{m^2 v \sqrt{u^2 + v^2}}{R^{1/3}}.
\]

The CFL stability condition used for computing the
time step is

\[ \Delta t = C_R \frac{\Delta x \Delta y}{(|R| + \sqrt{gh})(\sqrt{\Delta x^2 + \Delta y^2})} \]

with \( R \) indicating the resultant velocity at a grid node.
3.1. Filtering algorithm

The filtering algorithm for higher-dimensional flows is a direct extension of its one-dimensional counterpart outlined in Section 2.1. To encompass the two-dimensional nature, the flow variables are first filtered along the longitudinal direction and then along the normal direction. Since filtering is applied only at selected grid nodes, any additional computational effort by this approach is marginal.

3.2. Application

The definition sketch of the first test problem is shown in Fig. 6. The initial conditions in the 100 m × 100 m flow domain are \( h = 0.04 \) m, \( q_x = 0.06 \) m\(^2\)/s and \( q_y = 0 \) m\(^2\)/s. The bottom surface is assumed to be horizontal and frictionless. The grid spacings are 1 m × 1 m (i.e., \( i_{\text{last}} = j_{\text{last}} = 101 \)), and the Courant number was selected to be 0.6. The non-uniform boundary conditions at the upstream end (i.e., AB) are \( h(1,j) = 0.06 \) m, \( q_x(1,j) = 0.1 \) m\(^2\)/s, \( q_y(1,j) = 0 \) m\(^2\)/s, for \( 30 \leq j \leq 60 \).

The transient solution at \( t \geq 20 \) s of the present formulation was compared with the results of an ENO scheme. We had selected the ENO scheme since it has been documented that the solution of ENO schemes is more accurate than the TVD formulations [5]. The flow variables along the other three boundaries were interpolated from the interior nodes with the assumption that the wave profile did not propagate outside the flow.
domain during the computational period. Since the flow is supercritical, any error introduced by such an approach is minimal. The contour plot indicating the advance of the wave profile for all the three variables at $t \approx 20$ s is shown in Fig. 7. While the flow profiles of the two formulations are in close agreement, the Engquist filter was computationally faster by 32%.

In order to illustrate that the present filter can be directly incorporated into any existing two-dimensional code, we inserted it into our previously developed code that computes the flow in a symmetric transition. The definition sketch of the problem is shown in Fig. 8. The original code uses the FCT concept as described by Book et al. [6]. The FCT formulation in it was replaced with the present non-linear filter. To absorb the irregular boundary the flow equations were first recast onto a rectangular domain. This mapping, along with the boundary conditions are detailed by Jimenez and Chaudhry [16]. The Froude number of the incoming flow is 4.00. This problem has become a benchmark for code validation due to the availability of experimental data. Fig. 9 is the stationary profile of water surface along the centerline of the channel. The solutions are compared with the results of Molls and Chaudhry [17] who had solved the flow equations with effective stresses using ADI method. Since oscillations tend to appear in the ADI approach, they have smoothed the solution by selecting the optimum viscosity values through trial and error, a procedure not required in the present formulation. Fig. 10 shows the number of nodes at which the flow

![Fig. 6. Definition sketch for the two-dimensional problem.](image)

![Fig. 7. Profile of wave advance ($t \approx 20$ s) (— current model, --- ENO scheme).](image)

![Fig. 8. Definition sketch for flow in transitions.](image)

![Fig. 9. Stationary profile along the center line (— current model, --- FCT model, ○ [17]).](image)

![Fig. 10. Number of filtered nodes for flow in transitions (two-dimensional flow) (total number of nodes in the domain = 80 × 20).](image)
variables are filtered as a function of time. The filter is triggered only in the nodes (both along longitudinal and transverse flow directions) surrounding the wave front, consistent with the fact that the oscillations will tend to appear around the discontinuity. For very large flow domains, this accrues to the computational savings. As the results show for all the flow cases, the net amount of correction to the flow variables at these nodes is applied in a conservative manner, and no flow information is lost.

4. Conclusions

In this paper, we have discussed the advantages of using a non-linear filter for open channel flow equations. The obtained results for one- and two-dimensional flows were compared with those of other high-resolution schemes. We have directly applied the MacCormack discretization to the one-dimensional (Eqs. (1)–(3)) and two-dimensional (Eqs. (8)–(10)) flow equations, arriving at solutions that are comparable with those of FCT, ENO, TVD and ADI schemes, as appropriate. For flows with shocks, the advantages in using this non-linear filter are (i) the captured shock profile is monotone, (ii) the obtained solution is in close agreement with those of other high-resolution schemes, (iii) the present filter is simpler to code with portability to a wide range of codes and (iv) the computational speed is faster than other numerical schemes yielding similar results.

Acknowledgements

The authors would like to thank Dr. Thomas Molls of Southern Illinois University, Carbondale, IL, for providing his numerical data for flow in channel transitions.