True-to-mechanism model of steady-state two-phase flow in porous media, using decomposition into prototype flows

M.S. Valavanides 1, A.C. Payatakes *

Department of Chemical Engineering, Institute of Chemical Engineering and High Temperature Chemical Processes, University of Patras, GR 265 00 Patras, Greece

Received 29 August 1999; received in revised form 30 March 2000; accepted 31 August 2000

Abstract

A true-to-mechanism model is proposed, which considers steady-state two-phase flow in porous media (SS2PM) as a composition of two prototype flows, namely ganglion dynamics (GD) and connected-oil pathway flow (CPF). Coupling of the prototype flows is effected with the simple rule that the macroscopic pressure gradient is the same in both. For a given set of values of the flow system parameters, a domain of admissible flow combinations is obtained. The solution is determined by assuming that each point in this domain has equal probability of being 'visited'. This leads to unique values for the flow arrangement variables (FAV), the rate of mechanical energy dissipation, and the relative permeabilities. The new model accounts for the non-linearity of the flow as well as for the effects of all the system parameters (notably those affecting interfaces), and its predictions are in very good agreement with existing data.

Ó 2001 Published by Elsevier Science Ltd.

Keywords: Porous media; Two-phase flow; Relative permeabilities

1. Introduction

Two-phase flow in porous media (2PM) occupies a central position in enhanced oil recovery, the behavior of liquid organic pollutants near the source in contaminated soils, etc. Conventional fractional flow theory assumes that the relative permeabilities are only functions of the water saturation \( S_w \) (for a given porous medium). This approach has been based on Richards’s postulate that each fluid flows through its own separate network of connected pathways and that all portions of the disconnected non-wetting phase (say oil) are stranded [10,17].

However, experimental investigations [1,2,4] have shown that the disconnected oil contributes significantly (even exclusively) to the flow and that relative permeabilities are strong functions of a large number of other parameters

\[
\begin{align*}
  k'_i &= k'_i(S^w_i, Ca, r, \kappa, \cos \theta^A_r, \cos \theta^R_r, Co, Bo, x_{pm}, \text{flow history}), \\
  i &= w, o.
\end{align*}
\]

Here, \( Ca \) is the capillary number (defined as \( Ca = \tilde{\mu}_w \tilde{U}^w / \tilde{\gamma}_{ow} \), where \( \tilde{\mu}_w \) is the viscosity of water, \( \tilde{U}^w \) the superficial velocity of water, and \( \tilde{\gamma}_{ow} \) is the interfacial tension), \( r = \bar{q}/\bar{q}^w \) the oil/water flowrate ratio, \( \kappa = \tilde{\mu}_o/\tilde{\mu}_w \) the oil/water viscosity ratio, \( \theta^A \) and \( \theta^R \) the limiting values of the advancing and receding contact angles at nil velocity, \( Co \) the coalescence factor, \( Bo \) the bond number, \( x_{pm} \) a parameter vector composed of all the dimensionless geometrical and topological parameters of the porous medium affecting the flow (porosity, genus, coordination number, normalized chamber and throat size distributions, chamber-to-throat size correlation factors, etc.) and finally, ‘flow history’ denotes the way in which the steady-state conditions have been achieved, for example, whether through initial imbibition or drainage. Similar observations hold for the capillary pressure. In the following, a tilde is used to characterize any dimensional physical quantity.

Note, in particular, that \( k'_w \) and \( k'_o \) increase strongly as \( Ca \) increases (at fixed saturation), which means that the flowrate vs pressure gradient relation is strongly non-linear. The key question, then, is ‘What are the sources of non-linearity and of the other complex effects

---

* Corresponding author.

E-mail addresses: marval@iceht.forth.gr (M.S. Valavanides), acp@iceht.forth.gr (A.C. Payatakes).

1 Now with IRC Help-Forward, Xenofontos 5, GR 105 57 Athens, Greece.
Notation

General notes:

A tilde (\(\tilde{\cdot}\)) on top of a character denotes a dimensional variable. Absence of a tilde denotes that the variable is dimensionless.

Superscript A superscript attached to a variable denotes characterization on the macroscopic scale.

Subscript A subscript attached to a variable denotes characterization on the microscopic scale.

Variables with prime (\(S'\), \(\beta'\)) are the physically admissible values of the corresponding plain variables.

Latin characters

\(a\)  crowding effect factor
\(A\)  macroscopic cross-sectional area, normal to the macroscopic flow direction
\(A_{jk}^{\Phi}\)  reduced resistance of a \(jik\)-class cell in one-phase flow
\(A_{jk}^b\)  reduced bulk phase resistance of a \(jik\)-class cell for different flow configurations
\(A_{m}^b\)  mean of \(A_{jk}^b\) over all possible \(jik\) combinations, Eq. (30)
\(B_n^b\)  reduced interphasial resistance of any conducting cell of an \(n\) ganglion for different flow configurations
\(Bo\)  bond number
\(Ca\)  capillary number
\(C_0\)  pressure fluctuation factor, Eq. (27)
\(C_i^{\Lambda,G}\)  fractional distributions of blocked cells (side-gate unit cells) in any \(i\)-class ganglion
\(C_i^{C,G}\)  fractional distributions of core cells in any \(i\)-class ganglion
\(C_i^{E,G}\)  fractional distributions of end-gate cells in any \(i\)-class ganglion
\(Co\)  coalescence factor
\(D_T\)  uniform maximum pore-network depth
\(D_{ck}\)  class-\(j\) chamber diameter
\(Gm\)  ganglion mobilization number, Eq. (22)
\(g\)  gravity
\(g_{et}\)  reduced conductance of the effective porous medium
\(g_{et}^{\text{DOF}}\)  reduced effective conductance of the porous medium DOF region
\(g_{jk}^{\Phi}\)  conductance of a \(jik\)-class cell in one-phase flow (reduced and dimensional)
\(g_{jk,n}^{b}\)  reduced conductance of a \(b\)-type \((b = \text{E,G'}, \text{C,G'} \text{ or } \text{X,G'}) \) \(jik\)-class cell of a \(n\)-ganglion
\(g_0\)  reduced uniform conductance of all cells in the equivalent network
\(f_j\)  occurrence probabilities of class \(j\)
\(f_{ijk}\)  occurrence probabilities of any \(jik\)-class cell
\(f_{\max}\)  maximum ganglion size class
\(j_{jk}^{GC}(\tilde{z};0)\)  mean surface curvature of the o/w interphase, as a function of its position in a \(j\)-class chamber
\(j_{jk}^{GT}(\tilde{z};0)\)  mean surface curvature of the o/w interphase, as a function of its position in an \(i\)-class throat
\(j_{jk}\)  mean (along a \(jik\) cell) reduced o/w mean interphase curvature
\(\tilde{j}_{jk}\)  drainage curvature of the receding o/w interphase in an \(i\)-class throat
\(\tilde{j}_{\text{min},jk}\)  minimum curvature value of the o/w interphase advancing in a \(j\)-class chamber
\(\tilde{k}\)  absolute permeability
\(\tilde{k}^b\)  effective porous medium absolute permeability
\(k_e^*, k_w^*\)  relative permeabilities in oil and water
\(\ell\)  lattice network constant (node-to-node distance)
\(\tilde{l}_{jk}\)  \(n\)-class ganglion’s length, projected in the macroscopic flow direction
\(m_i^G\)  assigned mobilization probability to any \(n\)-class oil ganglion
\(n_i^G\)  reduced number density of the \(i\)-class ganglia
\(n_i^e\)  number density of the \(i\)-class ganglia characteristic distribution
\(N_g\)  network lattice dimensionality
\(N_{i}^G\)  total number of unit cells occupied by any \(i\)-class ganglion
\(p_{y}\)  normal distribution of \(y\)
\(p_{i}, p_{j}\)  actual pressures in water next to menisci
\(n_i^G\)  occurrence probability of any \(n\)-ganglion in the ganglion cells domain
\(p_{jk,n}\)  occurrence probabilities of the various ganglion cells in the ganglion cells domain
\(q_{jk}^{\Phi}\)  equivalent macroscopic flowrate of one-phase in the DOF region
\(q_{\text{CPF}}^{\Phi}\)  macroscopic flowrate of oil
\(q_{\text{DOF}}^{\Phi}\)  macroscopic flowrate of water
\(q_{\text{CPF}}^{w}\)  mean oil flowrate in CPF
\(q_{\text{DOF}}^{w}\)  mean oil flowrate in DOF
\(q_{\text{local}}^{w}\)  mean water flowrate in DOF
\(q_{w}^{c}\)  flowrate of an incompressible fluid through a cell
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{q}_{i,j}^G$</td>
<td>total flowrate through any $i$-class ganglion conducting cells</td>
</tr>
<tr>
<td>$r$</td>
<td>oil/water flowrate ratio</td>
</tr>
<tr>
<td>$S_w$</td>
<td>water saturation</td>
</tr>
<tr>
<td>$\bar{S}_w$</td>
<td>physically admissible value of water saturation, see also Eq. (51)</td>
</tr>
<tr>
<td>$\bar{S}_{i,G}$</td>
<td>specific oil saturation in any $i$-class ganglion</td>
</tr>
<tr>
<td>$\bar{S}_o,G$</td>
<td>average oil saturation in the ganglion cells domain</td>
</tr>
<tr>
<td>$S_{o, DOF}$</td>
<td>disconnected-oil saturation in the DOF region</td>
</tr>
<tr>
<td>$u_1$, $u_{0,i,j}$</td>
<td>bipolar coordinates</td>
</tr>
<tr>
<td>$\bar{u}_i^G$</td>
<td>mean velocity of an $i$-class ganglion’s mass center</td>
</tr>
<tr>
<td>$\bar{U}_w$</td>
<td>superficial velocity of oil</td>
</tr>
<tr>
<td>$\bar{U}_{w,G}$</td>
<td>oil superficial velocity in the domain of ganglion cells, Eq. (32)</td>
</tr>
<tr>
<td>$\bar{U}_{o,CPF}$</td>
<td>oil mean superficial velocity in CPF</td>
</tr>
<tr>
<td>$\bar{U}_{w,DOF}$</td>
<td>oil mean superficial velocity in DOF</td>
</tr>
<tr>
<td>$\bar{U}_{w,DOF}$</td>
<td>water mean superficial velocity in DOF</td>
</tr>
<tr>
<td>$\bar{V}$</td>
<td>macroscopic control volume</td>
</tr>
<tr>
<td>$\bar{V}_{uc}$</td>
<td>average (over all classes) volume of the unit cell</td>
</tr>
<tr>
<td>$\bar{V}_{i,G}$</td>
<td>average actual volume of an $i$-class ganglion</td>
</tr>
<tr>
<td>$L_{m,G}$</td>
<td>expected mean ganglion size of the mobilized ganglia</td>
</tr>
<tr>
<td>$\bar{L}_{m,G}$</td>
<td>expected mean ganglion size of the overall ganglion population</td>
</tr>
<tr>
<td>$\bar{W}$</td>
<td>reduced total rate of mechanical energy dissipation</td>
</tr>
<tr>
<td>$\bar{W}_{o,CPF}$</td>
<td>reduced rate of mechanical energy dissipation in CPF</td>
</tr>
<tr>
<td>$\bar{W}_{o,DOF}$</td>
<td>reduced rate of mechanical energy dissipation in DOF</td>
</tr>
<tr>
<td>$\bar{W}_{ij}$</td>
<td>class-$j$ throat width</td>
</tr>
<tr>
<td>$x$</td>
<td>reduced common pressure gradient</td>
</tr>
<tr>
<td>$X_{0,i,j}$</td>
<td>Cartesian coordinates</td>
</tr>
<tr>
<td>$\chi_{mic}$</td>
<td>reduced microscopic pressure gradient</td>
</tr>
<tr>
<td>$\chi_{pm}$</td>
<td>parameter vector composed of all the dimensionless geometrical and topological parameters of the porous medium affecting the flow (porosity, genus, coordination number, normalized chamber and throat size distributions, chamber-to-throat size correlation factors, etc.)</td>
</tr>
<tr>
<td>$y$</td>
<td>reduced variable</td>
</tr>
<tr>
<td>$\bar{z}$</td>
<td>macroscopic flow direction</td>
</tr>
<tr>
<td>$(-\bar{\nabla}p/\bar{\nabla}z)$</td>
<td>common pressure gradient</td>
</tr>
</tbody>
</table>

**Greek symbols**

- $\beta$ | the porous medium volume fraction occupied by the connected oil |
- $\beta'$ | physically admissible value of $\beta$ |
- $\beta_{i,j}$ | oil/water interfacial tension |
- $\Delta p_{\phi}$ | pressure drop along any $jik$-class unit cell during one-phase flow through it |
- $\Delta p_{\phi,jk,n}$ | pressure drop along any $jik$-class cell of an $n$-ganglion for different flow configurations |
- $\zeta$ | free parameter which regulates the steepness of the ganglion size distribution |
- $\eta^{\phi,CPF}$ | expected mean macroscopic connected-oil flow fraction |
- $\kappa$ | contact angle |
- $\hat{\varepsilon}_A, \hat{\varepsilon}_R$ | limiting values of the advancing and receding contact angles (at nil velocity) |
- $\hat{\phi}_A, \hat{\phi}_R$ | limiting values of the advancing and receding contact angles |
- $\bar{\mu}_o, \bar{\mu}_w$ | dynamic viscosity of oil and water, respectively |
- $\sigma$ | pore-network coordination number |
- $\sigma_a$ | variance of the distribution of the logarithm of the microscopic pressure gradient |
- $\phi$ | expected mean macroscopic flow quantity |
- $\chi^G$ | tortuosity of the spine of any ganglion |
- $\omega$ | fraction of all the ganglion cells over all the DOF region cells |
- $\Omega_{PA}$ | integration domain corresponding to all physically admissible solutions |

**Abbreviations**

- CPF | connected-oil pathway flow |
- C,G | core unit cell of a ganglion |
- DeProF | decomposition in prototype flows |
- DTF | drop traffic flow |
- DOF | disconnected-oil flow |
- E,G | end-gate unit cell of a ganglion |
- EMT | effective medium theory |
- FAV | flow arrangement variables |
- GCD | ganglion cells domain |
- GD | ganglion dynamics |
- LGD | large ganglion dynamics |
- PA | physically admissible |
- SFA | smooth field approximation |
- SGD | small ganglion dynamics |
- SSFD | steady-state fully developed |
- WSCD | water saturated cell domain |
- X,G | non-conducting unit cell of a ganglion |
- 2PM | two-phase flow in porous media
indicated by Eq. (1), and how can these phenomena be modeled?

2. Experimental observations on steady-state two-phase flow in porous media (SS2FPM)

Visual experimental studies [1,2] have shown that the disconnected oil contributes substantially to the overall flow of oil. The observed flow behavior has been roughly classified into four flow regimes, namely large ganglion dynamics (LGD), small ganglion dynamics (SGD), drop traffic flow (DTF), and connected-oil pathway flow (CPF). In the first three of these regimes, all of the oil flow is due to the motion of disconnected bodies of oil (ganglia or drops). Even in the fourth regime, CPF (which is achieved with relatively large $Ca$ values), an appreciable portion of the oil flowrate is due to the motion of droplets and ganglia in between the connected-oil pathways. In many cases, the overall flow is a mixture of two or more of the aforementioned flow patterns. For reasons of economy, in the present work, we often combine LGD and SGD into an overall flow regime, namely, ganglion dynamics (GD).

The complex behavior indicated by Eq. (1) becomes comprehensible once the active participation of ganglia and droplets in the flow is acknowledged and taken into account explicitly.

A theoretical model is developed here which accounts for the pore-scale mechanisms and the network wide cooperative effects, and which is sufficiently simple and fast for practical purposes. This model is based on the concept of decomposition in prototype flows (DeProF) and is an improved version of an earlier work presented in [15].

Briefly, we assume that the macroscopic flow can be decomposed into two prototype flows, namely CPF and disconnected-oil flow (DOF). The latter comprises LGD, SGD and DTF (although in the present work we omit DTF; the contribution of DTF will be included in a future work). Each prototype flow has the essential characteristics of the corresponding experimentally observed flow patterns, in suitably idealized form, and so the pore-scale mechanisms are incorporated in the prototype flows. The cooperative effects among ganglia are also incorporated in DOF making suitable use of effective medium theory (EMT). A key feature of the proposed model is that the two prototype flows are entwined by imposing the condition that the pressure gradient is the same in both. This is a condition that is supported both by theory and experiment [4]. We find that there is a domain of admissible combinations of the prototype flows that corresponds to any given set of

![Fig. 1](image.png)

Fig. 1. (a) Actual flow and its theoretical decomposition into prototype flows: CPF and DOF; (b) conceptual replacement of the actual DOF region with an effective porous medium layer, of absolute permeability $\tilde{k}^f$, through which flows "effective water", with a total flowrate equal to the sum of the flowrates of oil and water through DOF, Eq. (9). The effective permeability of this layer is defined by Eq. (10).
values of the system parameters. The solution is determined by assuming equal probability for each admissible combination and averaging. The solution depends on the values of all the system parameters (see RHS of Eq. (1)). A more detailed description follows.

Consider the general flow in Fig. 1(a). Connected pathways of oil (when they exist) occur in roughly ‘parallel’ pathways. In between these pathways, a fraction of the population of oil ganglia are mobilized by the water-induced pressure gradient and migrate downstream. The total flow is a mixture of two flow patterns, specifically CPF and DOF. The key difference between these flow patterns is that in DOF, numerous menisci are in motion and many ‘catastrophic’ (irreversible) flow phenomena are entwined by having common pressure gradient, \((-\partial \bar{p}/\partial z)\). Strictly speaking, this holds true for steady-state fully developed two-phase flow in porous media (SSFD2oFPM) [2,4] (and only approximately so in regions with weak saturation gradients).

Using this condition, we obtain

\[
\frac{\bar{q}_i}{A} = k_i \bar{K} \left( -\frac{\partial \bar{p}}{\partial z} \right) \quad i = o, \ w
\]  

as well as the following relation between the mobility ratio, \(\hat{\lambda}_i/\lambda_i\), and the flowrate ratio, \(r\):

\[
\frac{\bar{K}_i}{\lambda_i} = \frac{\bar{K}_w}{\lambda_w} = \frac{\hat{\lambda}_i}{\lambda_i} = r.
\]  

We define the reduced pressure gradient, \(x\), as

\[
x = \frac{\bar{K}}{\gamma_{ow} Ca} \left( -\frac{\partial \bar{p}}{\partial z} \right).
\]  

Assuming that in the CPF region oil flows with virtually one-phase flow (i.e., neglecting the possible lubrication effect of wetting films), we can use Darcy’s law to obtain

\[
\bar{U}^{o, \text{CPF}} = \frac{k}{\lambda_o} \left( -\frac{\partial \bar{p}}{\partial z} \right).
\]  

Assuming that there is no water present in the CPF region, a total water mass balance yields

\[
\bar{q}^{w, \text{DOF}} = \bar{q}^{w} = (1 - \beta) \bar{U}^{w, \text{DOF}} = \bar{U}^{w}
\]  

and a total oil mass balance yields

\[
\bar{q}^{o, \text{CPF}} + \bar{q}^{o, \text{DOF}} = \bar{q}^{o} = \beta \bar{U}^{o, \text{CPF}} + (1 - \beta) \bar{U}^{o, \text{DOF}} = \bar{U}^{o}.
\]  

An overall oil balance gives

\[
\beta + (1 - \beta) S^{o, \text{DOF}} = 1 - S^{w}.
\]  

In the final analysis, the macroscopic behavior of the flow is the manifestation of pore-scale flow phenomena and pore-network cooperative effects.

Scaling-up of the flow in the CPF region is relatively simple. Oil flow in the pores is virtually one-phase flow and, under creeping flow conditions, the hydraulic conductivity of each unit cell, Fig. 2, is a function of local geometrical parameters only. The permeability of
the CPF region can be obtained from the hydraulic conductances of the unit cells through application of EMT. This is the absolute permeability of the porous medium, and it is strictly a function of pore geometry and pore-network topology (or connectivity).

Scaling-up of the flow in the DOF region is more complex, because the two different fluids are simultaneously arranged within the pores separated by a multitude of advancing, receding and stranded menisci. Here, the hydraulic conductivity of any individual unit cell is not only a function of the appropriate local geometrical parameters, but also of the fluid arrangement within it, Fig. 3. Furthermore, even under creeping flow conditions, the hydraulic conductance in a unit cell will be a function of the velocity of the menisci, because of the dependence of the contact angles (advancing and
receding) on the local displacement velocity. In order to deal with this problem, we proceed as follows. First, we replace the DOF region with an effective porous medium, through which flows a one-phase fluid with flowrate, \( q^{1\Phi} \), that is equal to the sum of the flowrates of water, \( q^{w, DOF} \), and oil, \( q^{o, DOF} \), through the DOF region, Fig. 1(b). In order to fix ideas, we assume that this conceptual fluid has viscosity equal to that of water, and we will call it ‘water’. (This choice does not cause loss of generality, because the viscosity that is assigned to the conceptual fluid is taken into account properly.) Thus, we have

\[
q^{1\Phi} = q^{w, DOF} + q^{w, DOF}.
\]

(9)

Second, we define the effective permeability, \( k^{ef} \), of this conceptual medium by imposing Darcy’s law for one-phase flow

\[
q^{1\Phi}[\hat{A}(1-\beta)]^{-1} = \frac{k^{ef}}{\mu_w} \left( -\frac{\partial \hat{p}}{\partial \hat{z}} \right).
\]

(10)

Eqs. (9) and (10) along with (5)–(7) give

\[
\begin{align*}
\hat{k}^{ef} & = \frac{\hat{q}_w^{DOF} \hat{l}_{\hat{k}}}{\hat{e}} \\
& = \left[ Ca(1+r) \frac{\hat{g}_{ow}}{(-\partial \hat{p}/\partial \hat{z})} - \frac{\hat{k}^{ef}}{\hat{e}} \right] (1-\beta)^{-1},
\end{align*}
\]

(11)

where \( \hat{g}_w^{DOF} \) is, by definition, the effective hydraulic conductance. Eq. (11) shows that \( \hat{k}^{ef} \) is a function of the absolute permeability (of the actual porous medium) as well as of the pressure gradient, the capillary number, the flowrate ratio, the viscosity ratio, the interfacial tension and the FAV, \( \beta \).

Now, a basic tenet of the present theory is that the effective hydraulic conductance, \( \hat{g}_w^{DOF} \) (and therefore, \( \hat{k}^{ef} \), can be calculated from the hydraulic conductances of the unit cells composing the (actual) pore-network in the DOF region by pressing into service EMT. To this end, one needs to have the topology of the skeleton of the pore network, and the size distribution of the hydraulic conductances of the unit cells. The latter, as already mentioned, are functions not only of the unit cell geometry, but also of the flow arrangement in each unit cell and the probability of occurrence of each flow arrangement. These results are developed in the following section. Before proceeding, however, we must note that EMT in its conventional form is used for networks, each branch (here unit cell) of which has a random but constant conductance. In the present work, the conductance of each unit cell can only be considered at any given instant as momentarily constant. Furthermore, the flow arrangement in each unit cell and the probability of occurrence of each flow arrangement depend, as we shall see, on the pressure gradient itself. However, we expect that EMT is still valid in a narrow domain around the solution, because one can assume that in such a narrow domain all hydraulic conductances are constant and equal to the values that correspond to the pressure gradient given by the solution. (The results bear out this expectation.)

3.2. Microscopic scale

3.2.1. Model porous medium. Geometric and topologic characteristics

The microscopic scale analysis involves the specific geometric and topologic characteristics of the porous medium under consideration. In order to be able to validate the theoretical model by comparing its predictions with available experimental results, we carried out the analysis adopting the topological and geometric characteristics of the glass pore-network model that was used in the experiments of Avraam and Payatakes [2]. It should be noted that this choice does not constitute a constraint on the validity of the approach. A similar analysis can be made with 3-D pore networks, even networks with coordination number \( \sigma \) that varies from node to node (within reason of course). The pore geometry can also be different, but the basic approach would still be the same. As shown in [1], two-phase flow in 3-D networks differs from than in 2-D ones quantitatively, but not qualitatively.

The model pore network of [2] is of the chamber-and-throat type. The network skeleton is a square lattice with node-to-node distance \( \ell = 1221 \mu m \), and arranged so that the macroscopic flow direction is parallel to one family of diagonals. For this particular network, the lattice dimensionality is \( N_d = 2 \), and the coordination number is \( \sigma = 4 \). (For a cubic network, we would have \( N_d = 3 \) and \( \sigma = 6 \).) The maximum pore depth is nearly uniform and was experimentally determined to be \( D_p = 115 \mu m \). The chamber size (diameter) distribution and the throat size (width) distribution have discrete (composed of five classes each) nearly normal size distributions with mean values \( \langle D_C \rangle = 610 \mu m \), \( \langle W_T \rangle = 169 \mu m \) and standard deviations equal to 1/4 of the corresponding mean values. The occurrence probabilities \( f_j \) of each class along with the respective actual characteristic dimensions of the chambers (diameters), \( D_C \), and the throats (widths), \( W_T \), are given in Table 1.

It should be noted that the chamber and throat diameter values in Table 1 are somewhat different from those reported in [2]. This is because [2] reported

<table>
<thead>
<tr>
<th>( f_j ) (%)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_C ) (( \mu m ))</td>
<td>330</td>
<td>470</td>
<td>610</td>
<td>750</td>
<td>890</td>
</tr>
<tr>
<td>( W_T ) (( \mu m ))</td>
<td>111</td>
<td>139</td>
<td>169</td>
<td>195</td>
<td>222</td>
</tr>
</tbody>
</table>
nominal dimensions, i.e., the dimensions on the negative mask that was used in the etching of the glass model. The dimensions given here are the measured actual dimensions of the model pore network that was used in the experiments of [2].

The chambers are modeled as normal circular cylinders with their longitudinal axes oriented perpendicular to the network plane and located at the network nodes. The throats are modeled as normal elliptic cylinders with their longitudinal axes placed on the network bonds. The basic unit (building block) of the pore network is the unit cell that comprises of a throat connecting the corresponding sectors (quarters) of the chambers adjacent to it. A typical unit cell is depicted in Fig. 2. It is denoted as a \( jik \)-class cell, because it is composed of a \( j \)-class chamber and a \( k \)-class chamber connected with an \( i \)-class throat. The thick dashed lines represent the unit cell limits, and the dashed–dotted lines represent the network lattice. The solid matrix of the cell is shaded. \( V_{uc} \) is the average (over all classes) volume of the unit cell. All lengths are normalized with \( \ell \), the network lattice constant. The basic geometrical characteristics of the cell, i.e., the reduced chamber diameters, \( D_{Cj}, D_{Ck}, \) and the reduced throat width, \( W_{Tr} \), are also defined in Fig. 2. All other dimensions shown in this figure are explicit functions of \( D_{Cj}, D_{Ck} \) and \( W_{Tr} \). In the present work, we assume that there is no correlation between the classes of throats, chambers and cells. Thus, the occurrence probability of any \( jik \)-class unit cell, \( f_{jik} \), is given by

\[
f_{jik} = f_{j} f_{i} f_{k}.
\]

### 3.2.2. Reduced conductances for one-phase flow, \( g_{jik}^{\Phi} \)

When the problem of one-phase flow through this network is considered, one can calculate the conductances of each unit cell solely from its geometrical characteristics. These conductances represent the transfer function of each cell for one-phase creeping flow. All cells that belong to the same \( jik \)-class have the same one-phase conductance, \( g_{jik}^{\Phi} \). The reduced conductances, \( g_{jik}^{\Phi} \), are given by

\[
g_{jik}^{\Phi} = \frac{\mu}{\ell^3} = \frac{q_{uc}^{\Phi} \mu}{\Delta p_{jik}^{\Phi} (q_{uc}^{\Phi})^2 \ell^3} = \frac{q_{uc}^{\Phi} \mu}{A_{jik}^{\Phi} q_{uc}^{\Phi} \ell^3} = \frac{1}{A_{jik}^{\Phi}}
\]

\[
= \left[ 12 \frac{1}{D_{t}} (u_{0ij} + u_{0ik} - 2u_{t}) + \frac{64}{\pi} \frac{W_{Tj}^2 + W_{Tk}^2}{W_{Tj}^2 W_{Tk}^2} (1 - x_{0ij} - x_{0ik}) \right]^{-1},
\]

where \( \ell \) is the network lattice constant, \( \mu \) the dynamic viscosity of the fluid, \( q_{uc}^{\Phi} \) the flowrate through the unit cell, \( \Delta p_{jik}^{\Phi} (q_{uc}^{\Phi}) \) the corresponding pressure drop along the unit cell, and \( A_{jik}^{\Phi} \) is the reduced resistance of the \( jik \)-class cell. Details concerning the derivation of the transfer function \( \Delta p_{jik}^{\Phi} (q_{uc}^{\Phi}) \) are given in the Appendix A. The first term in the brackets on the RHS of Eq. (13) is the contribution of the flow in the two chambers, and the second term is the contribution of the flow in the throat.

It is clear that \( g_{jik}^{\Phi} \) is a function of only the geometrical characteristics of the \( jik \)-class cell. This is a typical characteristic of creeping one-phase flow.

### 3.2.3. Effective medium theory

The overall conductance of a network, such as the one previously described, can be calculated numerically by applying effective medium theory (EMT). The first step is to replace the original network with an equivalent network that has the same skeleton as the prototype, but is composed of uniform unit cells. In order to achieve the same overall conductance, the uniform cells of the equivalent network must have reduced conductances equal to \( g_{0} \), which is determined by solving the EMT equation [13]

\[
\sum_{jik} f_{jik} \left( \frac{g_{0} - g_{jik}^{\Phi}}{\sigma/2 - 1} g_{0} + g_{jik}^{\Phi} \right) = 0,
\]

where \( \sigma \) is the pore-network coordination number. The equivalence of these networks is expressed on the macroscopic scale: their macroscopic transfer functions relating superficial velocities and pressure gradients (Darcy’s law) are identical. When Eq. (14) is solved in terms of \( g_{0} \), the absolute permeability, \( k \), of the original network is analytically recovered through the relation

\[
k = g_{ef} \hat{\ell}^2,
\]

where \( g_{ef} \) is the reduced effective conductance of the equivalent network, and is related to \( g_{0} \) by (Burganos and Sotirchos [5])

\[
g_{ef} = \frac{\sigma}{2N_{d} g_{0}}.
\]

Note that for the network adopted here, and for all regular orthonormal networks, \( g_{ef} = g_{0} \).

Using Eqs. (13)–(16) and the data in Table 1, we obtain the theoretical value \( k = 8.89 \mu m^{2} \), which is in excellent agreement with the experimental value \( 8.90 \mu m^{2} \) reported in [2].

When the problem of one-phase creeping Newtonian flow in a cell is considered, the resistance to the flow is caused by viscous stresses only, and so the transfer function of the cell is linear. In the case of DOF, where two-phase flow in a cell is common, things are more complicated and non-linearity appears even in the case of very slow Newtonian flows. This case is treated below.

### 3.2.4. Reduced conductances for the disconnected-oil flow (DOF) domain

A microscopic scale representation of a typical DOF region is shown in Fig. 3. An oil ganglion having a
typical ‘cruising’ configuration [16,18] is shown at the center. All the cells that accommodate parts of this (or any other) oil ganglion are called ganglion cells and are outlined with a thick dashed line.

For the purposes of the present work, we will use the term ganglion in an extended way (rather than saying network-ganglion) to denote the entire part of the pore network that is composed of the ganglion cells of any given oil ganglion (see, for example, the region outlined with the thick dashed line in Fig. 3). The DOF region is partitioned into two sub-regions: the sub-region that comprises all ganglion cells and the sub-region that comprises unit cells containing only water. These will be called ganglion cell domain (GCD) and water saturated cell domain (WSCD), respectively. The fraction of all the ganglion cells over all the DOF region cells is denoted by \( \omega \), and will be called the GCD fraction. Thus, the fraction of all the water saturated cells in the DOF region equals \((1 - \omega)\). (As it will be shown below, the variable \( \omega \) will be eliminated in the system of DeProF equations presented in this work. We introduce it here for the sake of clarity in further developments. In a future extension of the present work in which drop traffic flow (DTF) will be included in the DOF domain, \( \omega \) will be shown to be the appropriate third flow-arrangement variable.)

The ganglion sizes have a distribution that depends on the flow conditions (for a given porous medium and fluids system). Determination of this distribution will be achieved along with the rest of the solution. Each oil ganglion is said to belong to a certain class \( i \) with \( i = 1, I_{\text{max}} \) based on the number of chambers it occupies. At any instant, the total number of unit cells occupied by any \( i \)-class ganglion (having reduced volume \( i \)) is \( N_{\text{G}}^i = \lfloor (2N_d - 1) + \rfloor \). The average actual volume of an \( i \)-class ganglion is given by \( V_{\text{G}}^i = N_{\text{G}}^i V_{\text{core}} \). \( I_{\text{max}} \) is the upper truncation class of the ganglion size distribution, and so no ganglia with volume greater than \( N_{\text{fun}}^i V_{\text{fun}} \) appear. It is important to note that \( I_{\text{max}} \) is an implicit function of \( x \), and is a derived parameter. In a typical situation, only a fraction of the ganglion population will be mobilized. Moving ganglia have a tendency to become aligned with the macroscopic flow direction, acquiring the so-called ‘cruising shape’ [16,18]. In the present work, this is taken into account by assuming that all ganglia have a zigzag spine that is aligned along the macroscopic pressure gradient. An important parameter associated with the ‘zigzag’ motion of the ganglion is the tortuosity of its spine, \( \chi^G \). This is also the ratio of the length of its actual migration path over the effective migration length in the macroscopic flow direction. This parameter depends on the porous medium geometrical and topologic characteristics and will be considered to be equal for all ganglion size classes. For the case under consideration, we set \( \chi^G = \sqrt{2} \).

Unit cells in which ganglion menisci exist are called gate unit cells in accordance with [14,16]. In the present work, we assume that all ganglia are ‘wormlike’ and aligned (as much as their zigzag shape allows) with the macroscopic flow direction. Among all gate unit cells of a mobilized ganglion, there are only two gate unit cells in which rheons take place (one cell for the ‘xeron’ and one cell for the ‘hygron’ [16]). Such cells will be denoted with the index ‘E,G’ (‘end-gate,ganglion’). We assume that the only potential end-gate unit cells are those located at the foremost part (for the hygron) and at the aft-most part of the ganglion (for the xeron); for example unit cells 1 or 7 for the xeron and 6 or 12 for the hygron in Fig. 3. All cells of any mobilized ganglion that are completely saturated with oil, e.g., cells 8, ... , 11 in Fig. 3 will be called ‘core cells’, and will be denoted with the index ‘C,G’ (‘core,ganglion’). Any \( i \)-class mobilized ganglion has \((i - 1)\) ‘C,G’ cells. The rest of the ganglion cells (which are neither end-gate cells nor core cells) contain oil and water separated by virtually immobile menisci and are called side-gate unit cells.

Ganglion migration induces a corresponding flow, \( q_{\text{G,J}} \), of oil in the core cells and an equal total flowrate of water and oil in the end-gate cells. This flow must be consistent with the mean velocity of the mass center of the ganglion, \( \bar{u}_G \), i.e.,

\[
\bar{u}_G = \frac{\bar{u}^G}{\chi^G}. \tag{17}
\]

The rest of the cells of a mobilized ganglion (side-gate unit cells) are considered as non-conducting, and they are denoted with the index ‘X,G’ (short for ‘non-conducting,ganglion’). Any \( i \)-class mobilized oil ganglion occupies \( 2i(N_d - 1) \) ‘X,G’ cells, and, of course, all of the \( N_G^i \) cells of any \( i \)-class stranded oil ganglion are ‘X,G’ cells. Therefore, the fractional distributions of the various flow configurations of any \( i \)-class ganglion cells (mobilized as well as stranded ganglia taken into account) are explicitly given by

\[
C_{X,G}^i = \frac{i(2N_d - 1) + 1 - m_G^i(i + 1))}{N_G^i},
\]

\[
C_{C,G}^i = m_G^i(i - 1)/N_G^i, \tag{18}
\]

\[
C_{E,G}^i = 2m_G^i/N_G^i,
\]

where \( m_G^i \) is the mobilization probability assigned to the \( i \)-class oil ganglia, and will be discussed below. It is easily verified that \( C_{X,G}^i + C_{C,G}^i + C_{E,G}^i = 1 \) given the aforementioned expression for \( N_G^i \).

The number density of the \( i \)-class ganglia, \( n_G^i \), \( i = 1, I_{\text{max}} \), is defined as the number of \( i \)-class ganglia per unit volume of all the ganglion cells, or, better, per GCD unit volume. Expressing the volume of ganglion cells in the DOF region in terms of the ganglion size distribution, we obtain
If all of the disconnected oil were redistributed into class-1 ganglia (singlets), then the number density of singlets would be \( \hat{n}_s^G = 1/(2N_d \hat{V}_{uc}) \). We define the reduced number density of the \( i \)-class ganglia, \( \{ \hat{n}_i^G, \ i = 1, \ I_{\text{max}} \} \), as \( n_i^G = \hat{n}_i^G / \hat{n}_s^G = 2n_i^G N_d \hat{V}_{uc} \). Clearly, \( n_i^G \) is equal to the ratio of the total number of \( i \)-class ganglia cells over the total number of all ganglion cells in the DOF region. All \( \{ n_i^G (i = 1, I_{\text{max}}) \} \) are unknown variables, which will be determined in the context of the DeProF model.

By definition, the GCD fraction, \( \omega \), in terms of the ganglion size distribution, is given by

\[
\omega = \frac{(1 - \beta) \hat{V} \sum_{i=1}^{I_{\text{max}}} \hat{n}_i^G \hat{V}_i^G}{(1 - \beta) \hat{V}} = \sum_{i=1}^{I_{\text{max}}} \hat{n}_i^G \hat{V}_i^G. \tag{20}
\]

A specific oil saturation, \( S_{n}^{G} \), corresponds to the \( i \)-class of ganglia. This saturation is easily determined by assuming that all core cells (\( C,G \) cells) are completely saturated with oil whereas all other ganglion cells are saturated half with oil and half with water. By making an oil volume balance over all cells of this ganglion class, one obtains \( S_{n}^{G} = n_n^G / N_n^G \). The average oil saturation in the GCD, \( S_{G}^{G} \), in terms of the reduced ganglion size distribution is given by

\[
S_{G}^{G} = \frac{\omega (1 - \beta) \hat{V} \sum_{i=1}^{I_{\text{max}}} \hat{n}_i^G \hat{V}_i^G S_{n}^{G}}{\omega (1 - \beta) \hat{V}} \Rightarrow S_{G}^{G} = \frac{1}{2} \sum_{i=1}^{I_{\text{max}}} \hat{n}_i^G = \frac{S_{n,\text{DOF}}}{\omega}. \tag{21}
\]

As in the case of \( \omega \), \( S_{G}^{G} \) is also a variable of temporary usefulness if one neglects oil droplets (as we do here). However, it will have a more substantial role to play in the extended model formulation in which DTF will be taken into account.

3.2.5. Probability of oil ganglion mobilization, \( m^{G} \)

In a typical situation, only a fraction of the oil ganglion population is mobilized. We assign a probability of mobilization, \( m^{G} \), to ganglion class-\( n \). This probability is estimated by applying, in a statistical sense, the generalized oil ganglion mobilization criterion ([14,16]) appropriately extended to account for the crowding effects that emerge in the presence of a dense population of oil ganglia.

Consider a typical ganglion that is placed within a pore network through which water flows from left to right driven by a macroscopic pressure gradient, \((- \nabla \bar{p})\), Fig. 4. In order to obtain a conservative estimate of the pressure gradient that is required to cause mobilization of the ganglion, we consider the case in which the end-mениisci offer maximum capillary resistance. Thus, the downstream meniscus is in a throat (denoted with \( i \)), while the upstream meniscus is in a chamber (denoted with \( j \)).

The curvature of the downstream meniscus, \( J_{d,i} \), is a function of the size of the throat (not to be confused with that of the \( i \)-class of throats) and the receding contact angle, \( \theta_{R} \). For simplicity, we use the limiting value of \( \theta_{R} \) (as the receding velocity go to zero), \( \theta_{R}^{0} \), irrespective of the actual velocity of the meniscus. The minimum possible curvature of the upstream meniscus, \( J_{\text{min},j} \), is a function of the size of the local chamber (not to be confused with that of the \( j \)-class of chambers) and the advancing contact angle, \( \theta_{A} \). Again, for simplicity, we use the limiting value of \( \theta_{A} \), namely \( \theta_{A}^{0} \), irrespective of the actual velocity of the meniscus. (The dependence of \( \theta_{R} \) and \( \theta_{A} \) on the corresponding meniscus velocity can be introduced readily; this would produce a small benefit

\[\begin{align*}
\theta_{A}^{0} & \quad \text{(Advancing contact angle)} \\
\theta_{R}^{0} & \quad \text{(Receding contact angle)} \\
J_{d,i} & \quad \text{(Downstream meniscus curvature)} \\
J_{\text{min},j} & \quad \text{(Minimum upstream meniscus curvature)}
\end{align*}\]

![Fig. 4. A ganglion at a 'hard-to-move' configuration in the time interval between two rheons. (Emphasis on the end-gate unit cells.)](image-url)
at the expense of substantially more cumbersome calculations.)

Following [14,16], the appropriate ganglion mobilization number is defined by

$$Gm \equiv \max_{ij} \frac{(\bar{p}_i - \bar{p}_j)}{\gamma_{cw} \left[ \bar{J}_{dx,i}(\theta_R^i) - \bar{J}_{dx,j}(\theta_R^j) \right]}$$  \hspace{1cm} (22)$$

where $\bar{p}_i$ and $\bar{p}_j$ are the actual pressures in the water next to the menisci with indices $i$ and $j$, respectively. In (22), index $i$ scans all the throats in the gate unit cells, whereas index $j$ scans all the chambers. According to [14,16], the criterion for ganglion mobilization is

If $Gm > 1 \Rightarrow$ mobilization,
If $Gm < 1 \Rightarrow$ stranding. \hspace{1cm} (23)

Calculation of the actual values of $\bar{p}_i$ and $\bar{p}_j$ requires detailed pore-network simulations as in [6,7,9]. Whenever ‘exact’ values of $\bar{p}_i$ and $\bar{p}_j$ are available, they should be used. Otherwise, a rough estimate of the pressure difference $(\bar{p}_i - \bar{p}_j)$ can be obtained by applying smooth field approximation (SFA).

For a solitary ganglion, Eq. (22) and use of SFA give

$$Gm \simeq \beta_{ij} \frac{Ca}{k}$$  \hspace{1cm} (24)$$

with $k = k_f^{\ell^2}$, and

$$\beta_{ij} = \max_{ij} \frac{\Delta L_{ij} \cos \theta_j}{\ell^2 \left[ \bar{J}_{dx,i}(\theta_R^i) - \bar{J}_{dx,j}(\theta_R^j) \right]}$$  \hspace{1cm} (25)$$

Here, $\Delta L_{ij} \cos \theta_j$ is the length of the line connecting points $i$ and $j$ projected along the macroscopic pressure gradient, Fig. 4; $I$ and $J$ are the values of $i$ and $j$, respectively, for which the RHS of (25) becomes maximum. (In the case of ‘wormlike’ ganglia that are aligned with the macroscopic pressure gradient, positions $I$ and $J$ usually correspond to the ‘nose’ and ‘tail’ of the ganglion, respectively. This is adopted in the present work as the standard case.)

If one introduces typical parameter values in (24) and (25) and applies the mobilization criterion (23), then it is easy to conclude that a typical ganglion remains stranded unless the value of $Ca$ becomes of the order of $(10^{-4})$ to $(10^{-3})$ or larger depending on ganglion size. However, ganglion mobilization becomes possible at much smaller $Ca$ values as in the case of dense populations of ganglia. As discussed in [14,16], in the case of populations of ganglia, Eq. (24) must be modified as follows:

$$Gm \simeq \beta_{ij} \frac{Ca}{k_f^{\ell^2}} \left( \text{population of ganglia} \right),$$  \hspace{1cm} (26)$$

where $k_f^{\ell^2}$ is the relative permeability to water. Here, $k_f^{\ell^2}$ accounts for the ‘crowding effect’ and it can take values much smaller than unity when the population of ganglia becomes very dense. Consequently, in the case of ganglion populations, ganglia can move even at $Ca \ll 10^{-3}$. On the pore-scale, the explanation is that the presence of a dense population of ganglia reduces the number of pathways that are available for water flow to relatively few. Thus, the interstitial velocity of water in these pathways is large even for small $Ca$ values. This, in turn, requires large pressure gradients in the water even for small $Ca$ values, and this causes mobilization of ganglia.

A further improvement in the calculation of $Gm$ is proposed here. Pore network calculations [7,9] have shown that the local pressure of water at any node, even under steady-state macroscopic two-phase flow, fluctuates rapidly and by orders of magnitude as a result of the cooperative effects of the motion of a multitude of menisci throughout the network. These fluctuations have an asymmetric effect on ganglion mobilization. Negative fluctuations in the value of $(\bar{p}_i - \bar{p}_j)$ do not affect any ganglion that would not be mobilized by the time-averaged value. On the contrary, a positive fluctuation may mobilize a ganglion that would not be mobilized by the time-averaged value. On the other hand, a mobilized ganglion is relatively unaffected by pressure fluctuations when its downstream meniscus is not in a throat, which is a considerable part of its lifetime. Thus, pressure fluctuations facilitate and promote ganglion mobilization. This can be taken into account by introducing a pressure fluctuation factor $C_n$ in Eq. (26) to obtain

$$Gm \simeq \beta_{ij} \frac{CaC_n}{k_f^{\ell^2}}$$  \hspace{1cm} (27)$$

in which $C_n$ can take values substantially larger than unity.

In order to estimate $C_n$ in the present work, we assume that the local (dimensionless) pressure gradient on the pore-scale, $x_{mic}$, fluctuates around the macroscopic pressure gradient, $x$, with a probability distribution $P(y)$, where $y = (\ln x_{mic} - \ln x)/\sigma_x$. We assume that the distribution $F(y)$ is normal with mean 0 and variance 1. The value of $\sigma_x$ is determined by setting $x_{mic} = ax$, $y = 3$, where ‘$a$’ is a ‘gain’ parameter. (The effect of the gain parameter will be examined using sensitivity analysis.) Clearly, the distribution of $x_{mic}$ is lognormal. Lognormal distributions are characteristic of populations of entities that interact through birth–death phenomena. Eqs. (25) and (26) [or (27)], and the criterion (23) form the basis for the estimation of the mobilization probability for the various size classes of ganglia.

In order to estimate the $n$-class ganglion mobilization probability, $m_{ij}^{n\ell^2}$, we examine, for all $ij$ combinations, whether the estimated pressure difference along the ganglion is larger than the critical pressure difference required to induce mobilization. Using Eqs. (23), (25) and (27), we examine whether the following inequality holds:
The LHS of the above inequality is the smooth-field approximation of the local pressure difference along the ganglion. \( \hat{L}^G_n \int_{-\infty}^{+\infty} \frac{\partial P(y)}{\partial x} dy \geq \gamma_{ow} \left[ \hat{J}_{dr,j}(\theta_R^j) - \hat{J}_{min,j}(\theta_A^j) \right] \)
\( \forall j, i = 1, \ldots, 5. \) (28)

The LHS of the above inequality is the smooth-field approximation of the local pressure difference along the ganglion. \( \hat{L}^G_n \) is the macroscopic pressure gradient and \( \hat{L}^G_n = n \hat{L}^G \) is the ganglion length projected on the microscopic flow direction. It should be noted that the pressure gradient is calculated along with the rest of the variables in the context of the present model, and it inherently accounts for the crowding effect.

\[ \gamma_{ow} \] is the oil ganglion mobilization free energy. It is important to note that the pressure fluctuation effect is relatively insensitive to the gain parameter \( a \), and so use of this parameter introduces little arbitrariness. In all our simulations below, we set \( a = 10 \).

3.2.6. Oil ganglion mean velocity, \( u_n^G \)

From a pore-scale linear momentum balance (see Appendix A), the mean reduced velocity for class-\( n \) ganglia, \( u_n^G(x) \), is obtained

\[ u_n^G(x) = \frac{\bar{u}_n^G(x)}{\bar{U}^G} = \frac{1}{\bar{U}^G} \left[ \frac{x}{k} f_n^G - \frac{1}{Ca} (n + 1) b_n^G \right] \times \left( 2 A_m^G + (n - 1) A_m^{C,G} \right)^{-1}, \] (29)

where

\[ A_m^G = \langle A_{jik}^b \rangle_{jik} = \sum_{j,i,k=1}^5 A_{jik}^b \phi_f \phi_k, \] (30)

\[ B_n^G = B_n^{C,G} = B_n^{E,G}. \] (31)

\( A_{jik}^b \) and \( B_n^G \) represent the reduced effects of the bulk phases and the interphases on the total pressure drop in the corresponding cell, \( \Delta p_{nc,i}^G \) [see, also Eq. (35)], when a total flowrate \( \bar{q}_n^G \) is maintained through the ganglion cell; index ‘\( jik \)’ specifies the geometry of each cell, whereas ‘\( b \)’ specifies the two-phase flow pattern in it. Index ‘\( n \)’ specifies the ganglion size class. Explicit expressions for \( A_{jik}^b \) and \( B_n^G \) are given in the Appendix A [see Eqs. (A.5) and (A.6)]. The mean of \( A_{jik}^b \) over all possible \( jik \) combinations, i.e., \( A_m^G \), is used to account for

Fig. 5. The mobilization probability of a class-3 oil ganglion, \( m_3^G \), as a function of the reduced macroscopic pressure gradient, \( x \). The results shown are for a class-3 ganglion, this being either a solitary ganglion (\( a = 1 \)) or a member of a dense population and for various values of the crowding effect coefficient, \( a \). For smaller oil ganglia, the curves are translated to the left, for larger ganglia, the curves are translated to the right.
Combining Eqs. (32) and (33) we obtain
\[
\tilde{U}^{o,G}(x) = \sum_{i=1}^{n_{\max}} n_i^{G} \tilde{V}_{uc}^G S_i^{G} m_i^{G} \tilde{u}_i^{G}(x) + \frac{1}{\omega} \sum_{i=1}^{n_{\max}} n_i^{G} \tilde{V}_{uc}^G S_i^{G} m_i^{G} \tilde{u}_i^{G},
\]
(32)

An oil mass balance in the DOF domain gives
\[
\omega(1-\beta)\tilde{U}^{o,G}(x) = (1-\beta)\tilde{U}^{o,DOF}(x) \Rightarrow \omega\tilde{U}^{o,G} = \tilde{U}^{o,DOF}.
\]
(33)

Combining Eqs. (32) and (33) we obtain
\[
\tilde{U}^{o,DOF}(x) = \frac{1}{2} \sum_{i=1}^{n_{\max}} n_i^{G} \tilde{V}_{uc}^G S_i^{G} m_i^{G}(x) \tilde{u}_i^{G}(x).
\]
(34)

3.2.7. Unit cell conductances in the DOF region

At any instant, the pressure drop along any cell in the DOF region depends not only on the local geometrical configuration \((jik)\text{-class})\), but also on its flow configuration, i.e., on whether this cell belongs to the ganglion cells domain ('C,G', 'E,G', or 'X,G' cell), or to the water saturated cells domain (one-phase flow in a water saturated cell). The conductance of cells containing a single phase is given by Eq. (13). We turn now to unit cells containing menisci. Along any ganglion cell, the total pressure drop has two contributions. One contribution is caused by the viscous forces that resist the motion of each bulk phase. The other contribution is caused by the capillary action of the menisci. In the Appendix A, the transfer function \(\Delta \tilde{P}_{jik}^{G}(\tilde{Q}_{uc}^G)\) is derived for all ganglion cells. The reduced conductances of the cells in the GCD domain (Fig. 3) are given by the general expression
\[
\tilde{g}_{jik,n}^{G}(x) = \frac{\tilde{Q}_{uc}^G \tilde{m}_n}{\Delta \tilde{P}_{jik}^{G}} = \left[ \frac{\tilde{g}_{jik}^{G}}{C_{G}^{G} 1 \tilde{u}_i^{G}(x) B_n^{G}} \right]^{-1}.
\]
(35)

Note that \(\tilde{u}_i^{G}(x)\) is the \(n\)-ganglion reduced mean velocity and depends on the pressure gradient \(x\), Eq. (29). Thus, the hydraulic conductances of the ganglion cells are functions of the pressure gradient. This is one of the main causes of the non-linearity of the flow. (Another source of non-linearity is, as we shall see below, that the flow arrangement is a sensitive function of the pressure gradient.)

3.2.8. Application of effective medium theory (EMT) in the disconnected-oil flow (DOF) region

The EMT equation, appropriately modified to account for the different size classes \((j, i, k)\), ganglion sizes \((n)\) and flow configurations \((b)\) of the unit cell conductances in the DOF region, takes the form
\[
\sum_{jik} \begin{cases} (1-\omega) f_{jik} \left( \frac{g_0 - g_{jik}^{\phi}}{\xi - 1} \right) g_0 + g_{jik}^{\phi} \\
+ \omega \sum_n \sum_b \tilde{p}_{jik,n}^{b} \left( \frac{g_0 - g_{jik,n}^{\phi}}{\xi - 1} \right) g_0 + g_{jik,n}^{\phi} \end{cases} = 0.
\]
(36)

The first term in the brackets corresponds to the water saturated cells in the DOF region, whereas the second to the ganglion cells. \(\tilde{p}_{jik,n}^{b}\) are the occurrence probabilities of the various ganglion cells in the ganglion cells domain (GCD), and are given by
\[
\tilde{p}_{jik,n}^{b} = C_{n}^{b} g_{jik,n}^{G}.
\]
(37)

\(g_0\) is the reduced conductance of the uniform cells of the network that is macroscopically equivalent to the actual network and flow arrangement in analogy with Eq. (14). A brief description of Eq. (36) is necessary at this point. Strictly speaking, EMT has been developed for networks with conductances that are not functions of the corresponding fluxes and, therefore of the gradient of the potential that drives the flow. In Eq. (36), the conductances \(g_{jik,n}^{G}\) are functions of the pressure gradient \(x\) as shown by Eq. (35). One might say that EMT is pressed into service to deal with a non-linear flow system. This, however, is permissible in the neighbourhood of the solution, because one can think of \(g_{jik,n}^{G}\) as constant over a sufficiently narrow neighborhood around the solution.

The effective permeability of this equivalent network, \(k^{e}\), is related to \(g_0\) through a relation similar to Eqs. (15) and (16)
\[
k^{e} \equiv g^{e} \tilde{e} = \frac{\sigma}{2N_{d}} g_0 \tilde{e},
\]
(39)

so that Eqs. (10) and (11) hold. Inserting Eq. (39) in (36), we obtain a strongly non-linear equation in \(x\)
\[
(1-\omega) \sum_{jik} f_{jik} \left( 1 + r \right) - \frac{1}{k} x g_{jik}^{\phi} (1-\beta) k \right)
\times \left( 1 + r - x \frac{g_{jik}^{\phi} (1-\beta) k}{k} \right)
+ \omega \sum_{jik} \sum_{b} \tilde{p}_{jik,n}^{b} \left( 1 + r - x \frac{g_{jik,n}^{\phi} (1-\beta) k}{k} \right)
\times \left( 1 + r - x \frac{g_{jik,n}^{\phi} (1-\beta) k}{k} \right) = 0.
\]
(40)

for \(j, i, k = 1, 5, n = 1, I_{\max}\) and \(b = \text{C,G'}, \text{E,G'}, \text{X,G'}\).
4. The DeProF equations

The basic equations of the previous analysis comprise a system of equations in reduced form

\[ U_{\text{wCPF}} = \frac{1}{K_p} x, \]  
\[ (1 - \beta) U_{\text{wDOF}} = 1, \]  
\[ \beta U_{\text{oCPF}} + (1 - \beta) U_{\text{oDOF}} = 1, \]  
\[ \beta + (1 - \beta) S_{\text{oDOF}} = 1 - S_{\text{w}}, \]  
\[ \sum_{i=1}^{I_{\text{max}}} n_i^G n_j^G = 2N_d, \]  
\[ \sum_{i=1}^{I_{\text{max}}} \sum_{j=1}^{I_{\text{max}}} \sum_{k=1}^{I_{\text{max}}} P_{ijk}^b \frac{(1 + r) - x\beta/k - xq_{ijk}^b(x)(1 - \beta)/k}{(1 + r - x\beta/k)(\sigma/2 - 1) + xq_{ijk}^b(x)(1 - \beta)/k} \times \frac{1}{(1 + r - x\beta/k)(\sigma/2 - 1) + xq_{ijk}^b(x)(1 - \beta)/k} = 0. \]  

The physical meaning of each one of Eqs. (41)–(47) is as follows. Eq. (41): Darcy’s law in the CPF region (the transfer function of the particular region); Eq. (42): overall water mass balance; Eq. (43): overall oil mass balance; Eq. (44): overall oil (or water) mass arrangement condition; Eq. (45): oil ganglion volume normalization condition; Eq. (46): pore-scale vs macroscopic oil ganglion mass balance in DOF; Eq. (47): EMT equation for the DOF region. Eqs. (42)–(44) are mass balance and flow arrangement equations on a macroscopic scale, whereas Eqs. (45) and (46) express the consistence between the microscopic and macroscopic scale in the DOF description. Eq. (47) implicitly represents the transfer function for the DOF region, whereas in Eq. (41) the effect of \( x \) is explicit and linear, in Eq. (47) the effect of \( x \) is implicit and strongly non-linear. Actually, the non-linearity of SSFD2\( \phi \)FPM is incorporated in Eq. (47). The above system of seven equations relates \( (I_{\text{max}} + 5) \) dependent variables, namely \( x, U_{\text{wCPF}}, U_{\text{oDOF}}, U_{\text{wDOF}}, S_{\text{oDOF}} \) and \( \{n_i^G, i = 1, I_{\text{max}}\} \), along with the two FAVs, namely \( S_{\text{w}} \) and \( \beta \).

Closure of the above system can be accomplished by imposing certain relations between the \( n_i^G \)s in the form of a pre-selected type of ganglion size distribution. Two arguments lead to the appropriate closure equations.

The distribution type should be in accordance with the available experimental [1,2] and theoretical [7,18] observations. In addition, an order-of-magnitude-examination of Eqs. (45) and (46) shows that these are consistent only with classes of sharply decreasing ganglion size distributions. Based on these arguments, we adopt the following distribution:

\[ n_i^G = \begin{cases} n_1^G, n_2^G = \text{parameters to be determined}, \\ n_i^G = \frac{3 - i}{I_{\text{max}}}, & 3 < i < I_{\text{max}}, \\ n_i^G = 0, & i > I_{\text{max}}. \end{cases} \]

Here, \( n_1^G, n_2^G \) are variables to be determined. In solving the problem, \( I_{\text{max}} \) takes the value for which all oil ganglia with \( i \geq I_{\text{max}} \) have probability of mobilization equal to unity based on the ganglion mobilization criterion, Eq. (23), (27) and (25). The rationale behind this is that large ganglia having \( Pr(\text{mobilization}) = 1 \) fission into smaller ones very rapidly [16], therefore they are very rare. Here, \( \zeta \) is a parameter that regulates the steepness of the size distribution for \( i \geq 3 \). During the solution of the DeProF equations, \( \zeta \) is allowed to take all permissible values in the interval \( 0 < \zeta < 1 \). (As permissible value of \( \zeta \) is defined one that leads to a physically admissible \( 2\phi \) flow.)

The closure scheme has a serious impact on the predictions of the algorithm. The scheme proposed here gives predictions that are substantially better than those that were obtained using a simpler closure scheme in Payatakes and Valavanides [15]).

5. Algorithm for the solution of the DeProF equations

5.1. Physically admissible solutions

Let us consider a system of SSFD2\( \phi \)FPM for which the following information is given: (a) porous medium structure \( (\mathbf{x}_m) \); (b) physicochemical parameters \( \{\gamma_\rho, \rho_o, \rho_w, \rho_\mathbf{m}, \rho_\mathbf{m}, \rho_\mathbf{m}, \rho_\mathbf{m}, \rho_\mathbf{m}\} \); (c) operational parameters \( Ca \) and \( r \). We know from experience that a certain steady-state fully developed flow will be established. What we need to predict is the flow arrangement and the pressure gradient that correspond to this steady-state.

To solve the problem, we proceed with the following overall method. First, we partition the domain \( \{S_{\text{w}}, \beta\} \) of all possible flow arrangement parameter values using sufficiently fine steps to obtain a 2-D grid. Then, we select any pair of values for the flow arrangement parameters \( \{S_{\text{w}}, \beta\} \) from this grid, and seek a solution to the system of the DeProF equations (see below). Now, if a solution to the DeProF equations that is compatible with the selected \( \{S_{\text{w}}, \beta\} \) values, exists, this is allowed as a physically admissible solution and is denoted as \( \{S_{\text{w}}, \beta\} \). Otherwise, the set of \( \{S_{\text{w}}, \beta\} \) values is rejected as a possible flow arrangement. The process is repeated until all the grid points have been so characterized. In the end, we have a domain of the \( \{S_{\text{w}}, \beta\} \) plane that
corresponds to the physically admissible solutions, \( \omega_{PA} \). A prime is used to denote physically admissible values of any quantity. [The symbol without a prime is reserved for the expected value of the quantity, Eq. (51).]

We claim that on a mesoscopic scale (say \( 10^5 – 10^9 \) pores), the actual flow at a given region of the porous medium ‘wanders’ over the domain of physically admissible solutions ‘visiting’ any one with equal probability, or frequency (ergodicity). The difference between any two physically admissible solutions consists in the different ganglion size distributions prevailing in this region at the time (namely in the values of \( n_{G1}^0, n_{G2}^0, \) and \( z \)). However, on the macroscopic scale, the flow is steady and uniform (fully developed). Experimental observations [1–4] are in accord with this claim.

Now, we turn to the method that is used to solve the DeProF equations for any given set of \( \{S_w, \beta\} \) values. The solution is obtained iteratively. We start by making an initial guess for the value of \( x \) and using this tentative value for \( x \), the system of linear Eqs. (41)–(46) is solved analytically to obtain tentative values for the other six dependent variables, namely \( U^{0,CPF}, U^{0,DOF}, U^{w,DOF}, S^{0,DOF}, n_{G1}^0 \) and \( n_{G2}^0 \). \( I_{max} \) is determined as discussed in the above in connection with Eq. (48). These values are used to calculate the values of \( \omega, g_{ijk,a}^0 \) and \( P_{ijk,a}^0 \) from Eqs. (21), (35) and (37), respectively. Then, the non-linear Eq. (47) is solved numerically and a corrected value of \( x \) is obtained. A few iterations usually effect convergence. Thus, the seven dependent variables are determined, the flow is fully characterized, and the transfer function

\[
x' = x'(Ca, r; \kappa, \theta_{1,0}^a, \theta_{0,1}^R, x_{pm}[S^{w, \beta}])
\]

for the examined steady-state two-phase flow in the porous medium is determined.

The reduced rate of mechanical energy dissipation in SS2\( \phi \)FPM, defined as \( \dot{W} = \dot{W}\dot{k}_{pm}/(\dot{\gamma}_{om}Ca)^2 \), is calculated as the weighted sum of the reduced rates of mechanical energy dissipation in each prototype flow

\[
\dot{W} = \beta r U^{0,CPF} + [1 + (1 - \beta) r U^{0,DOF}] x.
\]

A unique value of \( \dot{W}' = \dot{W}'(S^{w, \beta'}) \) is obtained for every physically admissible couple \( \{S^{w, \beta'}\} \).

Fig. 6. Reduced rate of mechanical energy dissipation, \( \dot{W}' \), for various values of the system parameters. For each set of system parameters values the two-phase flow visits a continuum of physically admissible flow arrangements [which correspond to the indicated domain on the \( (S^w, \beta) \) plane]. Each such flow arrangement has its own value of \( \dot{W}' \). A unique set of values for \( S^w \) and \( \beta \) are obtained by averaging over all possible values, Eq. (51). These are shown with larger ‘balls’.

\[
\kappa = 0.66 \quad Ca = 0.119 \times 10^{-6} \quad \kappa = 3.35
\]
Several examples of domains of physically admissible solutions along with the corresponding values of $W$ are shown in Figs. 6(a) and (b).

5.2. Expected macroscopic flow arrangement

By assuming that each physically admissible solution is visited with the same probability, or frequency (assumption of ergodicity), and averaging over their domain, $\Omega_{PA}$, a unique solution for the macroscopic flow is obtained. For any physical quantity, $\Phi'$, the corresponding expected mean macroscopic flow quantity, $\Phi$, is defined as

$$\Phi = \int_{\Omega_{PA}} \int \Phi' (S^w, \beta) dS^w \, d\beta' \bigg/ \int_{\Omega_{PA}} dS^w \, d\beta'. \tag{51}$$

The rationale behind relation (51) is as follows. For systems in stationary but non-equilibrium states, the macroscopically established arrangement corresponds to the minimum entropy production that is compatible with the externally imposed constraint, so long as the phenomenological coefficients of the system can be supposed to be constants (or almost constants) ([8]); in addition, these stationary states are stable with respect to internal changes. Nevertheless, the aforementioned theorem does not apply to the system we consider here, because the flow system is far-from-equilibrium, and the conditions under which the Onsager reciprocal relations hold are not met [2–4]. The DOF patterns (GD, DTF) are dissipative structures typical of such systems. The flow arrangement that is obtained by averaging over the entire domain of physically admissible solutions, according to Eq. (51), is proposed as the expected macroscopic solution, and the transfer equation becomes

$$\dot{x} = x(Ca, \, r, \, \kappa, \, \theta^0_A, \, \theta^0_R, \, x_{pm}). \tag{52}$$

The relative permeabilities are obtained readily as a byproduct of the calculation. Below we give a few examples of the use of the proposed theory.

6. Sample calculations

The results of several calculations that correspond to typical experiments in [2] are presented here. The geometrical and topological characteristics of the pore

\begin{align*}
\kappa &= 1.45 \quad Ca = 4.750 \times 10^{-6} \quad \kappa &= 3.35
\end{align*}

Fig. 6 (continued).
network that were used in the theoretical calculations are those of the model pore network of [2] so that direct quantitative comparisons can be made (see also Section 3.2).

Three systems of fluids identical to those used in the experiments of Avraam and Payatakes (1995) [2] were examined. Their physicochemical properties are listed in Table 2.

In all systems, the static advancing and receding contact angles were assumed equal to $\theta_A^0 = 45^\circ$ and $\theta_R^0 = 39^\circ$, respectively. The operational parameters take values in the regions $10^{-7} < Ca < 5 \times 10^{-6}$ and $0.1 < r < 12.0$. The effects of the viscosity ratio, $\kappa$, the capillary number, $Ca$, and the flowrate ratio, $r$, are examined separately.

In Figs. 6(a) and (b), the cloud of the small gray solid spheres represents the reduced total mechanical energy dissipation, $\tilde{W}'$, for every physically admissible combination of the flow arrangement variables $\{S^w, \beta\}$, and for various values of $\kappa$ ($\kappa = 0.66, 1.45, 3.35$), $Ca$ and $r$. Each diagram corresponds to a SS2$\phi$ flow experiment in [2]. The domain of the physically admissible flow arrangements is also shown as the support of the spheres on the $(S^w, \beta)$ plane. In each diagram, the large solid sphere indicates the expected reduced mean macroscopic total mechanical energy dissipation, $\tilde{W}$, and it corresponds to the expected mean macroscopic flow arrangement indicated by the large circle on the $(S^w, \beta)$ plane (the projection of the large sphere). We observe that for low intensity SS2$\phi$ flows the flow arrangement is mainly or completely DOF (here GD). As the flow intensifies by increasing $Ca$ or $r$, the domain of physically admissible solutions progressively expands and ‘moves’

![Fig. 7. Mean macroscopic flow arrangement variables (FAV) $S^w$ and $\beta$ for different values of the operational parameters $Ca$ and $r$. The strong non-linearity between $S^w$ and $r$ is clearly represented by the trace of the FAV in the $(S^w, r)$ plane. Spheres indicate the simulations corresponding to the experiments in [2].]
towards CPF-dominated configurations. We also observe that small \( \kappa \) values favor DOF, whereas large \( \kappa \) values increase the contribution of CPF.

In Fig. 7, the expected mean macroscopic flow arrangement \{\( S^w \), \( \beta \)\} is shown for several values of \( \kappa \), \( Ca \) and \( r \). The spheres represent the expected flow arrangements that correspond to the conditions of all the experiments in [2]. The gray curve is the trace of the expected flow arrangement line in the \((S^w, \beta)\) plane, and shows the strongly non-linear relation between \( S^w \) and \( r \). For low \( Ca \) values, two-phase flow takes place only through DOF (\( \beta \approx 0 \)), whereas for moderate to high \( Ca \) values, the flow consists of a mixture of CPF and DOF depending on \( r \). These results are in full agreement with the experimental observations reported in [2].

The capability of the DeProF method for quantitative predictions is shown in Figs. 8 and 9 through direct comparison of the theoretical values of the mechanical power dissipation, \( W \), and the relative permeabilities \( k^o \), \( k^w \), with the experimental ones, \( W^{\exp} \), \( k^o \), \( k^w \), for several different values of the system and operational parameters. In each diagram, the experimental results are shown with ‘solid’ symbols (triangles and circles), while the theoretical results are shown with thick lines. In Fig. 8, the contribution of each prototype flow, \( W^{CPF} \), \( W^{DOF} \), to the total mechanical power dissipation is also presented. In Figs. 8 and 9, it is clear that the predictions of the DeProF theory are in very good to excellent agreement with the experimental values. The predictive capability of the DeProF method may be improved somewhat with the inclusion of the DTF regime in the calculations. (This is part of current work.) Similarly good results were obtained for all the experimental systems reported in [2], but are not presented here because of space limitations. (Available on request.) In order to gain more insight into \( \eta^{CPF} \), we introduce, here, three quantities. The first is the connected-oil fraction, \( \eta^{CPF} \), and is defined by

\[
\eta^{CPF} = \frac{V^o}{V}
\]

where \( V^o \) is the volume of oil in the connected phase and \( V \) is the total volume of the system.

**Fig. 8.** Comparison between the DeProF predictions and the actual experiments in [2] as a function of the operational parameters \( Ca \) and \( r \), for the reduced theoretical macroscopic mechanical power dissipation. Solid triangles represent experimental values while open symbols represent the corresponding DeProF theory predicted values.
The other two quantities are the expected mean ganglion sizes of the mobilized and overall ganglion populations, $V_{mG}^m$ and $V_{G}^m$, defined, respectively, as

$$V_{mG}^m = \sum_{i=1}^{l_{\text{max}}} m_i^G p_i^G,$$

$$V_{G}^m = \sum_{i=1}^{l_{\text{max}}} p_i^G.$$

The theoretical dependence of $\eta^{0,\text{CPF}}$ on $r$, for various $\kappa$ and $Ca$ values, is given in Fig. 10. We see that for low intensity $2\theta$ flows, the main (or only) flow regime is DOF ($\eta^{0,\text{CPF}} \ll 1$). As the flow intensifies (large $Ca$ or $r$, and also larger $\kappa$), the contribution of CPF increases, but still a significant oil flow fraction occurs through DOF. Only in extreme cases (very large $Ca$ and/or $r$) does $\eta^{0,\text{CPF}}$ approach unity.

In Fig. 11, DeProF predictions of the expected mean ganglion sizes, $V_{mG}^m$ and $V_{G}^m$, are plotted as a function of $r$ for various $\kappa$ and $Ca$ values. We observe that LGD is predicted for small values of $Ca$ and $r$ as observed in [2,4]. As $Ca$ and/or $r$ increase, SGD sets in again as observed in [2,4]. Finally, indication of severe oil ganglion fragmentation during relatively intense two-phase flows in porous media indicates that an extension of this DeProF model to account for drop traffic flow (DTF) would be useful.

7. Conclusions

A true to mechanism theoretical model of steady-state two-phase flow in porous media has been...
developed based on the novel concept of DeProF. The actual pore-scale flow mechanisms and the associated network-wide cooperative effects are incorporated into two prototype flows (CPF, and DOF). The two prototype flows are interlocked through the condition of common pressure gradient (which is strictly valid for steady-state fully developed two-phase flow in porous media, and approximately so for steady-state two-phase flow in porous media). The effect of the porous medium geometry and topology, the physicochemical properties of the two phases and the pressure gradient are consistently incorporated in the modeling of all relevant aspects of disconnected-oil flow dynamics, including ganglion size distribution, ganglion mobilization probability and ganglion velocity. An appropriate flow analysis on a macroscopic scale, complemented by micro–macroscopic consistency relations, and use of EMT in appropriately extended form lead to a system of equations (DeProF equations). The DeProF equations can be solved readily, in part analytically and in part numerically. The solution is relatively fast (~3–5 min with a good workstation).

Unlike conventional fractional flow theory, the present model of steady-state two-phase flow in porous media predicts the strong non-linearity of the flow as well as the effects of all the system parameters. It is also sufficiently simple for practical use.

Oil flow during steady-state fully developed two-phase flow in porous media takes place (to a large extent, and in many cases completely) in disconnected form, through the motion of ganglia and/or droplets. The motion of disconnected oil runs completely counter to the conventional assumption, but it has been proven experimentally and has a simple theoretical explanation (generalized ganglion mobilization criterion along with the crowding effect).
Agreement between the present model of steady-state two-phase flow in porous media and the available experimental data is very good to excellent. The present model is self-consistent, robust and has no adjustable parameters.

Demonstration of the prediction capability of the present approach was made by comparing its predictions with the only available experimental data that are sufficiently detailed those reported in [2]. Those data were obtained with a planarian network of chambers and throats. However, the present model, as it stands, can be applied readily to non-planar pore networks, even networks with variable coordination number, with only straightforward adjustments.

Acknowledgements

This work was supported by the Institute of Chemical Engineering and High Temperature Chemical Process. The authors thank Dr. V. Mavratzas for useful discussions, and Dr. D. Avraam supplying details concerning the geometrical characteristics of the porous medium used in [2].

Appendix A

For the solution of the flow problems in the unit cell, two types of orthogonal coordinate systems are used. For each throat, a cartesian coordinate system is used and any respective coordinates are denoted with ‘x,’. In each chamber, a bipolar cylindrical system is used [11] and the respective coordinates are denoted with ‘u,’. The bipolar system is aligned so that its z-axis coincides with the longitudinal axis of the chamber and its poles reside at the intersection of the annulus of the cylinder with the longitudinal axes of the adjacent throats (Fig. 2).

A.1. Determination of the unit cell transfer functions for one-phase flow

The transfer function for one-phase flow in a cell, in the form $\Delta p_{jik}^{\phi}(\tilde{q}_{uc}^{\phi})$, is derived here by solving the appropriate one-phase flow problem. $\Delta p_{jik}^{\phi}$ is the pressure drop along a $jik$-class cell for flowrate $\tilde{q}_{uc}^{\phi}$. We consider creeping flow conditions in the throats, and Hele–Shaw flow in the chambers.

The pressure drop along any $jik$-class unit cell, $\Delta p_{jik}^{\phi}$, for a flowrate $\tilde{q}_{uc}^{\phi}$, is given by

$$\Delta p_{jik}^{\phi} = \tilde{q}_{uc}^{\phi} \frac{\hat{u}}{\bar{c}} A_{jik}^{\phi},$$

(A.1)

where

$$A_{jik}^{\phi} = \left[ \frac{12}{D_{T}} \left( u_{0j} + u_{0k} - 2u_{l} \right) \right. \right.$$  

$$+ \frac{64}{\pi} \frac{W_{0j}^{2} + D_{T}^{2}}{W_{0j}^{2} D_{T}} \left( 1 - x_{0j} - x_{0k} \right) \right]$$

(A.2)

is the one-phase flow hydraulic resistance of a $jik$-class cell

$$x_{0j} = \frac{D_{Cj}}{2} \sqrt{\frac{1 - z_{ij}^{2}}{1 + z_{ij}^{2}}} \xi_{ij} = W_{rj}/D_{Cj},$$

(A.3)

and $j, i, k = 1, \ldots, 5$.

The main assumptions on which the solution of the one-phase-flow-in-cell problem was based are the following: (a) Hele–Shaw flow in chambers induced by an equipotential pair of source and sink located at the poles of the bipolar coordinate system of each chamber; (b) interaction with incoming and outgoing streams from the two throats normal to the $jik$ cell direction in the $j$ and $k$ chambers is neglected; (c) pressure continuity along the equipotential surfaces $u_{0j}$ and $u_{0k}$ and $u_{l}$ (Fig. 2); (d) creeping incompressible one-phaseflow in the throat; (e) extension of the exact solution of the problem of one-phase flow in an infinite elliptic cylindrical tube [11] to that of the flow in a finite length throat; (f) determination of the total pressure drop along any $jik$ cell through integration of the pressure equation from surface $u_{l}$ to surface $u_{0j}$ in chamber $j$, then, from surface $u_{0j}$ to surface $u_{0k}$ in throat $i$, and finally, from surface $u_{0k}$ to surface $u_{l}$ in chamber-$k$ (Fig. 2). Details on the derivation of Eq. (A.2) are omitted for lack of space, but are available on request (at marval@iceht.forth.gr, or at acp@iceht.forth.gr).

A.2. Determination of the ganglion cell transfer functions

The total pressure drop along any gate unit cell has two contributions: the viscous forces that resist the motion of each bulk phase, and the capillary pressure caused by the meniscus. To determine the conductances of ganglion cells, we extend the analysis of the previous section by introducing the following approximations: (a) the bulk flow of each phase is handled as if we had one-phase flow (akin to the Washburn approximation); (b) the contribution of the flow of each phase in the throat to the total pressure drop is computed on a volume fraction basis; (c) the meniscus is a spherical cap, with the appropriate receding or advancing contact angles. Suppose that a flowrate, $\tilde{q}_{uc}^{G}$, is maintained in all conducting cells of an $n$-class ganglion, Fig. 3. A ganglion of size class $n$, as it migrates downstream with an average velocity $\tilde{u}_{G}^{n}$, induces an average flowrate
\( \dot{q}_{G, \text{m}}^{\text{G}} (\ddot{u}_{G}^{\text{G}}) \) through the conducting cells it occupies (‘C,G’ and ‘E,G’ cells), which is given by Eq. (17). The pressure drop, \( \Delta \rho_{\text{G}, \text{m}}^{b} \), along any of the \( \text{jik} \)-class unit cells of this ganglion is given by an expression similar to Eq. (A.1), but with an extra term to account for the extra pressure drop induced by the meniscus

\[
\Delta \rho_{\text{G}, \text{m}}^{b} = \frac{\mu_{G} q_{\text{m}}^{b}}{\ell} \left( A_{jik}^{b} + \frac{\ddot{u}_{G}^{b}}{\ell} B_{n}^{b} \right)
\]

\[
= \frac{\mu_{G} q_{\text{m}}^{b}}{\ell} \left( A_{jik}^{b} + \frac{\ddot{u}_{G}^{b}}{\ell} \mu_{G}^{2} B_{n}^{b} \right)
\]

\[
= \frac{\mu_{G} q_{\text{m}}^{b}}{\ell} \left( A_{jik}^{b} + \frac{\ddot{u}_{G}^{b}}{\ell} \mu_{G}^{2} B_{n}^{b} \right)
\]

\[
= \frac{\mu_{G} q_{\text{m}}^{b}}{\ell} \left[ A_{jik}^{b} + \frac{1}{C_{G}^{b}} \left( \ddot{u}_{G}^{b} \right) \right]^{-1} B_{n}^{b}
\]

(A.4)

for \( b = \text{‘C,G’, ‘E,G’ or ‘X,G’,} \) where

\[
A_{jik}^{C,G} = \frac{64}{\pi} \frac{W_{T}^{2} + D_{T}^{2}}{W_{T}^{2} D_{T}^{2}} \left( 1 - x_{0j} - x_{0k} \right)
\]

\[
+ \frac{12}{D_{T}} \left( u_{0j} + u_{0k} - 2u_{1} \right),
\]

\[
A_{jik}^{G,G} = \frac{32}{\pi} \left( 1 + 2 \right) \frac{W_{T}^{2} + D_{T}^{2}}{W_{T}^{2} D_{T}^{2}} \left( 1 - x_{0j} - x_{0k} \right)
\]

\[
+ \frac{6}{D_{T}} \left( 1 + \kappa \right) \left( u_{0j} + u_{0k} - 2u_{1} \right),
\]

\[
A_{jik}^{X,G} = 0
\]

and

\[
B_{n}^{C,G} = B_{n}^{E,G} = \frac{2}{n + 1} \left( \dot{J}_{jik}^{G} \right)_{\theta_{A}}^{b}
\]

\[
B_{n}^{X,G} = 0.
\]

A note should be made here concerning Eq. (A.6). For the ‘C,G’ cells (core cells) we could have considered simple one-phase flow. However, in order to facilitate application of EMT in the DOF region [see the derivation of Eq. (36)], for which purpose each ganglion should be considered as a chain of unit cells, we ‘redistribute’ the net capillary effect of the menisci in the two end-gate cells \( \left( \dot{J}_{jik}^{G} \right)_{\theta_{A}}^{b} \) (caused by contact angle hysteresis) equally to all of the conducting cells of the ganglion. That is why \( B_{n}^{C,G} = B_{n}^{E,G} \). The computation of \( \left( \dot{J}_{jik}^{G} \right)_{\theta_{A}}^{b} \) will be presented next.

As a first step we will derive the mean (along a \( jik \) cell) reduced \( \theta_{A} \) mean interphase curvature, \( J_{j}^{G} \). Assuming that the contact angle is \( \theta \), the mean curvature of the meniscus in a \( j \)-class chamber, \( J_{j}^{G \text{GC}} (\ddot{z}, \theta) \), is given by

\[
J_{j}^{G \text{GC}} (\ddot{z}, \theta) = \frac{\cos \theta + 1}{D_{T} + D_{Cj}} \left( \cos \theta - \sin \theta \frac{\ddot{z}}{\sqrt{\left( D_{C}/2 \right)^{2} - \ddot{z}^{2}}} \right) \frac{1}{7}
\]

(A.7)

where \( \ddot{z} \) is the position of the meniscus.

The curvature of the meniscus in a \( i \)-class throat, \( J_{i}^{G \text{GT}} (\ddot{z}, \theta) \), is given by

\[
J_{i}^{G \text{GT}} (\ddot{z}, \theta) = \cos \theta \left( \frac{1}{D_{T}} + \frac{1}{W_{T}} \right) \frac{1}{\ddot{z}} = \cos \theta \frac{W_{Tj} + D_{Tj} \frac{1}{\ddot{z}}}{\ell}.
\]

(A.8)

Then, the mean reduced curvature of a meniscus along a \( jik \)-cell, \( J_{jik}^{G} (\theta) \), is given by

\[
J_{jik}^{G} (\theta) = \left[ \int_{z=0}^{L_{j}} J_{jik}^{G \text{GC}} (\ddot{z}, \theta) \frac{d\ddot{z}}{\ddot{z}} + \int_{z=-\ddot{z}/2}^{L_{j}} J_{jik}^{G \text{GT}} (\ddot{z}, \theta) \frac{d\ddot{z}}{\ddot{z}} \right]
\]

\[
+ \int_{z=-\ddot{z}/2}^{-\ddot{z}} J_{jik}^{G \text{GT}} (\ddot{z}, \theta) \frac{d\ddot{z}}{\ddot{z}} + \int_{z=L_{j}}^{0} J_{jik}^{G \text{GC}} (\ddot{z}, \theta) \frac{d\ddot{z}}{\ddot{z}}
\]

(A.9)

which, after some simple algebra, gives

\[
J_{jik}^{G} (\theta) = \frac{1}{2} \left[ \cos \theta \frac{1}{D_{T}} \left( D_{T} + D_{Cj} \right) \frac{1}{\sqrt{1 - \ddot{z}^{2}}} \right. \frac{\ddot{z}}{\ddot{z}}
\]

\[
+ \left( D_{T} + D_{Cj} \right) \left[ 1 - \frac{\ddot{z}^{2}}{\ddot{z}_{ik}^{2}} \right] + \sin \theta \left( \ddot{z}_{ij} - \ddot{z}_{ik} \right) \frac{1}{\ddot{z}} \frac{1}{\ddot{z}}
\]

\[
+ \cos \theta \frac{W_{Tj} + D_{Tj}}{\ddot{z}} \left[ 2 - D_{Cj} \sqrt{1 - \ddot{z}^{2}} \right]
\]

(A.10)

The mean (over all possible \( jik \) combinations) value of \( J_{jik}^{G} (\theta) \), which is used in Eq.(A.6), is given by

\[
J_{jik}^{G} = \sum_{j, k=1}^{5} J_{jik}^{G}_{jik, f_{j}, f_{k}, f_{k}}.
\]

(A.11)

Therefore,

\[
\left[ J_{jik}^{G} \right]_{\theta_{A}}^{b} = J_{jik}^{G} (\theta_{A}) - J_{jik}^{G} (\theta_{A}) = \frac{\cos \theta_{R} - \cos \theta_{A}}{2D_{T}} \times \sum_{j, k=1}^{5} \left[ \left( D_{T} + D_{Cj} \right) \frac{1}{\sqrt{1 - \ddot{z}_{ij}^{2}}} \right. \frac{\ddot{z}}{\ddot{z}}
\]

\[
+ \left( D_{T} + D_{Cj} \right) \left[ 1 - \frac{\ddot{z}^{2}}{\ddot{z}_{ik}^{2}} \right] + \frac{W_{Tj} + D_{Tj}}{\ddot{z}} \frac{1}{\ddot{z}} \frac{1}{\ddot{z}}
\]

\[
\times \left( 2 - D_{Cj} \sqrt{1 - \ddot{z}^{2}} \right) f_{j} f_{k} f_{k}
\]

\[
+ \sin \theta_{R} - \sin \theta_{A} \sum_{j, k=1}^{5} \left( \ddot{z}_{ij} - \ddot{z}_{ik} \right) f_{j} f_{k} f_{k}
\]

(A.12)
The advancing and receding contact angles, $\theta_A$ and $\theta_R$, are functions of the velocities in the throats and chambers. The dependence of the dynamic contact angles on the local velocity is calculated, in accordance with the results of Hinkley et al. [12], using the experimental values of $\theta_A$ and $\theta_R$ from Avraam and Payatakes [2].

A.3. Determination of the ganglion mean velocity, $\bar{u}_n^G$

A mobilized oil ganglion migrates downstream in an episodic manner through a series of hygrons [9]. In our analysis, we have considered that all mobilized ganglia of the same class migrate with an average velocity, $\bar{u}_n^G$, which depends on the system parameters and on the macroscopic pressure gradient.

The flow of water induces a pressure difference across a ganglion head and tail. Using smooth field approximation, this pressure difference is estimated as (\ref{eq:pressure_difference}). Motion of the oil ganglion is maintained so long as this pressure difference balances the viscous forces in the bulk of the ganglion, plus the net capillary resistance. The latter is estimated using the transfer function of the conducting ganglion cells, $\Delta p_{ijk,n}^{CG}$ and $\Delta p_{ijk,n}^{EG}$.

A pore-scale linear momentum balance on a time average basis gives

$$\frac{\partial \bar{p}}{\partial z} \bar{I}_n^G = 2\Delta p_{ijk,n}^{CG} + (n-1)\Delta p_{ijk,n}^{EG}. \quad (A.13)$$

Both $\Delta p_{ijk,n}^{CG}$ and $\Delta p_{ijk,n}^{EG}$ are functions of $\bar{u}_n^G$. Therefore, inserting Eqs. (A.4), (A.5) and (A.6) in Eq. (A.13), and solving for $\bar{u}_n^G$, the mean reduced velocity of any class $n$ oil ganglion is given (on an average basis) by Eq. (29). It is important to note that the pressure gradient ($-\frac{\partial \bar{p}}{\partial z}$) is calculated from Eq. (4) using for $x$, the value obtained from the solution of the DeProF system of equations, Eqs. (41)-(47). Thus, the effect of the entire population of ganglia on the velocity of any individual ganglion is taken into account, albeit to a first approximation.

References