Optimal spacing and cultivation intensity for an industrialized crop production system

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Abstract

The overall design of an industrial crop production system should optimize the system’s global parameters, such as total area required, intensity of cultivation and general production schedule. The key element is the optimal intensity of cultivation for the prevailing climatic and economic environments. A single-state-variable vegetative crop model (e.g., lettuce) was used to optimize a crop production system for two economic environments: quota- and area-limited production. A strictly uniform weather (every hour of the season) was assumed initially. Continuous spacing was a central control element. The main conclusions of the analysis with our model are: (1) plants of all ages can grow together in a single climatic compartment; (2) spacing should be scheduled to maintain a constant canopy density; (3) optimal canopy density is an increasing function of available light and a decreasing function of temperature; (4) where produce-price is high relative to the prices of rent and energy, the optimal cultivation intensity for an area-limited operation is higher than for a quota-limited operation; the opposite is true where rent is expensive; and (5) the marginal price to be paid for supplementary light is smaller where available natural light is more plentiful. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Plant factory; Plant spacing; Cultivation intensity; Supplementary light

1. Introduction

Attempts to industrialize the growing of greenhouse crops are not new. Ever since the introduction of Ruthner’s towers (Hix, 1974, pp. 73–75), schemes have been proposed for ‘plant factories’ where essentially all operations, in particular environmental control and handling, were to be automated (Prince and Bartok, 1978; Prince et al., 1981; Okano et al., 1988; Nakayama, 1991; Takatsuji, 1991; Annevelink, 1992; Goto et al., 1994; Both et al., 1998).

Two elements were given particular emphasis: artificial light and spacing. Some of the schemes (Okano et al., 1988; Both et al., 1998) used supplementary lighting, while others (Prince and Bartok, 1978; Prince et al., 1981; Nakayama, 1991) relied entirely on artificial light, which allowed the stacking of growing beds one on top of the other. All schemes considered variable plant spacing as a means of saving on the cost of space: Prince and Bartok (1978) and Prince et al. (1981) considered automated mechanical spacing, which should match closely the natural growth rate, while others
considered a small number of discrete spacing events (usually two to three). Goto et al. (1994) introduced a measure of space utilization, which reaches the value of one under an ‘ideal’ spacing policy. In their view, the ideal spacing is that which allows each plant exactly the projected area which it attains when grown in isolation (without neighbors).

Artificial light and spacing are elements of the cultivation effort or intensity. A higher cultivation intensity is associated with more expenditure on control measures, per unit of produce. The

<table>
<thead>
<tr>
<th>Nomenclature</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>area of greenhouse for production rate of one plant per unit time (m$^2$/s/plant)</td>
</tr>
<tr>
<td>$a$</td>
<td>area per plant (m$^2$/plant)</td>
</tr>
<tr>
<td>$c$</td>
<td>area-related operational expenses [$/m^2/\text{plant}^2]$</td>
</tr>
<tr>
<td>$e$</td>
<td>weather vector ($\Gamma$)</td>
</tr>
<tr>
<td>$G$</td>
<td>net growth rate of crop [Eq. (7)] (kg[dm]/(m$^2$/s))</td>
</tr>
<tr>
<td>$g$</td>
<td>rate of gross photosynthesis for a horizontally exposed leaf (kg[dm]/(m$^2$/s))</td>
</tr>
<tr>
<td>$h$</td>
<td>light-interception function (dimensionless)</td>
</tr>
<tr>
<td>$I$</td>
<td>flux of artificial photosynthetic radiation (light) (mol[phot]/(m$^2$/s))</td>
</tr>
<tr>
<td>$J_A$</td>
<td>performance criterion on a unit area basis ($$/m$^2$/s)</td>
</tr>
<tr>
<td>$J_Q$</td>
<td>performance criterion on a plant basis ($$/plant)</td>
</tr>
<tr>
<td>$K$</td>
<td>($\equiv \lambda/\kappa$) light-to-produce price ratio (kg[dm]/mol[phot])</td>
</tr>
<tr>
<td>$L$</td>
<td>leaf area index (m$^2$/plant/m$^2$)</td>
</tr>
<tr>
<td>$n$</td>
<td>rate of production (plant/s)</td>
</tr>
<tr>
<td>$P$</td>
<td>flux of natural photosynthetic radiation (light) (mol[phot]/(m$^2$/s))</td>
</tr>
<tr>
<td>$R$</td>
<td>($\equiv \lambda/p$) light-to-rent price ratio (m$^2$/s/mol[phot])</td>
</tr>
<tr>
<td>$r$</td>
<td>respiration rate (kg[dm]/(kg[dm]/s))</td>
</tr>
<tr>
<td>$T$</td>
<td>crop temperature (K)</td>
</tr>
<tr>
<td>$t$</td>
<td>time (s)</td>
</tr>
<tr>
<td>$U_A$</td>
<td>equivalent area integral [Eq. (30)] (m$^2$/s/plant)</td>
</tr>
<tr>
<td>$U_Q$</td>
<td>equivalent area integral [Eq. (29)] (m$^2$/s/plant)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>extinction coefficient in the light interception function [Eq. (20)] (m$^2$/kg[dm])</td>
</tr>
<tr>
<td>$\beta$</td>
<td>coefficient in the respiration function [Eq. (22)] (kg[dm]/(kg[dm]/s))</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>exponent in the respiration function [Eq. (22)] (K$^{-1}$)</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>coefficient in the photosynthesis function [Eq. (21)] (kg[dm]/mol[phot])</td>
</tr>
<tr>
<td>$\eta$</td>
<td>coefficient in the photosynthesis function [Eq. (21)] (kg[dm]/(m$^2$/s))</td>
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<tr>
<td>$\theta$</td>
<td>reference temperature in the respiration function [Eq. (22)] (K)</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>value of additional unit of dry mass ($$/kg[dm])</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>unit price of light ($$/mol[phot])</td>
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<tr>
<td>$\Pi$</td>
<td>value added to a plant over the growing period ($$/plant)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>cost of rent [$/m^2/s)]</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time fraction of light cycle (dimensionless)</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>residence time ($\equiv t_f - t_i$) (s)</td>
</tr>
<tr>
<td>$\downarrow$</td>
<td>vector elements have different units</td>
</tr>
<tr>
<td>${}$</td>
<td>are used exclusively to list arguments of functions; usually only context-relevant arguments are used</td>
</tr>
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Notes: 
- $\text{dm}$ dry matter

(Okano et al., 1988; Both et al., 1998), considered a small number of discrete spacing events (usually two to three). Goto et al. (1994) introduced a measure of space utilization, which reaches the value of one under an ‘ideal’ spacing policy. In their view, the ideal spacing is that which allows each plant exactly the projected area which it attains when grown in isolation (without neighbors).

Artificial light and spacing are elements of the cultivation effort or intensity. A higher cultivation intensity is associated with more expenditure on control measures, per unit of produce. The
optimal cultivation intensity, which in general varies with time, may be quantified by the costate vector of the solution to the optimal control problem (Seginer, 1989). The optimal cultivation intensity depends on the local climate and economic environment, including any limiting factors such as production quotas or available capital and labor. In general, where climate is more favorable and wages are lower, the optimal cultivation intensity is lower. An important question related to the design of an industrialized facility is whether the intensity of cultivation should depend on the age of the crop. More specifically, how should the spacing and the crop environment change, if at all, as the plants grow.

The objective of the present study is to derive the optimal control trajectory for simple crops, such as lettuce or pot plants, assuming uniform weather conditions (such as in the tropics) and continuous (not batch) cultivation. The uniform weather assumption is appropriate for design purposes where short-season greenhouse crops or a plant factory are considered. We show first that, under these assumptions, optimal canopy density (leaf area index) and optimal cultivation intensity are independent of plant age. We then show how the optimal cultivation intensity and canopy density depend on the natural light level and on the price of energy, for quota- and area-limited production systems. Finally, we estimate the error resulting from the zero-order assumption that light is uniformly distributed over the 24 h of the day.

2. Model

Let us consider a production line for a vegetative crop in uniform weather. Seedlings of size (state) \( w_i \) (expressed as dry matter), entering the line at the initial time, \( t_i \), exit the other end at the final time, \( t_f \), ready for marketing at the final size, \( w_f \). It is assumed that the final state, \( w_f \), is fixed (market requirement) and that the final time, \( t_f \), is free.

The area which an individual plant occupies, \( a(t) \), changes with time as a result of continuous (in practice–successive) changes of spacing. The appropriate area \( a(t) \) is expected to increase monotonically as the plant grows. For every plant entering the system per unit time, there is an area allocated in the greenhouse (or plant factory), which can be evaluated by integrating \( a(t) \) from initial to final time.

\[
A = \int_{t_i}^{t_f} a(t) dt.
\]

At steady state, the greenhouse is occupied by plants of all sizes between \( w_i \) and \( w_f \), each age-class having the same number of plants, say \( n \) per day.

The rate of growth of the plants depends on their environment and their spacing. A single plant is viewed as occupying an area \( a(t) \) out of a canopy which uniformly covers the ground. The canopy is represented by a single variable, \( W \), its dry mass per unit ground area.

The growth rate of an individual plant, \( dw/dt \), is the difference between its gross photosynthesis and respiration, namely:

\[
dw/dt = a[h(L\{W\})g(e,u) - Wr(e,u)].
\]

where, loosely speaking, \( h(L\{W\}) \) represents the fraction of light intercepted by the canopy, \( L \) is leaf area index, and \( g(e,u) \) is gross photosynthesis of a horizontal sunlit leaf. The respiration rate per unit dry mass, \( r(e,u) \), is multiplied by \( W \), to produce the crop respiration rate. The functions \( g(e,u) \), and \( r(e,u) \) are responses of the plants to the greenhouse environment (light, temperature, \( \text{CO}_2 \) concentration, etc.), which, in turn, is a function of the outdoor environment, \( e \), and the control operations, \( u \).

The relationship between the single-plant variable, \( w \), expressed as dry mass per plant, and the crop variable, \( W \), expressed as dry mass per unit ground area, is:

\[
w = aW.
\]

3. Optimization

In order to optimize the design and operation of the dynamic system described in the previous
section, a performance criterion, which measures the success of the operation, must be formulated. In the present context, a suitable economic goal to be maximized is the net income of the grower. As this may depend on the economic environment under consideration, two cases will be considered: quota-limited and area-limited operation.

3.1. Quota- and area-limited operation

In the first situation, the limiting factor of the grower’s activity is a marketing quota. As a result, the grower would want to maximize the gain for every plant that he/she is permitted to market. Hence the performance criterion is formulated on a per plant basis and takes on the form:

$$J_Q = \Pi - \int_{t_i}^{t_f} a(c(u) + \rho) dt,$$

(4)

where the criterion, $J_Q$, is expressed in $$/plant, $\Pi$ is the value added to the plant while it was growing from its initial size $w_i$ to its marketable size $w_f$, $c$ is the area-related operational costs (heating, lighting, etc.), and $\rho$ is the cost of rent. The cost of the spacing operation (changing $a$) is ignored for simplicity and it is assumed that all prices remain constant with time.

When available area (or investment capital) is the limiting factor, the grower would wish to maximize the gain per unit greenhouse area. This can be achieved by formulating a new criterion, $J_A$, which is obtained by dividing $J_Q$ by the area integral, $A$. The result:

$$J_A = J_Q / A,$$

(5)

has the units $$/m^2 s). Utilizing Eqs. (4) and (5), and in view of Eq. (1), $J_A$ may be written as:

$$J_A = \frac{\Pi}{A - \rho - (1/A) \int_{t_i}^{t_f} a c(u) dt}.$$

(6)

It is evident from Eqs. (4) and (6) that $\Pi$ will have no effect on the optimization of a quota-limited operation, and $\rho$ will have no effect on an area-limited operation (Seginer, 1989). Of course, low $\Pi$ or high $\rho$ may make the operation uneconomic, but while the operation continues, the optimal control is independent of $\Pi$ in the first case and on $\rho$ in the second.

3.2. Transformation

The problem has been formulated as having a fixed final state, $w_f$, while the final-time is free. It is, therefore convenient to change the integration variable in Eq. (4) from $t$ to $w$. To this end, Eq. (2) may be written more compactly as:

$$dw/dt = a[h(W)g(e, u) - W r(e, u)]$$

$$\equiv aG(e, u, W),$$

(7)

leading to

$$dt = \frac{dw}{aG(e, u, W)}.$$

(8)

This can be used to modify $J_Q$ to

$$J_Q = \Pi - \int_{w_i}^{w_f} \frac{c(u) + \rho}{G(e, u, W)} dw,$$

(9)

where the integrand is not a function of the state, $w$. Maximization of $J_Q$ requires the minimization of the integral in Eq. (9) over the controls, namely over $a$ and $u$. This is obtained by minimizing the integrand separately for each value of the independent integration variable $w$. Since, for a given $w$, a one-to-one relationship exists between $a$ and $W$, the minimization over $a$ may be replaced by minimization over $W$, as the notation in Eq. (9) indicates.

Minimizing the integrand over $W$ for a given $w$, yields an expression which relates $W$ to $u$ and $e$. This relationship can be used to substitute for $W$ in the integrand, which, as a result, depends now only on $e$ and $u$. Minimizing next over $u$, the result is that the optimal value of $u$ is a function of $e$ alone, and since $e$ is assumed to be constant, the optimal value of $u$ is also constant. Finally, since $e$ and the optimal value of $u$ are both constant, the optimal value of $W$ is also constant (note, however, that the optimal value of $a$ is not constant; it
The constancy of the optimal cultivation intensity (represented by $u$), and of the optimal crop density (represented by $W$) means that they are independent of the age of the crop. This is an important general guideline, because it means (1) that young and old plants can grow optimally together in the same climatic compartment, and (2) that spacing should be scheduled to maintain as constant a crop density as practical. When growth is nearly exponential, the latter guideline means that spacings should take place at nearly constant time intervals (e.g. once a week), independent of the size of the plants.

The qualitative solution for the quota-limited operation (namely constant $u$ and $W$), is also valid for the area-limited operation (Appendix). The optimal pair $[u, W]$ may be different this time, because $J_Q$ is affected by $\rho$, but not by $\Pi$, while the opposite is true for $J_A$.

Since $e$, $u$ and $W$ are constant, Eq. (9) can be integrated to obtain:

$$J_Q = \Pi - \frac{c[u] + \rho}{G[e, u, W]}(w_f - w_i).$$

If the value, $\kappa$, of a unit of dry mass in the final produce is defined by:

$$\kappa \equiv \Pi/(w_f - w_i).$$

Eq. (9) may be written as:

$$J_Q = [\kappa - \frac{c[u] + \rho}{G[e, u, W]}](w_f - w_i).$$

Next, by substituting from Eq. (8) for $dt$ in Eq. (1), and recalling the constancy of $e$, $u$ and $W$, $A$ may be written as:

$$A = \int_{w_i}^{w_f} \frac{dw}{G[e, u, W]} = \frac{(w_f - w_i)}{G[e, u, W]},$$

which shows $A$ to be inversely proportional to $G$. Using this expression to substitute for $A$ in Eq. (5), and in view of Eq. (12), the result for $J_A$ is:

$$J_A = \kappa G[e, u, W] - c[u] - \rho.$$  

where $\kappa$ may be omitted from the expression for $J_Q$ and $\rho$ from $J_A$.

3.3. Residence time

Using Eq. (3) to substitute for $a$ in Eq. (8) and recalling that $W$ and $G$ are constant with time, it is possible to integrate the resulting equation to find the residence time, $\Omega = t_f - t_i$, of a plant in the production line.

$$\Omega \equiv t_f - t_i = \frac{W}{G} \ln \left( \frac{w_f}{w_i} \right).$$

Alternatively, using Eq. (13) to replace $G$ with $A$, the result is:

$$\Omega = \frac{W A}{w_f - w_i} \ln \left( \frac{w_f}{w_i} \right),$$

showing that the residence time is proportional to $WA$.

3.4. Optimization

Maximization of the relevant performance criterion means finding the optimal values of $u$ and $W$. While the optimal value of the control vector, $u$, may often lie on a border of the feasible region, this is not normally the case with the crop density, $W$. It is convenient, therefore, to start by finding the optimal $W$ in terms of $u$, via:

$$\partial J/\partial W = 0.$$  

From Eqs. (12) and (14), the maximization of $J_Q$ or $J_A$ with respect to $W$ ($u$ being constant) is equivalent to the maximization of $G[e, u, W]$ with respect to $W$. The optimization, in this case, yields the same value of $W$ for $J_Q$ and $J_A$. For an unconstrained maximum (which is normally valid), and utilizing Eq. (7), the optimum is obtained from:

$$\partial G/\partial W = g[e, u, \partial h]/\partial W - r[e, u] = 0.$$
For any specific function $h(W)$, Eq. (18) can be used to substitute for $W$ in Eqs. (12) and (14), making the performance criteria $J_Q$ and $J_A$ functions only of $u$. In the new form, $dJ/du = 0$ indicates the location of the extremum.

4. A simple example

To illustrate the effect of weather and prices on the optimal control policy, let us consider a simple problem, where only light and spacing are controlled. It is assumed that temperature, CO$_2$ concentration and perhaps other environmental factors, are maintained at some fixed level, external to our problem, at a cost which is included in the rent $\rho$.

4.1. Basic relationships

It is now required to specify the functions $c(u)$, $h(W)$, $g(e,u)$ and $r(e,u)$. These will be mostly based on Seginer et al. (1991), who used lettuce as their model crop.

The cost of electricity for the artificial light, $I$, is assumed to be proportional to the flux of light:

$$c(u) = \lambda I,$$

where the unit price of light is $\lambda$.

The light-interception function, $h(W)$, is assumed to be convex (upwards), to increase monotonically, starting at $h=0$ when $W=0$, and to approach 1 asymptotically as $W$ approaches $\infty$.

The simple function:

$$h(W) = 1 - e^{-aW}$$

satisfies these requirements.

Gross-photosynthesis, $g(e,u)$, is taken to depend only on light intensity, which is the sum of the natural and artificial photosynthetic radiation fluxes, $P$ and $I$.

$$g(e,u) = g(P+I) = \frac{e\eta(P+I)}{e(P+I)+\eta}.$$  \hspace{1cm} (21)

Finally, respiration, $r(e,u)$, is assumed to be an exponential function of temperature, $T$, which is fixed in advance,

$$r(e,u) = r(T) = \beta e^{\gamma(T-\theta)}.$$  \hspace{1cm} (22)

4.2. Optimization

With these relationships, the maximization of $J_Q$, Eq. (10), over $u$ and $W$, may be replaced by the maximization of:

$$Z_Q = \frac{G(e,u,W)}{c(u)/\rho + 1}$$

$$= \frac{(1 - e^{-aW}) \frac{e\eta(P+I)}{e(P+I)+\eta} - W\beta e^{\gamma(T-\theta)}}{RI + 1},$$

and the maximization of $J_A$, Eq. (14), may be replaced by the maximization of:

$$Z_A = G(e,u,W) - c(u)/\kappa$$

$$= [(1 - e^{-aW}) \frac{e\eta(P+I)}{e(P+I)+\eta} - W\beta e^{\gamma(T-\theta)}]$$

$$- KI.$$  \hspace{1cm} (24)

where

$$R \equiv \frac{\lambda}{\rho}$$  \hspace{1cm} (25)

and

$$K \equiv \frac{\lambda}{\kappa}$$  \hspace{1cm} (26)

are price ratios. For a given environmental control, $u$ ($I$ in the present case), the optimal canopy density [Eq. (18)] is independent of the mode of production (quota- or area-limited) and, in view of Eq. (20), it may be written as:

$$W = \frac{1}{\alpha} \ln \left( \frac{\alpha g(P+I)}{r(T)} \right).$$  \hspace{1cm} (27)

Eq. (27) shows that the optimal canopy density, $W$, and therefore the leaf area index, increases with light level (photosynthesis) and decrease with
temperature (respiration). In view of Eq. (3), the opposite trends are true for the spacing $a$, and therefore for $A$. Eq. (27) is valid as long as:

$$\alpha g(P + I) > r(T).$$

(28)

Eq. (27) can now be used to substitute for $W$ in Eqs. (23) and (24), making $Z_Q$ and $Z_A$ functions of $I$ alone. Maximizing $Z_Q$ and $Z_A$ with respect to $I$, optimizes the control, namely $I$.

Note that if the use of supplemental light is not justified, there is still an optimal canopy density, $W$. In general, if $u$, and therefore $c\{u\}$ of Eq. (23), are zero, maximization of $Z_Q$ or of $Z_A$ implies maximization of $G\{e,W\}$ (net production per unit area). This means satisfying Eq. (18) (with $u = 0$), independent of the cost of rent, $\rho$, or the added value of the crop, $\Pi$. In other words, if the only available control is spacing, the optimal density, $W$, is the same for the quota- and area-limited operations.

Once the optimal canopy density, $W$, and the control, $I$, are obtained, the performance criteria, Eqs. (10) and (14), can be calculated. Here we present the results in terms of alternative criteria, which approach $A$ as the control approaches zero. Using Eq. (13) to replace for $(w_l - w_i)/G$ in Eq. (10) and for $G$ in Eq. (14), the two new criteria are:

$$U_Q \equiv (\Pi - J_Q)/\rho = [RI + 1]A,$$

(29)

and

$$U_A \equiv \Pi/(J_A + \rho) = [1 - KIA/(w_f - w_i)]^{-1}A.$$  

(30)

Maximization of $J_Q$ and $J_A$ is equivalent to minimization of $U_Q$ and $U_A$, which are larger than $A$, unless $I = 0$. $U_Q$ and $U_A$ may be considered as ‘equivalent’ area integrals, which increasingly deviate from $A$ as the cost of utilized light, namely $\lambda I$, increases (while $\rho$ and $\kappa$ remain constant).

4.3. Comparison between quota- and area-limited production

It should be clear from the difference between Eqs. (12) and (14), that the optimal operation of a quota-limited production system is, in general, not the same as the optimal operation of an area-limited system. Since, however, the optimal solution of the first case depends on $\rho$, and the other on $\kappa$, there may be combinations of these prices which produce the same optimal solution for both systems. To find these combinations, recall that for a given $e$ and $u$ the optimal $W$ is the same for the two production systems [Eq. (18)]. Hence, for a given climate, $G\{e,u,W\}$ may be replaced by $G\{u\}$, which makes $J_Q$ and $J_A$ functions of $u$, with $\rho$ and $\kappa$ as parameters. It should, therefore, be possible to search for the pairs $[\rho, \kappa]$ which produce the same optimal control for the two production systems.

From Eqs. (23) and (24):

$$Z_Q = \frac{G(P + I)}{RI + 1}$$

(31)

$$Z_A = G(P + I) - KI.$$  

(32)

If the optimal supplementary light level is positive, it must satisfy:

$$dZ_Q/dI = 0$$

(33)

$$dZ_A/dI = 0,$$

namely

$$RG(P + I) - (RI + 1)dG/dI = 0$$

(35)

$$dG/dI - K = 0.$$  

(36)

Suppose that $R$ (rent) is given, what is the value of $K$ (produce) that will result in the same optimal solution for both economic environments? With given $R$, one finds first the optimal solution for the quota-limited operation. Then, by eliminating $dG/dI$ between Eqs. (35) and (36), one finds:

$$K = \frac{RG(P + I)}{RI + 1},$$  

(37)

where $I$ and $G\{P + I\}$ are the optimal values for the quota-limited operation. Eq. (37) is a relationship between $K$ and $R$ for which the optimal intensity of cultivation is the same for both economic environments. Note that this relationship depends
on the climate, represented here by $P$.

5. Square-wave weather

Up to this point the weather was assumed to be strictly constant, not only along the season but also within the day. Since both photosynthesis and respiration are non-linear functions of the environment, it is desirable to find the order of magnitude of the error introduced by this zero-order approximation. A simple first-order approximation would be a periodic step function with the same light integral as the constant light. Viewing the process with a daily resolution, the weather remains constant, while the state (size of plants) and spacing change once a day.

This view makes the periodic problem similar to the original problem, with the exception that the daily integral of $G(e,u,W)$ may be somewhat different. Consequently, the optimal values of $W$ and $u$ are still constant with the age of plants, their values being different from the original ones only to the extent that the non-linearity of $G(e,u,W)$ responds differently to the uniform and periodic disturbances.

The response of $G$ to light is convex (upwards), except at very low light (to be mentioned again under supplemental light). Therefore, the benefit from adding light at night (dark period) is higher than from adding it during the day (light period). The optimal policy would be to first add light at night (if justified), until the day level is reached, and only then increase the light level further on a whole day basis. Hence, the non-linearity effect will be felt most strongly under the pure natural light square-wave sequence.

To estimate the error involved in assuming a uniform day, it is necessary to calculate corrected values of $G$ for use in Eqs. (31) and (32). These are obtained from:

$$G = G_d(I_d)\tau_d + G_n(I_n)\tau_n,$$

where

$$I = I_d\tau_d + I_n\tau_n,$$

(38)

$\tau_d$ is the fraction of time when natural light level, $P$, is high (d for day), and $\tau_n$ is the fraction of time when there is no natural light (n for night).

Further approximations may include more realistic light distribution and different electricity prices for different periods of the day.

6. Results and discussion

Optimal solutions were calculated for the specific relationships of Eqs. (19)–(22), in conjunction with the following, somewhat arbitrary values of the various parameters (mostly based on Seginer et al., 1991, for lettuce).

Crop parameters: $\alpha = 25$ m$^2$/kg[dm], $\epsilon = 1.25 \times 10^{-3}$ kg[dm]/mol[phot], $\eta = 1.4 \times 10^{-6}$ kg[dm]/(m$^2$ s), $\beta = 0.43 \times 10^{-6}$ s$^{-1}$, $\gamma = 0.069$ K$^{-1}$, $\theta = 25^\circ$C.

Production parameters: $w_i = 0.4$ g[dm]/plant, $w_f = 30$ g[dm]/plant.

6.1. Natural light

The dependence of the optimal solution on natural light, $P$, and temperature, $T$, namely an uncontrolled uniform environment, is shown in Fig. 1. This figure is not a function of the production parameters and of the prices. Supplementary light is not used because, presumably, its price is too high. Note that the highest value on the ordinate, namely a continuous light flux of $P = 400$ mmol[phot]/(m$^2$ s), is equivalent to about 17 MJ [solar]/(m$^2$ day), a representative daily outdoor winter solar radiation in Israel.

Three families of curves are shown in the $P$–$T$ plane of Fig. 1. Iso-lines for $W$, with maximum canopy density at high light and low temperature; iso-lines for $A$, with maximum area requirement at low light and high temperature; and iso-lines for $AW$, which is proportional to the residence time, $\Omega$ [Eq. (16)], with a minimum at high light and high temperature. At very low light levels, the condition of Eq. (27) is no longer satisfied, resulting in $W = 0$, $A = \infty$ and $AW = \infty$. Useful production ($J > 0$) will only be possible at considerably higher light levels.

It may seem unexpected that the minimum residence time is not necessarily associated with the
lowest temperature (respiration). Recall, however, that when environmental control (here, light) is not applied (as in the case of Fig. 1), the optimization maximizes $G$ [Eq. (23)] and, therefore, minimizes $A$ [Eq. (13)]. There is no requirement that $\Omega$ must be minimized.

$A$, $W$ and $\Omega$ are important design, operational and planning parameters, respectively. $A$ determines the size of the facility, since it represents the floor area required for the production of one plant per unit time (e.g. 3 m$^2$ to produce one lettuce head each day at Point I in Fig. 1); $W$ tells the grower how to space the plants (e.g. 0.10 kg[dm]/m$^2$ at Point I); and $\Omega$ determines the time from planting to marketing (e.g. $AW=0.3\rightarrow \Omega=44$ days at Point I).

A rough equivalence between canopy density, $W$, and leaf area index, $L$, for lettuce, may be obtained by utilizing a typical value of about 60 m$^2$[leaf]/kg[dm] for the specific leaf ratio. This converts the range of 0–0.16 kg[dm]/m$^2$[ground] for $W$, to a range of 0 to 9.6 m$^2$[leaf]/m$^2$[ground] for $L$.

6.2. Supplemental light

Fig. 2 is a cross-section of Fig. 1 along the line $T=20^\circ C$. The vertical dotted line on the left of Fig. 2 is where $G=0$, $W=0$ and $A=\infty$ (bottom of Fig. 1). New information, not presented in Fig. 1, are the curves tagged by $R$, the light-to-rent price ratio and by $K$, the light-to-produce price ratio. These are the marginal values of these ratios which justify the use of the first increment of supplementary light. The $R$ curve is relevant to the quota-limited operation and the $K$ curve to the area-limited operation.

Both curves decrease asymptotically to zero as the level of natural light increases, but while the $R$ curve approaches infinity as the natural light
The second derivative of the criterion is:

$$d^2 Z_A/dI^2 = d^2 G/dI^2,$$  \tag{42}

which is negative, indicating a maximum, only where $G(P+I)$ is convex upwards. When $P$ is below the inflection point, either no light should be added, or a finite flux (not just an increment of light). The exact amount depends on $P$ and on $K$ and can be calculated if so desired.

Fig. 3 shows the optimal level of supplementary light and its effect on $W$, $A$ and $U$, as a function of a range of price ratios, $R$, for the quota-limited production system. The climate under consideration is $P=100$ μmol[phot]/(m$^2$ s) and $T=20^\circ$C (Point II in Fig. 1), so that the values of $W$, $A$ and $R$ at $I=0$ are identical to those for $P=100$ μmol [phot]/(m$^2$ s) in Fig. 2.

The distance between the curves for $A$ and $U$ shows the contribution of the cost of light to the equivalent area integral. At the right-hand margin of Fig. 3, $U$ is almost three times larger than $A$, dwindles, the $K$ curve stops abruptly when it reaches a maximum (at about 83 μmol[phot]/(m$^2$ s) for 20°C). The behavior at high natural light levels is as expected, since photosynthesis response to light [Eq. (21)] is convex (upwards) and the returns diminish as the light level increases. The odd behavior of the $K$ curve at low light levels is due to a barely observable inflection point in $G$ (not $g$) as a function of $P$. The function $G(P)$ is concave upwards below the inflection point (here $P=83$ μmol[phot]/(m$^2$ s)) and convex above it. This peculiarity is the result of simultaneous changes of $g$ and optimal $W$ [Eqs. (21) and (27)] as the light level changes.

Searching for the extremum of $Z_A$ with respect to $I$, Eq. (32) leads to:

$$dZ_A/dI = dG/dI - K = 0,$$  \tag{40}

or

$$K = dG/dI.$$  \tag{41}

The second derivative of the criterion is:
meaning that the expenditure for light is almost twice the expenditure for rent. A smaller value of $U$ means higher net income. Fig. 3 shows that a 50% reduction in the price of light justifies the increase of supplementary light from zero to 200 $\mu$mol/[phot]/(m$^2$ s). This results in a significant improvement in terms of the criterion $U$. The improvement is smaller for higher natural light levels (not illustrated).

6.3. Quota- versus area-limited operation

Generally speaking, the optimal cultivation intensity (here supplementary light) is different for the quota- and area-limited operations. When the cost of rent is high (low $R$), the optimal cultivation intensity of the quota-limited operation is high (to reduce expenditure on rent). Conversely, when the price of produce is high (low $K$), the optimal cultivation intensity of the area-limited operation is high (to take advantage of the high price). There are, however, pairs of price-ratios which produce exactly the same optimal cultivation intensity for both situations [Eq. (37)]. These combinations are indicated by curves in Fig. 4 for $T=20^\circ$C and several levels of natural light. When the price of light is zero (origin of figure) the optimal level of supplementary light is infinite. As it increases, less and less light is used, until at a certain high price, supplementary light is no longer justified. This point is indicated by $X$ in the figure. As already mentioned, beyond this point (at an even higher price for light) the effective control reduces to just spacing. It becomes independent of prices, and there is no difference between the quota- and area-limited operation. When no natural light is available (opaque growth chamber), artificial light is a must if the plants are to grow. Therefore, the $X$ on the curve for $P=0$ is at infinity.

If the point $[R,K]$, which represents the actual price system, is far from the appropriate curve in Fig. 4, there is, intuitively, a bias towards one of
the production systems. For instance, if the price of the produce, $k$, decreases ($K$ increases), the quota-limited grower has no way to adjust to it, while the area-limited grower would respond by decreasing the cultivation intensity (by applying less light). It is likely, therefore, that in a reasonably free economic environment the price ratios will not be too far from the climatically appropriate curve. Suppose that the prices of rent (cost of land and construction) and of energy are the same ($R$) in two locations, which differ considerably in terms of available natural light. Where would one expect to find lower produce prices? Consider, for example, the vertical line for which $R=2 \, \text{m}^2 \, \text{s/mmol}$ (Fig. 4). As the level of natural light increases from 200 to 400 $\mu\text{mol} \, \text{[phot]}/(\text{m}^2 \, \text{s})$, $K$ increases and hence the price of produce decreases by about 30%.

If the prices are kept artificially at the bottom right corner of Fig. 4 (low rent and high produce price), the income to the growers is high and the optimal cultivation intensity (supplementary light) for the area-limited operation would be considerably higher than for the quota-limited operation. The opposite is true for the top-left corner.

6.4. Square-wave weather

The preceding sample results were calculated on the assumption that light is uniformly distributed over the 24 h of the day. An estimate of the error resulting from this assumption was calculated by comparing the optimal values of $W$ and $A$ for uniform and square-wave light regimes having the same light integral. Using Eq. (38) to estimate the production under a 12 h/12 h (light/dark) regime, and plotting the results in Fig. 5 on a logarithmic scale, the relative error is seen to increase with available light. The uniform-light approximation overestimates $W$ (at most 7% in Fig. 5) and underestimates $A$ (at most 30%).

6.5. Model improvements

The most critical assumptions of the model itself were made in the state equation [Eq. (2)]. In
particular, it is the assumption that gross photosynthesis is proportional to 
$h(L(W))$, independent of the age of the crop (here represented by $w$, the size of an individual plant). Since the morphology and physiology of plants both change with age, a more realistic assumption would be 
m(h, w) = h(w/a)$. This, however, results in a more complicated analysis.

A second major assumption is the uniform weather, which is only a useful approximation in the tropics. In other climates, successive crops will have different residence times and an adjustment will have to be made. The two extreme cases seem to be:

1. A constant rate of production. In this case $nA$, the total area, will have to be calculated for the slowest rate of growth (winter) and some area will be wasted during the other seasons. An alternative would be to control light by supplementation and shading, so that the daily light integral is constant (Albright, 1995).

2. Maximum area utilization. In this case, as $A$ and the optimal control change, the production rate, $n$, has to be adjusted to maintain a constant total area $nA$.

These cases and others can be optimized, too, if required.

The crop model, as described by Eqs. (20)–(22), has five parameters ($\alpha$, $\epsilon$, $\eta$, and $\lambda$; $\theta$ is arbitrary), whose values may vary from one production system to another. It is conceivable that adjustments of the original parameters can be made based on data collected on location. An improved crop model may be used, for instance, to re-evaluate $A$ and change the production rate, $n$, accordingly. For example, if the model with the current parameter values under-estimates the residence time, $G$ could be decreased by a fraction, until Eq. (15) balances the observed residence.
time. This corrected $G$ can then be used to evaluate an improved estimate of $A$, via Eq. (13). More refined tuning of the model, if at all necessary, and an improved estimate of $W$ may be more difficult to achieve under commercial conditions.

7. Conclusions

This study was concerned with the optimization of the main design elements of an industrialized crop production system, with emphasis on supplementary lighting and variable spacing. A simple vegetative crop and uniform weather were assumed, and two economic environments were considered: quota-limited and area-limited operation.

The main conclusions arising from the current model are as follows:

1. Properly spaced young and old plants can grow optimally together in a single climatic compartment.
2. Spacings should be scheduled to maintain a nearly constant canopy density (leaf area index).
3. Optimal canopy density is an increasing function of available light and a decreasing function of temperature.
4. The area required to produce plants at a certain rate decreases rapidly as more light becomes available.
5. The marginal price to be paid for supplementary light is generally smaller where more natural light is available.
6. Where rent is relatively more expensive than the other inputs, the optimal cultivation intensity is higher for a quota-limited operation that for an area-limited operation.
7. Where produce price is relatively more expensive than the other inputs, the optimal cultivation intensity is higher for the area-limited operation.
8. Representing the daily weather by a square wave is significantly more accurate than assuming uniform weather over the 24-h day.

These qualitative conclusions, derived here from a model-based analysis, are mostly in agreement with practice. Some conclusions refine previous intuitive statements. For instance, Goto et al. (1994) assumed that the ideal spacing is that which allows each plant exactly the projected area which it attains when grown in isolation. Our results indicate that the optimal spacing depends on the light level.

More accurate quantitative recommendations require more comprehensive and accurate crop and greenhouse models, or else more on-location calibration data.

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Appendix A

Although the problem of the area-limited case is not a regular optimal control problem, because the right-hand side of Eq. (6) contains a ratio of two integrals, it can be converted to such a problem as follows. First, Eq. (6) is transformed by changing the independent variable, as for the quota-limited case, from $t$ to $w$. Then two state variables, $x$ and $y$, are defined to be the current values of the integrals in the numerator and denominator of the modified Eq. (6). The resulting state equations are:

$$\frac{dx}{dw} = \frac{c(u)}{G(e, u, W)} \tag{A1}$$

and

$$\frac{dy}{dw} = 1/G(e, u, W), \tag{A2}$$

with initial values $x_i = x_{\{w_i\}} = 0$ and $y_i = y_{\{w_i\}} = 0$. The final values, $x_f = x_{\{w_f\}}$ and $y_f = y_{\{w_f\}}$ are free.

Using the final values of these state variables, the performance criterion may be written as:

$$J_A = -\rho + (P - x_f)/y_f, \tag{A3}$$

which is another representation of Eq. (6). In this form the optimal control problem is regular
and has the following properties: (1) there are no final conditions on the state variables; (2) the right-hand sides of the differential equations depend only on the control and on the constant climate (not on the independent variable, nor on the state variables); and (3) the criterion is a function of only the free final values of the state variables. It follows from these properties that the costates (Pontryagin et al., 1962) are constant and hence the optimal control vector is also constant.

References