Revision and clarification of “A general hydrodynamic theory for mixed-phase microphysics”

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Abstract

Some problems of the interpretation and the application of previously published general hydrodynamic theory of mixed-phase microphysics are addressed. Some restrictions in the applicability of the previous work, especially regarding aggregation, are removed using new definitions, and some minor errors are corrected. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Böhm (1992a,b,c) presented a general parameterization for solid and liquid hydrometeors along with semiempirical solutions for the hydrodynamics of a mixed population of particles. The solutions are based on boundary layer theory and were supplemented by a solution from potential flow theory in order to improve the collision efficiencies of particles of markedly different size (e.g., riming and aerosol impaction, Böhm, 1994). The results presented formerly are in close agreement with numerical and experimental results from the literature on fall speeds and collision efficiencies of hydrometeors. In the meantime, various applications in cloud models revealed that some definitions of the collision efficiencies of particles of complex shapes remain unclear. In the present contribution, a solution to these problems is presented, and most of the questions that arose in the last few years are answered. A refined approach to the general theory is presented, especially regarding its application for aggregation. Please see Böhm...
(1992a,b,c) for results. His results remain fully valid for the present contribution and, hence, are not reproduced in what follows.

The application of the formulae from Böhm (1992a) (henceforth, Part I) is clear for single hydrometeors in free fall, but not quite clear in the case of the hypothetical particle involved in the collision efficiencies in Böhm (1992b,c) (henceforth, Part II and Part III, respectively). In principle, we need all the shape characteristics of the hypothetical particle in the stopping problem discussed in Part II. The procedures applied in Part II and Part III have not been documented in sufficient detail anywhere, hence, numerous questions arose during the past few years.

In Section 2, the theory is briefly revised and the new approach presented, clarifying its application in aggregation models. In Section 3, the application of the general theory is documented step by step, including some numerical considerations in Section 3.5. A short summary is found in Section 4.

2. A brief revision and a new approach

2.1. Fall speed, drag and boundary layer thickness

The application of the semiempirical formulae for the fall speed and other hydrodynamic characteristics of a single particle with well-defined shape characteristics is quite clear from Part I. Confusion may arise due to the fact that Böhm (1990) differs from Part I in some minor cases. However, the formulae in Part I are generally correct.

One question may arise about the boundary layer thickness in connection with the fall speed according to Part I. For the calculation of the fall speed, the boundary layer thickness at the separation point is used implicitly, but it does not need to be calculated explicitly. Indeed, for prolates, the boundary layer thickness is not even defined directly, i.e., from boundary layer analysis. The boundary layer thickness at the separation point could, if desired, generally be calculated indirectly, from the pressure drag coefficient $C_{DP}$, i.e., from (3), (7) and (8) in Part I, namely

$$\delta = \delta_0 \frac{r}{\sqrt{N_\text{Re} T}}$$  \hspace{1cm} (1)

with the dimensionless parameter

$$\delta_0 = \sqrt{\frac{6k}{C_{DP}}} \text{ (at the point of separation)},$$  \hspace{1cm} (2)

while $r$ denotes the particle radius or characteristic length, $N_\text{Re}$ the Reynolds number, $k$ the viscous shape factor, and $T$ a dimensionless factor for oblate spheroids (set to unity for prolates). All these symbols are defined in Section 3 below.

Note that in the present contribution $\delta_0$ is generally defined regarding $r$, as in Eq. (1), and not regarding the diameter, as in (2) and (3) of Part I. The definitions of $\delta_0$ in Part I are not consistent (compare his Eq. (2) with Eq. (19), where the factor 2 implies that $\delta_0$ is related to the diameter instead of $r$). Regarding the radius, for axisymmetric
flow at the point of separation, $\delta_0 = 9.06$ (defined directly, from boundary layer analysis).

2.2. The boundary layers for the collisional problem

At first, the reader should note that the symbol $d$ used in (13) of Part II and the same symbol used in (2) of Part I do not refer to the same quantity. In Part I, $d$ denotes the boundary layer thickness of a single particle at the waist, while in Part II, $d$ denotes the sum of the boundary layer thicknesses at the stagnation point of the two colliding particles. Hence, the boundary layer thickness at the forward stagnation point is renamed to $\delta_s$ in the present contribution. In contrary to the boundary layer thickness at the point of separation (i.e., at the waist), $\delta_s$ is needed explicitly in order to determine the collision efficiencies (compare Section 2.1). The boundary layer thickness generally is defined according to Eq. (1), where $I(\alpha)$ is set to unity for axial ratios $\alpha > 1$ (prolates or columns). Note, however, that the factor $\delta_0$ at the stagnation point of a sphere, as presented in Part II (Eq. (6)), is inaccurate, while clear definitions for oblates with $\alpha < 1$ are missing in Part II.

The correct formula for all axial ratios for the boundary layer thickness at the stagnation point is given by Eq. (1), where $\delta_s = 3.60$ for $\alpha \leq 1$, $\delta_s = 4.54$ for $\alpha > 1$ (defined regarding the radius), while $I$ is defined by Eq. (11) in Section 3.1 below. The initial separation for the stopping problem is defined as the sum of the two boundary layer thicknesses at the stagnation point, $\delta_s = \delta_{11} + \delta_{12}$ (see Section 3.2).

2.3. Inertial drag and shape characteristics of the hypothetical particle

The hypothetical particle postulated in Böhm (1990) does not physically exist, and characteristic properties of the two-body system are used instead. For some parameters, the quantity applicable for the hypothetical particle is clear and discussed in Part II: the mass $m^*$, the initial separation $\delta^*$ and the initial speed $V^*$ (compare Section 3.2 below). One parameter, however, the inertial drag coefficient of the hypothetical particle, $C^*_U$, is not well defined yet, since it depends on its shape characteristics.

For the results presented in Böhm (1990), in Parts II and III, and in Böhm (1994), generally, the shape characteristics of the smaller particle were used. This approach was justified, because these studies either considered particles with equal shape characteristics (coalescence in Part I and aggregation of equally shaped plates in Part III) or particles of markedly different size (riming and impaction in Part III and in Böhm, 1994). In the former case, the choice of the shape characteristics is clearly a priori, while in the latter case the much smaller particle 2 may be viewed as penetrating the boundary layer of the larger particle 1, i.e., the hypothetical particle of the stopping problem is nearly equivalent to the particle 2.

When dealing with aggregation in general, however, particles of markedly different shape and of similar size and fall speed may collide. Here, the shape characteristics of the hypothetical particle, based on the two-body system, are not readily known. As an alternative to the general use of the shape characteristics of the smaller particle, some weighted mean of the characteristics of the two particles seems appropriate, especially
for the area ratio $q$ (related to the porosity, see Böhm, 1989 for definition), which is one of the parameters that determine the forces needed to remove the air barrier between the colliding crystals. Any simple mean of the axial ratios $\alpha_i$, however, would lead to an unrealistic axial ratio $\alpha^*$ near unity (sphere) when columns collide with plates of similar size.

In order to find comparable characteristics for all particle combinations, we can redefine prolate bodies according to Böhm (1989). On one hand, he showed that for columns, the drag and the fall speed determined according to the assumption of equivalent circular disks generally are in fair agreement with laboratory experiments and field observations. On the other hand, for all the results presented in Part III and in Böhm (1994), the expected value of the intersection of the two cross-sections, $A^*$, are based on equivalent circular definitions of the two cross-sections.

The best measure for the shape and size of a body, regarding its hydrodynamics, appears to be the radius, i.e., the characteristic length, $r$, which is well defined for both colliding hydrometeors as well as for the hypothetical particle. For the latter, the radius $r^* = (A^*/\pi)^{1/2}$ is defined as the radius of the equivalent circular representation of the cross-section $A^*$. For columns or irregularly shaped aggregates as well, the equivalent circular definition of its cross-section presented to the flow shall be used, i.e., for all particle shapes

$$r_c = \sqrt{A/\pi}$$

$$\alpha_c = \min\left\{\alpha, 1\right\} \frac{r}{r_c}$$

$$q_c = \begin{cases} \frac{\pi q/4}{q} & \text{for } q > 1 \\ q & \text{for } q \leq 1 \end{cases}$$

$$q_c = q \left[1 - \max\{q - 1, 0\} \left(1 - \frac{\pi}{4}\right)\right].$$

The second representation of $q_c$ in Eq. (5) may be used if one single equation with no branching condition is required (e.g., for vectorizing computers). The operators min or max on a set of values $s = \{x_1, \ldots, x_n\}$ in the present contribution are used in the common sense, i.e., according to the implication

$$\left(\min \max\right)^i_j = \left(\min \max\right)^i_j\{x_1, \ldots, x_i, \ldots, x_n\} = x_j \Leftrightarrow x_j \leq x_i, \forall j \neq i; i, j \in [1, n].$$

For particles that are axisymmetric in the direction of the flow at infinity (e.g., drops), $r_c = r$, $\alpha_c = \alpha$ and $q_c = q$. For columns, Eq. (3) results in $r_c = (4\alpha/\pi)^{1/2} r$. Eq. (5) originates from the special definition $q = 4/\pi > 1$ for columns (compare with Part I). For rimed columns, $q < 4/\pi$ and $q_c < 1$.

Based on Eqs. (3)–(5), the shape characteristics of the hypothetical particle or of the two-body system are determined according to a comparison only between plates or a suitable theoretical equivalent ($\alpha_c \leq 1$ for all particles). For predictable results, all the shape parameters of the hypothetical particle must be determined based on the same measure. Above, the characteristic length was found to be the best measure of a shape.
parameter $\xi$ ($\xi = q, \alpha$). Hence, $\xi^*$ of the hypothetical particle is defined as the mean of $\xi_1$ and $\xi_2$, weighted by the radius (or characteristic length), i.e.,

$$\xi^* = r^* \left( \frac{\xi_1}{r_{c1}} + \frac{\xi_2}{r_{c2}} \right) \ (\xi = q, \alpha),$$

where the indices 1 and 2 refer to particle 1 (collector) and particle 2, respectively. Based on the primary shape parameters $r$, $\alpha$ and $q$, all other parameters needed for the inertial drag coefficient (e.g., shape factor, pressure drag, etc.) are well defined by the formulæ presented in Part I. However, for the drag coefficients needed for the collision efficiency, these other parameters need not be calculated explicitly (see Section 3.2, Eq. (18)).

For markedly different particle sizes or for identical particle shapes, Eq. (6) basically yields the same results as presented in the previous studies. For similarly sized particles of different shape, however, (aggregation), the characteristics according to Eq. (6) may considerably differ from the characteristics of the smaller particle. To the knowledge of the author of the present contribution, no detailed experimental nor numerical results on the aggregation between particles of markedly different shape are available, that would allow a verification of Eq. (6). Please note that Keith and Saunders (1989) for their aggregation experiment used a two-dimensional cylindrical collector (i.e., an “infinitely” long cylinder), for which $r_c$ is undefined. Hence, the theoretical arguments presented here are the only indicators for its quality. Fortunately, the inertial effects are less significant for aggregation than for coalescence, because the shape characteristics of ice crystals and snowflakes vary statistically, even for a population with equal particle size or mass (see Part III).

2.4. Flow symmetry for the collision efficiencies

Considering Section 3, only one quantity involved in the collision efficiency for similarly sized particles of different shape is not yet clearly defined: the symmetry of the flow field, i.e., the parameter $\xi$ in the final equations for the collision efficiency $E$, as presented below in Section 3.3. However, the basic concept of the present solution from boundary layer theory is based on the flow field about the collector, which is falling at a greater speed. Hence — even though the flow field about the collector 1 undoubtedly is disturbed by a comparatively large particle 2 — the symmetry of the flow field must be defined by the symmetry of the collector.

3. Application of the theory

3.1. Terminal fall speed

The terminal fall speed is based on the Davies or Best number, $X$, which is defined by (17) and (18) in Part I, or according to the single equation

$$X(m, \alpha, q) = \frac{8mg \rho_a}{\pi n^2 \max\{\alpha, 1\} \max\{q^{1/4}, q\}},$$

(7)
where \( m \) denotes the particle mass, \( g \) the acceleration of gravity, \( \rho_0 \) and \( \eta_0 \) the density and viscosity of air, respectively. The Davies number is shifted for the transition to turbulent flow according to (23) in Part I. The axial ratio of large raindrops may be calculated from the equivalent spherical diameter \( d_e \) (in cm) by the linear fit of Pruppacher and Klett (1978)

\[
\alpha = \min\left\{1, 1.05 - 0.655d_e\right\}. \tag{8}
\]

Next, the viscous shape factor may be determined according to (11)–(14) in Part I or from the single equation

\[
k(\alpha) = \min\left\{\max\{0.82 + 0.18\alpha, 0.85\}, \left(0.37 + \frac{0.63}{\sqrt{\alpha}}\right)\frac{1.33}{\max\{\ln \alpha, 0\} + 1.19}\right\}. \tag{9}
\]

Based on Eq. (9), we find the Oseen drag coefficient according to (16) in Part I, while the pressure drag coefficient is given by (7) and (8) in Part I or by

\[
C_{dp,5} = \max\{0.292k\Gamma, 0.492 - 0.200/\sqrt{\alpha}\} \tag{10a}
\]

\[
C_{dp} = \max\{1, q(1.46q - 0.46)\}C_{dp,5}, \tag{10b}
\]

which is applicable to all relevant combinations of shape characteristics. For \( \alpha \leq 1 \), \( \Gamma \) is given by (5) and (6) of Part I, or for any axial ratio \( \alpha \) by

\[
\Gamma(\alpha) = \max\{1, \min\{1.98, 3.76 - 8.41\alpha + 9.18\alpha^2 - 3.53\alpha^3\}\}. \tag{11}
\]

Eq. (10a) is valid for spheroids only (oblate or prolate), while Eq. (10b) also includes columns with \( q > 1 \). Based on \( X \) from Eq. (7), as discussed above, and from \( k \) and \( C_{dp} \) according to Eqs. (9), (10a) and (10b), we proceed according to the equations given in Part I: First, the auxiliary parameters \( \gamma \) and \( \beta \) from (26) and (21), respectively, then the Reynolds number \( N_R \), according to (20) and (27), and finally, the terminal fall speed from (22) in Part I.

### 3.2. Initial conditions for the stopping problem

First, the characteristics of the hypothetical particle are determined: The reduced mass of the two-body system,

\[
m^* = \frac{m_1m_2}{m_1 + m_2}, \tag{13}
\]

and the radius or characteristic length,

\[
r^* = \sqrt{A^*/\pi} = \frac{r_1r_2}{r_{e1} + r_{e2}}, \tag{14}
\]

which is defined as the equivalent circular radius of the expected value of the intersection

\[
A^* = E(A_1 \cap A_2) = \frac{A_1A_2}{\pi\left(r_{e1} + r_{e2}\right)^2}. \tag{15}
\]
of the two cross-sections $A_1$ and $A_2$ with their equivalent circular radii $r_{e1}$ and $r_{e2}$ (compare with Part II and Böhm, 1994). By the aid of Eq. (14), the primary shape characteristics $q^*$ and $\alpha^*$ of the hypothetical particle are found from Eqs. (3)–(6) in Section 2.3 above.

The Reynolds number of the hypothetical particle, $N_{Re}$, is based on its initial velocity

$$V^* = |V_1 - V_2|$$

and on $r^*$ according to Eq. (14). Consequently, the initial separation for the stopping problem is found from the boundary layer thickness at the stagnation point, $\delta_{s1}$ and $\delta_{s2}$, according to Eq. (2) in Section 2.2 above, while $\delta_{s1}$ and $\delta_{s2}$ again are determined according to the initial velocity $V^*$, i.e.,

$$\delta^* = \delta_{s1} V_1/V^* + \delta_{s2} V_2/V^*.$$  (17)

The user might consider to determine both the coefficient of viscous and of inertial drag of the hypothetical particle and of particle 1, $C_{DL,i}$ and $C_{DV,i}$ ($i = 1,^*$), backwards, from the Reynolds and Davies number, namely

$$C_{DL,i} = \frac{C_{D,i} - C_{DV,i}}{\frac{X(m, \alpha, q)}{N_{Re}(r_i, V_i)} - \frac{24k(\alpha)}{N_{Re}(r_i, V_i)} (i = 1,^*),}$$

where $X(m, \alpha, q)$ and $k(\alpha)$ are defined by Eqs. (7) and (9), respectively, and

$$N_{Re} = \left( \frac{rV}{\eta} \right).$$

For particle 1, this procedure is justified. The hypothetical particle, however, is not moving at its terminal fall speed. Hence, its Davies number is not regularly defined, and the inertial drag coefficient $C_{DI}$ must be determined according to (25) in Part I. For a particle at terminal fall speed (e.g., particle 1), $\beta$ and $\gamma$ are given by (21) and (26) in Part I (compare Section 3.1 above), while for the hypothetical particle, $\beta$ can only be determined approximately, from (20) in Part I, namely

$$\beta^* = \sqrt{\frac{N_{Re} C_{DI}}{6k}}.$$  (20)

According to Böhm (1990), on neglecting the damping term for the matching of theories, the error in $C_D$ or $X$ is expected to be $< 16\%$ ($< 8\%$ for spheres). Hence, the relative error in $\beta$ due to the approximate nature of Eq. (20) is $< 4\%$ ($< 2\%$ for spheres, compare Eq. (21) in Part I). The slope $|\partial C_{DI}/\partial \beta| < 0.7$ for all shapes, and, hence, the resulting relative error in $C_{DI}$ is $< 3\%$ ($< 1.5\%$ for spheres). From Eqs. (13)–(20), all the characteristics of the hypothetical particle and the initial conditions for the stopping problem are defined.

### 3.3. Collision efficiency from boundary layer theory

In Parts II and III, the formulae for the collision efficiency were analysed independently for axisymmetric and for plane, two-dimensional flow (compare Eq. (9) in Part II
to Eq. (5) in Part III. For a clear documentation of the use of the symbols defined in Section 3.2, the equations are recapitulated here. However, the continuity equation for stagnation flow can be generally (valid for both symmetries) represented by

$$\frac{\partial u}{\partial x} + (j - 1) \frac{u}{x} + \frac{\partial v}{\partial y} = 0$$

(21)

and the velocity distribution of the boundary layer near the forward stagnation point by

$$U(x) = \frac{b}{j} x, \quad V(y) = by,$$

(22)

where $j = 2$ for axisymmetric and $j = 1$ for plane, two-dimensional flow. Based on the formulations in Eqs. (21) and (22) (equivalent to Part II for axisymmetric flow), we can find the general solution, valid for both symmetries, namely

$$E_b = \left( \frac{x_c}{r_1 + r_2} \right)^j \left( \frac{y_s}{\delta^*} \right)^{2/j} = \left[ H \ln \left( \frac{F}{R} + \frac{1 + G}{F} \frac{\sinh F}{\sinh F} \right) - G \right]^{2/j},$$

(23)

where $x_c$ denotes the initial horizontal offset and

$$F = \sqrt{G^2 + \frac{C_{D,1} V_{11}^2}{C_{D,1} \left( V^* \right)^2}},$$

$$G = \frac{6k^* \eta_b}{\rho_b \tau^* V^* C_{D,1}^*},$$

$$H = \frac{2m^*}{\rho_a A^* C_{D,1}^* \delta^*},$$

$$V_{11} = \frac{2 + j}{4} V_1.$$

The reader should note that $E_b$ is the collision efficiency and not every collision necessarily leads to coagulation. Hence, the coagulation efficiency $E_c \leq E_b$. The velocity $V_{11}$ corresponds to $V_1$ in the axisymmetrical case ($j = 2$) and is representative for the fluid inertial effects of the flow about particle 1. It denotes the expected value of the velocity component along the boundary layer of the latter. It is integrated from the velocity distribution near the surface according to potential flow theory (stagnation flow), based on the assumption of a homogeneous distribution of the initial horizontal displacement of the centers of the two particles (similar to $A^*$, see Böhm, 1990 for details).

3.4. Supplement to the collision efficiency from potential flow theory

For markedly different particle sizes, the collision efficiency according to Eq. (23) is improved by the efficiency from potential flow theory according to Böhm (1994). The reader should note that the definitions of the parameters $\Delta_i$ and $\Delta_j$ and of the collision
efficiency \( E \) in Böhm (1994, his Eqs. (12), (13) and (15)) contain errors. From his Eq. 10 and from the definition of the collision efficiency in Eq. (23) above, we find

\[
E = E_n E_p = E_n \left[ \left( \frac{\Delta t_b}{c_s} + \frac{1}{\Delta x} \sinh \frac{\Delta t_b}{c_s} \right) e^{-t_b/c_s} \right]^{-1},
\]

where \( c_s \) and \( t_b \) are given by Eqs. (7) and (14) from Böhm (1994), respectively. The definitions of \( \Delta_x \) and \( \Delta_y \) should read

\[
\Delta_x = \sqrt{2bc_y/j + 1},
\]

\[
\Delta_y = \sqrt{2bc_y - 1}.
\]

While the error in Eqs. (12) and (13) of Böhm (1994) appeared in his published formulae only and were not applied to his results, the erroneous exponent in his definition of the collision efficiency was also applied to the results presented there. However, \( E_p \) merely introduces to the collision efficiency a sharp cut-off at decreasing size of the collected particle. Fortunately, the size at which \( E \) sharply drops to zero is defined by the correct Eqs. (3) and (6) of Böhm (1994), while the wrong exponent in \( E_p \) only alters the steep slope of this cut-off, a slope that rapidly diverges to infinity with either the exponent of 1 or 2. Hence, this error has marginal effects on the results of Böhm (1994), and corrected figures need not be reproduced here. In most cases, his results are slightly but not significantly improved.

3.5. Numerical considerations

When the argument \( F/H \) of the hyperbolic functions in Eq. (23) increases to \( F/H \gg 1 \), the formula may result in a floating point overflow. Hence, for increasing \( F/H \), Eq. (23) should be replaced by

\[
\lim_{V^* \to 0} \left( \frac{F}{H} \right) = \lim_{F/H \to \infty} E = H \ln \left( \frac{F + G}{2F} \right) + F - G, \quad \text{for } \frac{F}{H} \geq 5.
\]

The collision efficiency according to Eq. (23) or Eq. (27) diverges when \( V^* \) according to Eq. (16) approaches zero, i.e., when the undisturbed fall speeds of the two particles are (nearly) equal. In this case, the collision efficiency according to Eq. (23) eventually results in a division by zero. However, \( V^* \) may be limited to a small minimal value \( V_{\min}^* \) before the calculation of the efficiency, i.e.,

\[
V^* = \max \{ |V_1 - V_2|, V_{\min}^* \}.
\]

The value \( V_{\min}^* = 10^{-10} \text{ m s}^{-1} \), e.g., limits \( E \) to a well-defined, high value. If the collision kernel

\[
K \propto EV^* = E_{\max} \{ |V_1 - V_2|, V_{\min}^* \}
\]
is based on the same $V^*$, as defined in Eq. (28), then the effects on the relevant physical quantity, the collision kernel, are negligible (i.e., within the numerical accuracy of the floating point calculations), even for vanishing $V^*$.

Similar to Eq. (23), the general formulation of Eq. (24) results in a floating point overflow for $bc_y \to 0.5$, i.e., for $\Delta_y \to 0$ (division by $\Delta_y$ and hyperbolic functions of diverging arguments). However, for $bc_y < 1$, $E_p$ may generally be calculated according to the simplified expression

$$
\lim_{\Delta_y \to 0} (E) = E_y \lim_{\Delta_y \to 0} (E_p) = E_y \left[ \frac{2\Delta_y - e^{(-\Delta_y - 1)\gamma/c_z}}{1 + \Delta_y} \right] \approx (\pm 0.2\% \text{ for } bc_y \leq 1).
$$

(30)

4. Summary

The various applications of the general hydrodynamic theories for mixed-phase microphysics according to Böhm (1990, 1992a,b,c, 1994) revealed that some questions about the theory still remain unanswered and some definitions in the former literature are unclear or even erroneous. Most of the questions communicated to the author have been addressed and clarified in the present contribution, along with a new approach that improves the definitions and the results for the collisional problem in general and for aggregation in particular. A complete and well-defined set of equations for the hydrodynamic theories has been presented, that definitively rectifies the errors in the former presentations. Based on the clear definitions given in the present contribution, any application of the general hydrodynamic theories should be easy and straightforward.

References


