Comparative statics predictions for the cross-effects of central dominance changes in risk with quasilinear payoffs

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Abstract

We examine cross-effects for central dominance changes in the random payoff to one activity on the willingness of a risk averter to undertake or expand a second. This willingness is enhanced for quasilinear payoffs when activities are cost substitutes.

Keywords: Central dominance; Cross-effects; Quasilinear payoffs

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1. Introduction

Intuition suggests that, when a change in uncertainty about one payoff causes a decision maker to curtail the associated activity, the marginal value of a second activity increases. We confirm this intuition for activities that are cost substitutes in quasilinear payoff functions when the payoff to the second activity is nonstochastic and the decision maker is risk averse.

The problem of a decision maker maximizing the expected utility of a quasilinear payoff has been investigated in an extensive literature on the effects of changes in price uncertainty on the supply decision of a competitive, single-product firm initiated by Sandmo (1971) and pursued by Ishii (1977), Meyer and Ormiston (1985), Black and Bulkley (1989), and Dionne et al. (1993a,b). The model can also be interpreted as one in which a risk averse taxpayer, facing an uncertain income-tax rate, can shelter some income from taxation at a cost that depends on the amount of income sheltered.

We introduce the option of undertaking a second activity, such as supplying a second product or...
sheltering income from a second source, and we focus attention on the cross-effects of changes in risk on this second activity. For the general class of central dominance changes in risk introduced by Gollier (1995), we find that the decision to initiate or to expand the second activity in response to a change in uncertainty about the payoff to the first turns critically on the presence of cost substitutability in the payoff function relating the two activities.

In the next section, we set out the decision model and state preliminary results for the comparative statics effects of central dominance changes in risk with one choice variable. In Section 3, we state two propositions concerning the cross-effects of these changes in risk on the decision to initiate or expand a second activity. Applications and conclusions are presented in Section 4.

2. The model and preliminary results

We examine quasilinear payoff functions of the form

\[ z = z^0 + \alpha x + \beta y - c(x,y), \]  

where \( z^0 \) is an exogenous constant, \( x \) and \( y \) are nonnegative choice variables, and the payoff coefficients \( \alpha \) and \( \beta \) are exogenously determined. The cost function \( c(x,y) \) is assumed to be twice continuously differentiable and convex, with uniform modularity in the sense that the cross-partial derivative \( c_{xy} \) is uniformly signed. Note that the two choice variables are payoff substitutes (complements) in the sense that the cross-partial derivative \( z_{xy} \) is negative (positive) if and only if they are cost substitutes (complements) and \( c_{xy} \) is positive (negative), so that a higher value for one choice variable reduces (raises) the marginal payoff to the other by raising (reducing) its marginal cost.

The objective of the decision maker is to maximize expected utility

\[ V(x,y) = \int U(z) \, dF(\alpha), \]  

where \( F(\alpha) \) denotes the probability distribution function for the coefficient \( \alpha \), and the utility function \( U(\cdot) \) is strictly increasing and concave, reflecting risk aversion \((U'' < 0)\) or risk neutrality \((U'' = 0)\).\(^1\) The coefficient \( \beta \) is assumed to be nonstochastic.

The Kuhn–Tucker conditions for the optimal choices \( x^* \) and \( y^* \) are\(^2\)

\[ V_x(x^*,y^*) = \int U'(\alpha - c_x) \, dF(\alpha) \leq 0 \quad \text{and} \quad x^*V_{xx}(x^*,y^*) = 0 \]  

\[ V_y(x^*,y^*) = (\beta - c_y) \int U' \, dF(\alpha) \leq 0 \quad \text{and} \quad y^*V_{yy}(x^*,y^*) = 0. \]  

Since \( \beta \) is nonstochastic, a positive choice for \( y^* \) must cause its marginal payoff to vanish, so that

\(^1\)Note that the limits of integration encompassing the support set for \( \alpha \) are suppressed in Eq. (2). Also, primes on \( U(\cdot) \) are used to denote derivatives while, for other functions, variables used as subscripts serve this purpose.

\(^2\)We assume that \( c(x,y) \) is strictly convex in both \( x \) and \( y \), and that either \( U(z) \) is strictly concave or \( c(x,y) \) is strictly convex in \( x \) and \( y \) jointly. We also assume that an optimal decision exists with \( x^* > 0 \) and second-order conditions satisfied.
\[ z_s = \beta - c_s = 0. \] In contrast, \( \alpha \) being stochastic implies that a positive supply \( x^* \) strikes an optimal balance between positive and negative marginal payoffs, \( z_s = \alpha - c_s \).

To state our comparative statics results we focus on the class of central dominance changes in uncertainty identified by Gollier (1995) and applied by Dionne and Gollier (1996), which cause every risk averse decision maker to reduce the choice of \( x^* \) when the choice variable \( y \) is absent.

**Definition.** The probability distribution \( F(\alpha) \) centrally dominates \( F(\alpha) \) around \( r \) if and only if there exists a scalar \( \gamma \) such that
\[
\int \alpha (\tau - r) \, dF(\tau) \leq \gamma \int \alpha (\tau - r) \, d\tilde{F}(\tau)
\]
for all possible values of \( \alpha \).

The following Lemma restates in the present context the fundamental result established by Gollier, that every risk averse or risk neutral decision maker chooses a lower value for \( x \) under \( F \) than under \( F \).

**Lemma 1.** If \( x^* \) is positive, then \( V(x^*, y^*) \) decreases when \( F(\alpha) \) is replaced by a distribution \( \tilde{F}(\alpha) \) that centrally dominates \( F(\alpha) \) around \( c_s(x^*, y^*) \).

Given the second-order condition \( V_{ss}(x^*, y^*) < 0 \), it follows from this Lemma that any risk neutral or risk averse decision maker for whom \( y \) is not a choice variable always reduces \( x^* \) in response to a central dominance shift in the distribution for \( \alpha \).

This type of change in uncertainty cuts across the boundaries of nth-order stochastic dominance shifts and nth-moment preserving spreads, but includes only subsets of each of these. Thus, for example, any centrally dominant mean preserving spread is an increase in risk that leads every risk averter to reduce the activity whose payoff is subject to risk, but there are mean preserving spreads that do not satisfy central dominance, and for each of these there are some risk averters who would increase the activity.

Our focus is on the cross-effects of changes in uncertainty, and we are interested in corner solution cases. To this end, we introduce the curves \( I^s \) and \( I^v \) to denote the two sets of activity levels \((x,y)\) that satisfy \( V_s(x, y) = 0 \) and \( V_s(x, y) = 0 \), respectively. In the next Lemma we establish that the relative position of the two curves in the \((x, y)\) plane depends on whether the expected utility function \( V(x, y) \) is supermodular \((V_{xy} \geq 0)\) or submodular \((V_{xy} \leq 0)\).

**Lemma 2.** The slopes of the level sets \( I^s \) and \( I^v \) have the same sign as \( V_{xy}(x, y) \), and at any point of intersection \( I^s \) has a steeper slope than does \( I^v \) in absolute value, that is, \( \left| \frac{dy}{dx} \right|_{V_{xy}=0} \geq \left| \frac{dy}{dx} \right|_{V_{xy}=0} \).

**Proof.** Totally differentiating \( V_s(x, y) = 0 \) we obtain \( \frac{dy}{dx} \big|_{V_{xy}=0} = -V_{ss}/V_{xy} \), which has the same sign as \( V_{xy} \) given the second-order condition \( V_{ss} < 0 \). Similarly, we obtain \( \frac{dy}{dx} \big|_{V_{xy}=0} = -V_{sy}/V_{ss} \), which also has the same sign as \( V_{xy} \) given the second-order condition \( V_{sy} < 0 \). (Recall that, in this instance, the cross-partial derivative is being evaluated at points \((x, y)\) along curve \( I^s \), where \( V_s = 0 \), implying that \( y \) is conditionally optimal.) To establish that \( I^s \) is more steeply sloped in absolute value than \( I^v \) at any point of intersection, observe that
\[
\frac{dy}{dx}_{|dy=0} - \frac{dy}{dx}_{|dy=0} = -(V_{xy}V_{yy} - V_{x}^{2})/V_{xy}V_{yy}.
\]

Given the second-order conditions \(V_{xx}V_{yy} - V_{xy}^2 > 0\) and \(V_{yy} < 0\) relevant at a point of intersection for the \(I^x\) and \(I^y\) curves, the difference on the right has the same sign as \(V_{xy}\), thereby completing the proof. \(\square\)

This Lemma shows that the sign of the cross-partial derivative

\[
V_{xy} = (\beta - c_{y}) \int U''(\alpha - c_{y}) \, dF(\alpha) - c_{xy} \int U' \, dF(\alpha)
\]

plays a decisive role in determining the configuration of an equilibrium. Along the level set \(I^y\), where the marginal payoff \(z_y = \beta - c_{y}\) equals zero, the first term on the right-hand side of Eq. (5) vanishes. From Lemma 2 it follows that, at points where the two level sets \(I^x\) and \(I^y\) intersect, the signs and relative magnitudes of their slopes are dictated solely by the sign of \(c_{xy}\). Since \(c_{xy}\) is uniformly signed, we conclude that, when \(\beta\) is nonstochastic, the level sets for \(x\) and \(y\) can intersect only once. This single-crossing property is exploited in the next section to derive predictions for the cross-effects of central dominance changes in risk.

3. Cross-effects of central dominance changes in risk

When the two choice variables are cost independents in the sense of being neither complements nor substitutes in the cost function, the \(I^y\) level set is horizontal. In that event, a change that shifts the \(I^y\) level set affects only the optimal choice for \(x\), leaving the optimal choice for \(y\) unchanged. Thus, introducing or changing risk about \(\alpha\) while leaving \(\beta\) nonstochastic has no affect on the decision to diversify by initiating the second activity, or to expand that activity, when the two are cost independents.

Turning to cases in which the two activities are either substitutes or complements in the cost function, our first result concerns cases in which the decision maker initially undertakes only activity \(x\), whose payoff coefficient \(\alpha\) is stochastic. We show that, in response to a central dominance change in the probability distribution for \(\alpha\), a risk averse decision maker may initiate activity \(y\), whose payoff is nonstochastic, if the two activities are cost substitutes. Since activity \(x\) declines when \(F(\alpha)\) changes to a centrally dominant distribution, the marginal payoff to \(y\) increases under cost substitutability, and may do so by enough to make diversification profitable when it was not before the change in risk.

**Proposition 1.** Assume that a risk averse decision maker is initially active with respect to \(x\) \((x^* > 0)\), but inactive with respect to \(y\) \((y^* = 0)\). If \(F(\alpha)\) is replaced by a distribution \(\hat{F}(\alpha)\) that centrally dominates \(F(\alpha)\) around \(c_{xy}(x^*, y^*)\), then the decision maker reduces \(x^*\), and maintains \(y^* = 0\) if \(c_{xy}\) is nonpositive, but may initiate \(y^* > 0\) if \(c_{xy}\) is positive.

**Proof.** Recall that the cross-partial derivative \(V_{xy}\), given in Eq. (5), has the same sign as \(-c_{xy}\) at points where the level sets \(I^x\) and \(I^y\) intersect. Moreover, by Lemma 2, at any point of intersection, both sets are negatively (positively) sloped when \(c_{xy}\) is positive (negative), while \(I^x\) has a steeper slope than does \(I^y\) in absolute value. Initially, the two level sets either do not intersect, or intersect at...
$y^* = 0$. With $\beta$ nonstochastic, level set $I^x$ is independent of $F(\alpha)$, while by Lemma 1, the $I^y$ curve shifts to the left when the centrally dominant distribution $\hat{F}(\alpha)$ replaces $F(\alpha)$. The alternative cases are illustrated in Fig. 1. These are the only possible configurations since the slopes of $I^x$ and $I^y$ must have the same sign at any point of intersection. It follows that $y^* = 0$ is maintained if the activities are cost...

![Diagram of cost complements and substitutes](image)
complements and \( c_{xy} \) is negative, but the decision maker may initiate \( y^* > 0 \) if the activities are cost substitutes and \( c_{xy} \) is positive. \( \square \)

Our second result refers to cases in which the decision maker is already diversified, engaging in both activities. When the activities are cost complements, a central dominance change in the distribution \( F(\alpha) \) leads to a reduction in both activities. However, when the two activities are cost substitutes, the decision maker takes further advantage of diversification by expanding activity \( y \) whose payoff remains nonstochastic.

**Proposition 2.** Assume that a risk averse decision maker is initially active with respect to both \( x \) and \( y \) (\( x^* > 0 \) and \( y^* > 0 \)). If \( F(\alpha) \) is replaced by a distribution \( \tilde{F}(\alpha) \) that centrally dominates \( F(\alpha) \) around \( c(x^*, y^*) \), then the decision maker reduces \( x^* \), and if \( c_{xy} \) is positive (negative), increases (reduces) \( y^* \).

**Proof.** Since the decision maker is initially active with respect to \( y \), we have \( \beta - c_y = 0 \), so that, as noted, the first term in the expression for \( V_{xy} \) in Eq. (5) vanishes and, thus \( V_{xy} \) has the same sign as \(- c_{xy}\). While recognizing that the two level sets initially intersect, one can apply the argument of the preceding proof in the present context to establish that \( y^* \) increases (decreases) when the activities are cost substitutes (complements) and \( c_{xy} \) is positive (negative). \( \square \)

Together, Propositions 1 and 2 highlight the role of cost substitutability in determining the effect of changes in uncertainty on the value of diversification with quasilinear payoffs. When the two activities are cost complements, changes in uncertainty about the payoff to activity \( x \) that lead every risk avenger to reduce \( x \) also diminish the value of exploiting diversification. However, when the two activities are cost substitutes, the value of diversification is enhanced by such changes in uncertainty.

In the preceding, we have assumed that \( \beta \) is nonstochastic. When \( \beta \) is stochastic, but the decision maker is risk neutral, the problem becomes trivial because optimal choices depend only on the mean values for \( \alpha \) and \( \beta \). The analysis is complicated, however, when the decision maker is risk averse. Even if the random payoffs are stochastically independent, both level sets shift in response to a change in the distribution for \( \alpha \), and there is no link between them to ensure that they shift in opposite directions, which is necessary for definite predictions whether the expected utility function is supermodular or submodular.\(^3\)

4. Conclusions

One example that fits our model with quasilinear payoffs is the entrepreneurial competitive firm facing output-price uncertainty \( F(\alpha) \) while deciding on supply of the product \( x \) and on the supply of a

\[^3\text{Hadar and Seo (1990) are able to obtain a prediction concerning cross-effects for changes in uncertainty in the portfolio problem with stochastically independent asset returns. With linear cost } c(x, y) = x + y, \text{ and with } z^* \text{ interpreted as initial wealth, our model conforms to the three-asset portfolio problem when two of the asset returns are stochastic. In their Theorem 5, Hadar and Seo establish conditions under which a change in the independent risk } F(\alpha) \text{ must cause an increase in the share of wealth invested in one of the other two assets, but do not obtain a specific prediction for the cross-effect involving a particular asset.} \]
second product $y$ whose price is $\beta$. A second example is the taxpayer with an exogenous income $\bar{x}$ facing tax-rate uncertainty $F(\alpha)$ while deciding on an amount of income $x$ to leave exposed to taxation when sheltering income is possible, although costly, and a second income $\bar{y}$ taxed at rate $\beta$ may also be sheltered. In both examples, any change in uncertainty from $F(\alpha)$ to a centrally dominant distribution causes every risk averse or risk neutral decision maker to reduce activity $x$, which reduces the marginal cost of the second activity $y$ if the two are cost substitutes. In this event, the decision maker may choose to initiate activity $y$ if it was initially deemed unprofitable, and will increase $y$ if it was initially positive.

In the cases we have considered, two facts have been exploited to establish comparative statics predictions for cross-effects. First, at points of intersection, the relative slopes of the level sets satisfying the first-order conditions for $x$ and $y$ depend predictably on the presence or absence of cost substitutability; in particular, cost substitutability alone determines whether the expected utility function is supermodular or submodular. Second, the optimal choice of $y$ is independent of $F(\alpha)$, so no change in $F$ has any influence on the level set for $y$. Unfortunately, both of these properties are lost when the second payoff is stochastic and the decision maker is risk averse, and as a consequence definite comparative statics predictions are not possible for the general case of a risk averse decision maker facing two stochastic payoff opportunities.

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References


Note that, in this second example, the choice variables $x$ and $y$ are the amounts left unsheltered. Thus, in the payoff function (1), $z^* = \bar{x} + \bar{y}$, the tax rates are $-\alpha$ and $-\beta$, while the cost function $c(\bar{x} - x, \bar{y} - y)$ increases with the amount of income sheltered, and so decreases with both $x$ and $y$. 