Best-response potential games

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Abstract

Best-response potential games are introduced and characterized. Relations with other classes of potential games are indicated. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In potential games, introduced by Monderer and Shapley (1996), information concerning Nash equilibria can be incorporated into a single real-valued function on the strategy space. All classes of potential games that Monderer and Shapley defined share the finite improvement property: start with an arbitrary strategy profile. Each time, let a player that can improve deviate to a better strategy. Under the finite improvement property, this process eventually ends, obviously in a Nash equilibrium.

The purpose of this paper is to introduce and study best-response potential games, a new class of potential games. The main distinctive feature is that it allows infinite improvement paths, by imposing restrictions only on paths in which players that can improve actually deviate to a best response. The definition of best-response potential games, along with some notational matters, is given in Section 2. A characterization of these games is provided in Section 3. Relations with the potential games of Monderer and Shapley (1996) are indicated in Section 4. Section 5 contains a discussion and motivation for the concept of best-response potential games.
2. Best-response potential games

A strategic game is a tuple \( \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \), where \( N \) is the finite player set, for each \( i \in N \) the set of player \( i \)’s strategies is \( X_i \), and \( u_i : \prod_{i \in N} X_i \rightarrow \mathbb{R} \) is player \( i \)’s payoff function.

For brevity, we define \( X = \prod_{i \in N} X_i \) and for \( i \in N \): \( X_i = \prod_{j \in N \setminus \{i\}} X_i \). Let \( x \in X \) and \( i \in N \). With a slight abuse of notation, we sometimes denote \( x = (x_i, x_{-i}) \).

**Definition 2.1.** A strategic game \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is a best-response potential game if there exists a function \( P : X \rightarrow \mathbb{R} \) such that

\[
\forall i \in N, \forall x_{-i} \in X_{-i}: \quad \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) = \arg\max_{x_i \in X_i} P(x_i, x_{-i}).
\]

The function \( P \) is called a (best-response) potential of the game \( G \).

In other words, a game \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) is a best-response potential game if there exists a coordination game \( \langle N, (X_i)_{i \in N}, (P_i)_{i \in N} \rangle \) where the payoff to each player is given by function \( P \) such that the best-response correspondence of each player \( i \in N \) in \( G \) coincides with his best-response correspondence in the coordination game.

Analogous to Monderer and Shapley (1996), one obtains the following results.

**Proposition 2.2.** Let \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a best-response potential game with best-response potential \( P \).

1. The Nash equilibria of \( G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) and \( G = \langle N, (X_i)_{i \in N}, (P_i)_{i \in N} \rangle \), the coordination game with all payoff functions replaced by the potential \( P \), coincide.
2. If \( P \) has a maximum over \( X \) (e.g., if \( X \) is finite), \( G \) has a Nash equilibrium.

3. Characterization

This section contains a characterization of best-response potential games, similar to the main result of Voorneveld and Norde (1997).

Let \( \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle \) be a strategic game. A path in the strategy space \( X \) is a sequence \((x^1, x^2, \ldots)\) of elements \( x^k \in X \) such that for all \( k = 1, 2, \ldots \) the strategy combinations \( x^k \) and \( x^{k+1} \) differ in exactly one, say the \( i(k) \)th, coordinate. A path is best-response compatible if the deviating player moves to a best response:

\[
\forall k = 1, 2, \ldots: \quad u_{i(k)}(x^{k+1}) = \max_{y_i \in X_i} u_{i(k)}(y_i, x^k_{-i(k)}).
\]

Best-response compatible paths have restrictions only on consecutive strategy profiles, so by definition
the trivial path \((x^1)\) consisting of a single strategy profile \(x^1 \in X\) is best-response compatible. A finite path \((x^1, \ldots, x^m)\) is called a best-response cycle if it is best-response compatible, \(x_1 = x_m\), and \(u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})\) for some \(k \in \{1, \ldots, m - 1\}\).

Define a binary relation \(\sqsubseteq\) on the strategy space \(X\) as follows: \(x \sqsubseteq y\) if there exists a best-response compatible path from \(x\) to \(y\), i.e., there is a best-response compatible path \((x^1, \ldots, x^m)\) with \(x^1 = x\), \(x^m = y\). Notice that \(x \sqsubseteq y\) for each \(x \in X\), since \((x)\) is a best-response compatible path from \(x\) to \(x\). The binary relation \(\sim\) on \(X\) is defined by \(x \sim y\) if \(x \sqsubseteq y\) and \(y \sqsubseteq x\).

By checking reflexivity, symmetry, and transitivity, one sees that the binary relation \(\sim\) is an equivalence relation. Denote the equivalence class of \(x \in X\) with respect to \(\sim\) by \([x]\), i.e., \([x] = \{y \in X | y \sim x\}\), and define a binary relation \(<\) on the set \(X\) of equivalence classes as follows: \([x] < [y]\) if \([x] \neq [y]\) and \(x \sqsubseteq y\). To show that this relation is well-defined, observe that the choice of representatives in the equivalence classes is of no concern:

\[
\forall x, \tilde{x}, y, \tilde{y} \in X \quad \text{with} \quad x \sim \tilde{x} \quad \text{and} \quad y \sim \tilde{y} : x \sim y \iff \tilde{x} \sim \tilde{y}.
\]

Notice, moreover, that the relation \(<\) on \(X\) is irreflexive and transitive.

A tuple \((A, <)\) consisting of a set \(A\) and an irreflexive and transitive binary relation \(<\) on \(A\) is properly ordered if there exists a function \(F: A \to \mathbb{R}\) that preserves the order \(<\):

\[
\forall x, y \in A: \quad x < y \implies F(x) < F(y).
\]

**Theorem 3.1.** A strategic game \(G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N}\rangle\) is a best-response potential game if and only if the following two conditions are satisfied:

1. \(X\) contains no best-response cycles;
2. \((X, <)\) is properly ordered.

**Proof.** (\(\Rightarrow\)): Assume \(P\) is a best-response potential for \(G\). Suppose that \((x^1, \ldots, x^m)\) is a best-response cycle. By best-response compatibility, \(P(x^k) \leq P(x^{k+1})\) for each \(k = 1, \ldots, m - 1\). Since \(u_{i(k)}(x^k) < u_{i(k)}(x^{k+1})\) for some \(k \in \{1, \ldots, m - 1\}\), it follows that for such \(k\): \(P(x^k) < P(x^{k+1})\). Conclude that \(P(x^1) < P(x^m) = P(x^1)\), a contradiction. This shows that \(X\) contains no best-response cycles.

To prove that \((X, <)\) is properly ordered, define \(F: X \to \mathbb{R}\) by taking for all \([x] \in X : F([x]) = P(x)\). To see that \(F\) is well-defined, let \(y, z \in [x]\). Since \(y \sim z\) there is a best-response compatible path from \(y\) to \(z\) and vice versa. But since the game has no best-response cycles, all changes in the payoff to the deviating players along these paths must be zero: \(P(y) = P(z)\).

Now take \([x]\), \([y]\) \(\in X\) with \([x] \neq [y]\). Since \(x \sqsubseteq y\), there is a best-response compatible path from \(x\) to \(y\), so \(P(x) \leq P(y)\). Moreover, since \(x\) and \(y\) are in different equivalence classes, some player must have gained from deviating along this path: \(P(x) < P(y)\). Hence \(F([x]) < F([y])\).

(\(\Leftarrow\)): Assume that the two conditions hold. Since \((X, <)\) is properly ordered, there exists a function \(F: X \to \mathbb{R}\) that preserves the order \(<\). Define \(P: X \to \mathbb{R}\) by \(P(x) = F([x])\) for all \(x \in X\). Let \(i \in N, x_{-i} \in X_{-i}\).
Let $y_i \in \arg\max_{x_i \in X_i} u_i(x_i, x_{-i})$ and $z_i \in X_i \backslash \{y_i\}$.

- If $u_i(y_i, x_{-i}) = u_i(z_i, x_{-i})$, then $(y_i, x_{-i}) \sim (z_i, x_{-i})$, so $P(y_i, x_{-i}) = F([(y_i, x_{-i})]) = F([(z_i, x_{-i})])$.
- If $u_i(y_i, x_{-i}) > u_i(z_i, x_{-i})$, then $(z_i, x_{-i}) \subset (y_i, x_{-i})$. By the absence of best-response cycles, not $(y_i, x_{-i}) \subset (z_i, x_{-i})$. Hence $[(z_i, x_{-i})] < [(y_i, x_{-i})]$, which implies $P(z_i, x_{-i}) = F([(z_i, x_{-i})]) < F([(y_i, x_{-i})]) = P(y_i, x_{-i})$.

The above observations imply that $y_i \in \arg\max_{x_i \in X_i} P(x_i, x_{-i})$. This concludes the proof that

$$\forall i \in N, \forall x_{-i} \in X_{-i}, \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) \subseteq \arg\max_{x_i \in X_i} P(x_i, x_{-i}).$$

Let $y_i \in \arg\max_{x_i \in X_i} P(x_i, x_{-i})$ and $z_i \in X_i \backslash \{y_i\}$. Suppose $u_i(z_i, x_{-i}) > u_i(y_i, x_{-i})$. Then $(y_i, x_{-i}) \not\subset (z_i, x_{-i})$. By the absence of best-response cycles, not $(z_i, x_{-i}) \subset (y_i, x_{-i})$. Hence $[(y_i, x_{-i})] < [(z_i, x_{-i})]$, which implies $P(y_i, x_{-i}) = F([(y_i, x_{-i})]) < F([(z_i, x_{-i})]) = P(z_i, x_{-i})$, contradicting $y_i \in \arg\max_{x_i \in X_i} P(x_i, x_{-i})$. This finishes the proof that

$$\forall i \in N, \forall x_{-i} \in X_{-i}, \arg\max_{x_i \in X_i} u_i(x_i, x_{-i}) \supseteq \arg\max_{x_i \in X_i} P(x_i, x_{-i}).$$

Conclude from (1) and (2) that $P$ is a best-response potential for the game $G$. □

If the strategy space $X$ is countable, i.e., $X$ is finite or there exists a bijection between $\mathbb{N}$ and $X$, the proper order condition is redundant.

**Theorem 3.2.** Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. If $X$ is countable, then $G$ is a best-response potential game if and only if $X$ contains no best-response cycles.

**Proof.** If $X$ is countable, $X_i$ is countable. It follows from Lemma 2.2 in Voorneveld and Norde (1997), which is essentially Theorem 2.6 in Fishburn (1979), that $(X_i, \prec)$ is properly ordered. The result now follows from Theorem 3.1. □

### 4. Relations with other potential games

Monderer and Shapley (1996) introduce exact, weighted, ordinal, and generalized ordinal potential games. The relations between these classes of games (indicated by E, W, O, and G, respectively) and best-response potential games (indicated by BR) are indicated in Fig. 1. For easy reference, their definitions are as follows. A strategic game $(N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is

- an **exact potential game** if there exists a function $P:X \rightarrow \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$ and all $y_i, z_i \in X_i$:
  $$u_i(y_i, x_{-i}) - u_i(z_i, x_{-i}) = P(y_i, x_{-i}) - P(z_i, x_{-i}).$$
- a **weighted potential game** if there exists a function $P:X \rightarrow \mathbb{R}$ and a vector $(w_i)_{i \in N}$ of positive numbers such that for all $i \in N$, for all $x_{-i} \in X_{-i}$ and all $y_i, z_i \in X_i$:
• an ordinal potential game if there exists a function $P:X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$ and all $y, z_i \in X_i$:

$$u_i(y, x_{-i}) - u_i(z, x_{-i}) \geq w_i(P(y, x_{-i}) - P(z, x_{-i})).$$

• a generalized ordinal potential game if there exists a function $P:X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$ and all $y, z_i \in X_i$:

$$u_i(y, x_{-i}) - u_i(z, x_{-i}) > 0 \Longleftrightarrow P(y, x_{-i}) - P(z, x_{-i}) > 0.$$

Since Monderer and Shapley already indicated the relations between their classes of games, we stress the relation with best-response potential games.

That an ordinal potential game is a best-response potential game follows immediately from their definitions. Example 4.1 indicates that a generalized ordinal potential game is not necessarily a best-response potential game. Example 4.2 indicates that a best-response potential game is not necessarily a generalized ordinal potential game. Example 4.3 indicates that the intersection of the set of best-response potential games and generalized ordinal potential games properly includes the set of ordinal potential games, i.e., there are games which are both a best-response and a generalized ordinal potential game, but not an ordinal potential game.

**Example 4.1.** The game in Fig. 2a has a generalized ordinal potential as given in Fig. 2b. However, a best-response potential (and ordinal potential) would have to satisfy $P(T, L) = P(B, L) > P(B, R) > P(T, R) > P(T, L)$, which is a contradiction.
Example 4.2. The game in Fig. 3a has a best-response potential as given in Fig. 3b. However, a generalized ordinal (or ordinal) potential would have to satisfy $P(T, M) > P(B, M) > P(B, R) > P(T, R) > P(T, M)$, a contradiction.

Example 4.3. The game in Fig. 4a has a best-response and generalized ordinal potential as given in Fig. 4b. However, an ordinal potential would have to satisfy $P(T, M) > P(B, M) > P(B, R) > P(T, R) = P(T, M)$, a contradiction.

5. Discussion

There are several reasons for introducing best-response potential games. In the first place, they are based on a simple insight: to determine Nash equilibria, what matters are best responses. It is quite natural, in trying to find out whether a finite game has a Nash equilibrium, to look at the best situation a player can achieve by changing his strategy choice. This idea is at the root of fictitious play (Brown, 1951). Moreover, this is exactly what Milchtaich (1996) does to prove the existence of an equilibrium in his congestion games.

Best-response potential games differ from the potential games of Monderer and Shapley in an important aspect: they allow the presence of infinite improvement paths even in finite games. The games of Monderer and Shapley have equilibria because one could look at an improvement path and notice that it stopped somewhere. Best-response potential games give sufficient conditions for the existence of equilibria even if infinite improvement paths exist, as is the case in Example 4.2.
References


