Heterogeneous or homogeneous quantity competition

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Abstract

Every homogeneous goods Cournot model is payoff equivalent to a heterogeneous goods model. We characterize when the converse is true for linearly separable and multiplicatively separable models. These results have implications that blur the distinction between process and product innovations. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Heterogeneous goods models often look remarkably similar to homogeneous goods models with higher marginal costs. Take, for example, two firms producing differentiated products, the inverse demand functions of which are linear and given by 
\[ P(x, Y) = 1 - x - Y/2, \]
where \( x \) and \( Y \) denote the quantities produced by a firm and its rival respectively. Suppose further that each firm faces the quadratic cost function 
\[ C(x) = x^2/2. \]
If the firms compete in quantities, then this model is payoff equivalent to a heterogeneous goods model where the inverse demand at each firm is given by:
\[ P(x + Y) = 1 - (x + Y)/2, \]
and cost is \( \hat{C}(x) = x^2 \).

Recognition of this equivalence provides some useful insights. For example, product innovations that lead to greater product differentiation can have similar effects to cost shocks that raise the marginal cost of production. Just as product differentiation reduces competitive pressures, so do higher marginal costs.

In this note we are interested in the cases when heterogeneous goods models can be transformed into payoff equivalent homogeneous goods models (and vice versa) by changing the inverse demand
and cost functions. If two models are payoff equivalent, then the behavior of the firms in the two models will be the same.

The first result of the paper, Proposition 1, shows that every homogeneous goods model is payoff equivalent to a heterogeneous goods model with lower total and marginal costs. Consequently, one might expect that we will be able to transform heterogeneous goods models into homogeneous goods models with higher marginal costs. While this is true for some models, like the linear demand example given earlier, such a transformation is not generally possible. We explore two interesting classes of heterogeneous goods demand curves: linearly separable and multiplicatively separable. In Propositions 2 and 3 we characterize when such models can be transformed into payoff equivalent homogeneous goods models. We also discuss whether the resulting model will have higher or lower total and marginal costs.

In Section 3 we give some examples where the results of this note are applicable.

2. Heterogeneous versus homogeneous

Let \( N = \{1, \ldots, n\} \) denote the set of firms in the industry. Each firm \( i \in N \) is endowed with a twice continuously differentiable cost function \( C_i(x) \) where \( x \) is the output level chosen by the firm. Cost functions may differ across firms. The cost function of a firm is required to be independent of the output levels chosen by other firms. We refer to this restriction as Condition A:

**Condition A.** (Independence of cost functions): The cost function \( C_i(x) \) of each firm \( i \in N \) is independent of the output levels chosen by other firms.

Condition A, while quite natural, does have important implications about our ability to transform heterogeneous goods models into homogeneous goods models.

In a homogeneous goods model all firms face the same price which is determined by a twice continuously differentiable inverse demand function \( P(Q) \) where \( Q \) is the total industry output. The law of demand requires that \( P(Q) < 0 \).

A homogeneous goods model is any \( G \) of the form \( G = (P(Q), (C_i(x))_{i \in N}) \), where \( P(Q) \) is the inverse demand function and for each \( i \in N \), \( C_i(x) \) is firm \( i \)'s cost function.

The profit function of firm \( i \) in a homogeneous goods model \( G \) is implicitly defined by \( \pi_i^G(x, Y) = P(Q)x - C_i(x) \) where \( x \) denotes the quantity choice of firm \( i \) and \( Y \) denotes the sum of the quantity choices of rival firms. Thus, \( Q = x + Y \) is the total industry output.

In a heterogeneous goods model, own output level and the sum of rival output levels affect own price differently. We concentrate on heterogeneous goods models with symmetric twice continuously differentiable inverse demand functions of the form \( P(x, Y) \) where once again \( x \) is the quantity produced at the firm in question and \( Y \) is the sum of the outputs at rival firms.

We impose the natural restriction that own output changes affect own price more than do output changes at other firms. This restriction and the law of demand are summarized as:

**Condition B.** (Greater own price effects): \( \partial P(x, Y)/\partial x < \partial P(x, Y)/\partial Y < 0 \) for all \( (x, Y) \).

The linear demand function given in the introduction satisfies Condition B as does the demand function \( P(x, Y) = e^{-bQ} - dx^2 \) when \( b, d > 0 \).
A heterogeneous goods model is any \( D \) of the form \( D = (P(x, Y), (C_i(x))_{i \in N}) \) where \( P(x, Y) \) is a symmetric inverse demand function satisfying Condition B, and for each \( i \in N \), \( C_i(x) \) is the cost function of firm \( i \).

The profit function of firm \( i \) in a heterogeneous goods model is implicitly defined by \( \pi_i^D(x, Y) = P(x, Y)x - C_i(x) \).

In a quantity setting model, whether heterogeneous or homogeneous, firms choose quantities simultaneously and independently and then prices are determined by the inverse demand functions so that quantity demanded at each firm just equals the quantity chosen by each firm.

Let \( G \) be a homogeneous goods model and let \( D \) be a heterogeneous goods model. We say that \( G \) and \( D \) are payoff equivalent if and only if for each \( i \in N \), \( \pi_i^G(x, Y) = \pi_i^D(x, Y) \) for all \( (x, Y) \).

The following Lemma describes a relationship that must hold between the demand and cost functions in homogeneous goods model and a payoff equivalent heterogeneous goods models. It gives us an idea of the types of models that can be transformed from heterogeneous goods models to payoff equivalent homogeneous goods models and vice versa.

**Lemma 1.** Let \( D = (P(x, Y), (\hat{C}_i(x))_{i \in N}) \) be a heterogeneous goods model and \( G = (P(Q), (C_i(x))_{i \in N}) \) be a homogeneous goods model and let both models satisfy Condition A. Then \( D \) and \( G \) are payoff equivalent if and only if there is a function \( f(x) \) which is independent of \( Y \) and satisfies \( f'(x) = \partial P(x, Y)/\partial x - \partial P(x, Y)/\partial Y < 0 \), \( P(x, Y) = P(Q) + f(x) \) and \( \hat{C}_i(x) = C_i(x) + f(x)x \) for each firm \( i \in N \) and all \( (x, Y) \).

**Proof.** (Only-if) If \( D \) and \( G \) are payoff equivalent, then for all \( (x, Y) \) we have \( \hat{C}_i(x) = [P(x, Y) - P(x + Y)]x - C_i(x) \). Differentiating both sides with respect to \( Y \) and using Condition B we find that:

\[
\frac{\partial P(x, Y)}{\partial Y} = P'(Q) \text{ for all } (x, Y), \text{ where } Q = x + Y. \tag{2.1}
\]

It then follows by the fundamental theorem of calculus that \( P(x, Y) = P(Q) + f(x) \) and the function \( f(x) \) does not depend on \( Y \).

Now, since \( f(x) = P(x, Y) - P(Q) \) for all \( (x, Y) \) and both \( P(x, Y) \) and \( P(Q) \) are differentiable it follows that \( f(x) \) is differentiable. Differentiating \( f(x) \), then using (2.1) and Condition B we get:

\[
f'(x) = \frac{\partial P(x, Y)}{\partial x} - \frac{\partial P(x, Y)}{\partial Y} < 0. \tag{2.2}
\]

(If) Just write down the profit functions. \( \square \)

Lemma 1 actually provides a simple way to transform any homogeneous goods model into a payoff equivalent heterogeneous goods model. Choose a differentiable function \( f(x) \) that satisfies \( f'(x) < 0 \) everywhere and adjust costs and demand in offsetting ways. The next proposition makes use of this technique.

**Proposition 1.** Every homogeneous goods quantity setting model \( G = (P(Q), (C_i(x))_{i \in N}) \) is payoff equivalent to a heterogeneous goods quantity setting model with lower total and marginal costs, one of which is given by \( D = (P(x, Y), (\hat{C}_i(x))_{i \in N}) \) where \( P(x, Y) = P(Q) - x \) and \( \hat{C}_i(x) = C_i(x) - x^2 \).
Proposition 3. A multiplicatively separable inverse demand function is not so easy. Take for example the demand function \(P(x, Y) = e^{cx}/xe^{cY}\), where \(c < 0\). This inverse demand function satisfies Condition B, but it is not clear how to transform it into a homogeneous goods model. We will find from Proposition 3 that no such transformation exists.

We give results for inverse demand functions that are linearly separable and multiplicatively separable.

A heterogeneous goods inverse demand function is called linearly separable if it can be described as \(P(x, Y) = g(Y) + h(x)\), where \(g(Y)\) does not depend on \(x\) and \(h(x)\) does not depend on \(Y\).

Linear demand curves like the example given in the introduction satisfy this condition. Notice that since linear separability does not require that \(g(Y)\) and \(h(x)\) are linear functions, \(P(x, Y) = a - bY - cx - dx^2\) is also linearly separable.

The next Lemma is used to prove Propositions 2 and 3.

Lemma 2. If \(D = (P(x, Y), (\hat{C}_i(x))_{i \in N})\) is a heterogeneous goods model and \(G = (P(Q), (C_i(x))_{i \in N})\) is a payoff equivalent homogeneous goods model and both models satisfy Condition A, then \(\partial^2 P(x, Y)/\partial Y^2 = \partial^2 P(x, Y)/\partial Y \partial x\).

Proof. Since \(D\) and \(G\) are payoff equivalent, we have from Lemma 1 that \(f'(x) \equiv \partial P(x, Y)/\partial x - \partial P(x, Y)/\partial Y\). Differentiating this expression with respect to \(Y\) and setting it equal to zero yields the desired result. \(\square\)

The next proposition characterizes the class of linearly separable heterogeneous goods models that are payoff equivalent to homogeneous goods models.

Proposition 2. A linearly separable heterogeneous quantity setting model with \(P(x, Y) = g(Y) + h(x)\) is payoff equivalent to a homogeneous goods quantity setting model if and only if \(g(Y)\) is linear.

Proof. (Only if) Since \(P(x, Y)\) is linearly separable it follows that \(\partial^2 P(x, Y)/\partial Y \partial x = 0\) for all \((x, Y)\). But then it follows by Lemma 2 that \(\partial^2 P(x, Y)/\partial Y^2 = g''(Y) = 0\), which implies \(g(Y)\) is linear.

(If) Let \(D = (P(x, Y), (\hat{C}_i(x))_{i \in N})\) be a heterogeneous goods model where \(P(x, Y) = a - bY + h(x)\). Then \(P(Q) = a - bQ\) and \(C_i(x) = \hat{C}_i(x) - [h(x) + bx]x\) defines a payoff equivalent homogeneous goods model. \(\square\)

We now turn our attention to multiplicatively separable models. A heterogeneous goods inverse demand function is called multiplicatively separable if it can be described as \(P(x, Y) = g(Y)h(x)\) where \(g(Y)\) does not depend on \(x\) and \(h(x)\) does not depend on \(Y\). We assume that \(g(Y)\), \(h(x)\), \(-g'(Y)\), \(-h'(x)\) are all positive.

The inverse demand function \(P(x, Y) = e^{cx}/xe^{cY}\) discussed earlier is multiplicatively separable. The inverse demand function \(P(x, Y) = (e^{cy} - d/c)(e^{cx})\) with \(d > 0 > c\) is also multiplicatively separable.

Proposition 3. A multiplicatively separable heterogeneous goods quantity setting model with \(P(x,
Proof. (Only if) Let \( P(x, Y) = g(Y)h(x) \) with \( g(Y) > 0, h(x) > 0, g'(Y) < 0, \) and \( h'(x) < 0 \) for all \( (x, Y) \). Condition B and Lemma 2 in this example give respectively:

\[
g(Y)h'(x) < g'(Y)h(x) \quad \text{for all } (x, Y); \quad \text{and}
\]

\[
g''(Y)h(x) = g'(Y)h'(x) \quad \text{for all } (x, Y).
\]

From (2.4) we get \( h'(x)/h(x) = g''(Y)/g'(Y) = c \) (a constant) for all \( (x, Y) \). This implies that \( h''(x) = ch'(x) \) for all \( x \) and \( h'(x) = ch(x) + b \), where \( b \) is a constant.

We next show that \( b = 0 \). Substituting \( h'(x) = ch(x) + b \) into (2.4) and rearranging we find that 

\[
g''(Y) = cg'(Y) + bg'(Y)/h(x) \quad \text{for all } (x, Y).
\]

Since \( h'(x) < 0 \) and \( g'(Y) \neq 0 \), it follows that this equality cannot hold for all \( (x, Y) \) unless \( b = 0 \).

Since \( b = 0 \), we have that \( h'(x) = ch(x) \). Substituting this into (2.3) and (2.4) we get:

\[
g'(Y) = cg(Y), \quad \text{and}
\]

\[
g''(Y) = cg'(Y).
\]

Eq. (2.6) implies that \( g'(Y) = cg(Y) + d \) where \( d \) is a constant. For consistency with (2.5) we need that \( d > 0 \). The general solution in this setting is well known to be of the form: \( g(Y) = B e^{cY} - d/c \) and \( h(x) = A e^{cx} \). For our purposes we require that \( A, B, d, \) and \( -c \) are positive.

(If) Let \( D = (P(x, Y), \tilde{C}(x))_{x \in K} \) be a heterogeneous goods model with \( P(x, Y) = (B e^{cY} - d/c)(A e^{cx}) \). Then \( P(Q) = AB e^{cQ} \) and \( C_i(x) = \tilde{C}(x) + x(d/cA e^{cx}) \) defines a payoff equivalent homogeneous goods model.

This Proposition rules out demand functions like \( P(x, Y) = e^{cY}/x e^{cY} \) discussed earlier.

**Remark 1.** If we move beyond linearly separable and multiplicatively separable heterogeneous goods models we can find many models that are payoff equivalent to a homogeneous goods model. Starting with any inverse demand function \( P(Q) \) and using Lemma 1 we can construct a payoff equivalent heterogeneous goods model using a function \( f(x) \) which satisfies \( f'(x) < 0 \). The demand function \( P(x, Y) = e^{-bQ} - d x^2 \) discussed after the definition of Condition B is obtained from \( P(Q) = e^{-bQ} \) by using \( f(x) = -d x^2 \). This model is neither linearly separable or multiplicatively separable.

**Remark 2.** It is interesting to know whether or not the transformation from heterogeneous to homogeneous goods model increases or decreases marginal and total costs. In general this depends on the transformation used.

In the linear demand case of \( P(x, Y) = a - bY - cx \) with \( c > b > 0 \), a natural transformation is to \( P(Q) = a - bQ \). This implies that \( \tilde{C}(x) = C(x) + (c - b)x \). Clearly total and marginal costs are higher in the resulting homogeneous goods model.
However, in the multiplicatively separable case of \( P(x, Y) = AB e^{cQ} - d/cA e^{cx} \), a natural transformation is to \( P(Q) = AB e^{cQ} \). This yields \( \hat{C}_i(x) = C_i(x) + d/cA e^{cx} \) which is less than \( C_i(x) \) for all \( x \). Furthermore, \( \hat{C}_i(x) = C_i(x) + dA e^{cQ[A/c + x]} \) which is greater than \( C_i(x) \) if and only if \( x > |A/c| \). The homogeneous goods model has lower total cost always and higher marginal cost only for sufficiently large quantities \( x > |A/c| \).

So in general whether or not the transformation will yield higher or lower total and marginal costs seems ambiguous.

**Remark 3.** Our results link heterogeneous goods models to payoff equivalent homogeneous goods models and vice versa. On a behavioral and welfare level for the firms we cannot distinguish between the two models. However, clearly there are observable differences between the two models. For example, the models will have different equilibrium prices.

### 3. Examples

In this section we give some examples where the results of the previous section prove useful.

**Higher costs yield higher profits:** Inverse demand in a homogeneous goods market is linear and composed of \( n \) firms: \( P(Q) = a - Q \). We consider two cost curves: \( C(x) = x/2 \) and \( \hat{C}(x) = 0 \). We ignore the subscript in the cost curves since we will assume now that all firms face the same costs. In the first case marginal costs are linear and increasing and in the second case marginal costs are zero. One might expect that firms should be better off in the case with lower costs. However, it turns out that for \( n \geq 4 \) the firms prefer the model with higher marginal and total costs. The equilibrium profit as a function of the number of firms is \( \pi_i(n) = 3a/2(n+2)^2 \) when \( C(x) = x/2 \), and \( \hat{C}(x) = 0 \).

Higher marginal costs reduce competition in a way that is similar to a differentiated products model. In fact the model with increasing marginal costs is equivalent to a model with zero costs and differentiated products. The model has \( P(x, Y) = a - Y - \frac{3}{2}x \), and \( \hat{C}(x) = 0 \).

**Process or product innovation:** Process innovation lowers costs of production. Product innovation changes the product and thus some demand parameters. Consider a heterogeneous goods model with \( P(x, Y) = a - \gamma Y - bx \), and cost function \( C_i(x) = cx + dx^2 \).

A process innovation that lowers \( c \) and a product innovation that raises \( a \) both lead to payoff equivalent models. Also, a process innovation which simultaneously lowers \( c \) and raises \( d \) may be indistinguishable from a product innovation that raises \( a \) and lowers \( b \).

Bonanno and Haworth (1998) compare strategic choice of process or product innovations in a duopoly model of vertical differentiation. In contrast to a model of horizontal differentiation, a model of vertical differentiation has the property that if prices are the same at the two firms, only the firm with the higher quality good will sell anything. In a model of horizontal differentiation, both firms sell when prices are the same. Still our results have implications for this model.

In the case of quantity competition their derived inverse demand curves are: \( f_h(q_h, q_l) = k_h - k_h q_h - k_l q_l \), and \( f_l(q_h, q_l) = k_l - q_h - q_l \), where \( k_i \) represents the quality of the product at firm \( i \) (\( i = l, h \)) and \( q_h \) and \( q_l \) are the quantities produced. Letting \( c_h \) and \( c_l \) denote the constant marginal costs at firm \( h \) and \( l \) respectively, the authors assume that \( k_h > k_l \) and \( c_h > c_l \).

A process innovation at firm \( h \) lowers \( c_h \) and a product innovation at firm \( h \) raises quality \( k_h \). Notice
here that a product innovation is just like a process innovation that lowers $c_h$ and raises the slope of the marginal cost curve at firm $h$.

We see now that the distinction between process and product innovations can be quite subtle.

References