A method for solving multi-region models

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Abstract

The paper develops a two-step method for solving multi-region general equilibrium models. The method has the advantage that the number of regions can be arbitrarily large. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In order to understand international linkages between economies and the transmission of shocks across economies, since the early 1990s an extensive literature has developed studying two-country general equilibrium models.1 They are often applied to individual country data, even though the models themselves only contain two countries making up the entire world. In some papers an aggregate of countries is constructed, such as “Europe”. While more consistent with the model, it has the disadvantage that in the process of aggregation a significant amount of information is lost. In many applications one would like to apply the model to a large set of actual countries, or maybe a large set of regions within a country. We are then also able to ask questions about differences across countries with regards to business cycle moments and to study cross-country relationships between variables (e.g. cross-country saving–investment correlations).

In this paper I develop a method for solving general equilibrium multi-region models, where region can be broadly interpreted as corresponding to any type of geographical area (e.g. cities, counties, states, provinces, countries, or continents). The sum of all these regions either encompasses the world

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1See Baxter (1995) for an overview.
or an area that is relatively closed, so that general equilibrium analysis is appropriate. The method most commonly used to solve two-country general equilibrium models is that developed by King et al. (1987). Although it is an approximate numerical method, it has turned out to be quite precise for most applications. The method is based on linearization (or log-linearization) of all first order conditions, leading to an approximate linear solution of decision variables as a function of a set of state variables. The next section extends this technique to multi-region models. One can always stack all region-specific equations and solve the whole model directly with the KPR technique. When the number of regions is large (and the number of state and control variables is large) this can be computationally difficult or even impossible. Here we develop a two-step method of solving general equilibrium multi-region models which has the advantage that the number of regions can be arbitrarily large.

2. Computational method

There are $J$ regions, inhabited by representative agents. We will refer to the sum of all the regions as the “world”. Their size is defined by population, which in region $i$ is a fraction $w_i$ of world population. Let $c_{it}$ and $s_{it}$ denote, respectively, the vectors of $c$ control and $s$ state variables in region $i$. When relevant, these variables are defined in per capita terms. There are $c$ first order conditions associated with the control variables, and $s$ equations describing the dynamic process of the state variables (accumulation of assets, capital, etc.). These equations also include global good and asset prices, indicated by the vector $p$. In a perfect risksharing model (complete markets) this vector consists only of the shadow prices of world resources for each traded good. In a model in which there are $g$ traded goods and $a$ traded assets, the vector $p$ consists of $g + a - 1$ relative prices.

First order conditions and state difference equations are, respectively,

$$E_{t-1} f(c_{it}, s_{it}, c_{it-1}, s_{it-1}, p_{t-1}) = 0$$ \hspace{1cm} (1)

$$h(s_{it}, c_{it-1}, s_{it-1}, p_{t-1}, \epsilon_{it}) = 0$$ \hspace{1cm} (2)

where $\epsilon_i$ is a vector of stochastic innovations. The innovations have expectation zero, but may be drawn from different distributions in different countries. The expectation in the first order conditions is conditional on the period $t-1$ information set, $\{s_{i,t-1}, i = 1, ..., J\}$.

It is assumed that $f(.)$ and $h(.)$ have the same functional form for all regions, allowing us to easily aggregate. This assumption can be relaxed somewhat, an issue to which I will return in the next section.

After linearization around the steady state, redefining variables as their deviation from steady state, the equations can be written as

$$F_1 E_{t-1} x_{it} + F_2 x_{it-1} + F_3 p_{t-1} = 0$$ \hspace{1cm} (3)

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1Prices of non-traded goods and assets are solved from the corresponding market equilibrium condition for each region and substituted back into the rest of the model.

2As is well known, this imposes no restriction on the complexity of the dynamics as second and higher order difference equations can always be written in the form of a first order difference equation.

3Often there is no steady state with respect to asset holdings. In that case linearization is done around the initial holdings of the assets. See for example Baxter (1995).
\begin{equation}
H_1s_{it} + H_2x_{it-1} + H_3p_{t-1} + H_4\epsilon_{it} = 0
\end{equation}

where

\[ x_{it} = \begin{pmatrix} c_{it} \\ s_{it} \end{pmatrix}. \]

The first step towards the solution is to solve for the global prices as a function of the global state of the world. This is done by aggregating over Eqs. (2)–(4), with a weight \( w_i \) for country \( i \). Let \( x^g = \sum_{i=1}^c w_i x_i \). After taking the expectation of the aggregated state differential equations, the global system becomes:

\begin{align*}
F_1E_{t-1}x^g_i + F_2x^g_{t-1} + F_3p_{t-1} &= 0 \\
H_1E_{t-1}s^g_i + H_2x^g_{t-1} + H_3p_{t-1} &= 0
\end{align*}

To this we need to add the global asset and good market equilibrium conditions. With \( g \) traded goods and \( a \) traded assets, by Walras’ Law we need to impose only \( a + g - 1 \) of these equilibrium conditions. After linearization of the \( g \) global good market equilibrium conditions, aggregation yields a set of \( g \) equations \( Gx^g = 0 \). If \( a_i \) is the vector of net asset demands by country \( i \) for \( a - 1 \) assets, the global asset market equilibrium conditions are \( a_i = 0 \), or \( a_i = 0 \), where \( a_i \) is part of the vector \( x^g_i \). In the perfect risksharing model we only need to impose the \( g \) global good market equilibrium conditions.

Combining \( a + g - 1 \) good and asset market equilibrium equations with (5) and (6), solving for the price vector and \( E_{t-1}x^g_i \), we get

\begin{align*}
E_{t-1}x^g_i &= Dx^g_{t-1} \\
p_{t-1} &= Px^g_{t-1}
\end{align*}

where \( D \) is a \( (c + s) \times (c + s) \) matrix, and \( P \) is a \( a + g - 1 \times (c + s) \) matrix. From (7), global control variables can be solved in the standard way as a function of global state variables. Using the decomposition \( D = V\Delta V^{-1} \), decision rules are solved by setting the rows of \( V^{-1}x^g \) corresponding to the zero and explosive eigenvalues equal to zero. The solution takes the form \( c^g_i = C^g s^g_i \). Together with (5) and (7) this delivers \( E_{t-1}s^g_i = S^g s^g_{t-1} \), and the equilibrium price vector \( p_{t-1} = P^g s^g_{t-1} \). In the perfect risksharing model we only need to impose the \( g \) global good market equilibrium conditions.

In the second part of the solution, country specific control variables are solved as a function of the country specific and the global state of the world. Combining the country specific equations (3) and (4) with the global price solution \( p_{t-1} = P^g s^g_{t-1} \), and the difference equation for the global state variables \( E_{t-1}s^g_i = S^g s^g_{t-1} \), we obtain the system

\[ SP^* = P\hat{C} \quad \text{and} \quad S^* = -H_1^{-1}[H_1\hat{C} + H_2P^*], \quad \text{where} \quad \hat{C} = \begin{pmatrix} C^g \\ I_{(c+s),s} \end{pmatrix}. \]

\[ S^*M = \begin{pmatrix} F_1 & 0_{s,s} & -F_2 & -F_3P^* \\ H_1 & 0_{s,s} & -H_2 & -H_3P^* \\ 0_{(c+s),(c+s)} & I_{(c+s),s} & 0_{(c+s),s} \end{pmatrix} \begin{pmatrix} C^g \\ I_{(c+s),s} \end{pmatrix}. \]
Using the decomposition $M = WAW^{-1}$, decision rules are solved by setting the rows of $W^{-1} \begin{pmatrix} x_t^g \\ s_t^g \end{pmatrix}$ corresponding to zero and explosive eigenvalues equal to zero. The solution takes the form

$$c_{it} = C \begin{pmatrix} s_{it}^g \\ s_t^g \end{pmatrix}$$

This solves the model. Combined with (3) and its global counterpart, the equation of motion of the state vector can be derived:

$$s_{it} = s_{i,t-1} + Ss_{i,t-1} + S \varepsilon_{it}$$

$$s_{it}^g = Ss_{i,t-1}^g + S \varepsilon_{it}^g$$

### 3. Final remarks

One limitation of the solution method outlined above is that it assumes that the first order conditions and accumulation equations have the same functional form across regions. Implicitly we assume that preference and technology parameters are identical. This is necessary in order to be able to aggregate in the first step towards the solution. The method can be extended though in order to allow for parameter heterogeneity as long as the extent of heterogeneity is limited. For example, when in one of the first order conditions the capital stock is multiplied by a parameter that differs across countries, we are unable to aggregate. Each of the region-specific capital stocks will then need to enter the solution for the global price vector in the first step. With a significant extent of heterogeneity the method described here is no longer useful. One needs to stack all the region-specific equations and solve the whole system simultaneously in one step. This can become difficult or impossible when the number of regions and equations is large as the matrix that needs to be diagonalized quickly becomes enormous. It is worth noting that the solution method described here does allow for heterogeneity with respect to size and the stochastic features of the innovations such as standard deviations and cross-region correlations.

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