Distributional properties of the uniform rule in economies with single-peaked preferences

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Abstract

We investigate the distributional properties of the uniform rule for the problem of allocating an infinitely divisible commodity among a group of agents with single-peaked preferences. First, we show that the uniform rule is the only subsolution of the convex no-envy and Pareto solution. Next, we show that the uniform rule offers the greatest upper bound when there is too little of the commodity, and the least lower bound when there is too much of the commodity.

Keywords: Economies with single-peaked preferences; Convex no-envy; Greatest upper bound; Least lower bound; Uniform rule

JEL classification: D63; D70

1. Introduction

We consider the problem of allocating an infinitely divisible commodity among a group of agents whose preferences are single-peaked: each agent has a most preferred level of consumption and moving away from that level, in either direction, makes him worse off. The uniform rule (Benassy, 1982) is the best-known solution to this class of problems, which has been studied from various perspectives: strategy-proofness by Sprumont (1991), consistency by Thomson (1994b), monotonicity by Thomson (1994a,1995,1997), separability by Chun (1999), equity by de Frutos and Massó (1995) and Schummer and Thomson (1997), and others (for surveys, see Klaus (1998) and Thomson (forthcoming)).
These studies impose one of two distributional requirements in their characterizations of the uniform rule: no-envy (Foley, 1967) requires that every agent should prefer what he receives to what any other agent receives, and the equal division lower bound requires that every agent should prefer what he receives to equal division.

In this paper, we study these distributional requirements. First, we investigate the implications of strengthening no-envy to convex no-envy (Kolm, 1973): every agent should prefer what he receives to any convex combination of what all agents receive. We show that the uniform rule is the only subsolution from the intersection of the convex no-envy solution and the Pareto solution.

Then, we investigate properties of solutions related to the equal division lower bound. Given an economy in which the amount $\Omega$ of the commodity is available and a single-valued solution, we define the upper bound to be the maximal number $\beta \in [0,1]$ with the property that for all $\beta \in [0,1]$ such that $\beta \leq \beta$, each agent prefers what he receives to $\beta \Omega$, and the lower bound to be the minimal number $\beta \in [0,1]$ with the property that for all $\beta \in [0,1]$ such that $\beta \geq \beta$, each agent prefers what he receives to $\beta \Omega$. When there is too little of the commodity, agents want to have a large portion of the commodity, and prefer a solution that offers a large upper bound. On the other hand, when there is too much of the commodity, agents want to have only a small portion of the commodity, and prefer a solution that offers a small lower bound. Therefore, we ask the existence of solutions that offer the greatest upper and the least lower bounds. Here we show that the uniform rule offers the greatest upper bound when there is too little of the commodity, and the least lower bound when there is too much of the commodity.

The paper is organized as follows. Section 2 contains some preliminaries and introduces the uniform rule. Section 3 characterizes the uniform rule on the basis of convex no-envy. Section 4 investigates the upper and lower bounds. Section 5 concludes.

2. Preliminaries

Let $N \equiv \{1, \ldots, n\}$ be a set of agents, indexed by $i$. A social endowment $\Omega \in \mathbb{R}_+$ of some infinitely divisible commodity has to be allocated among the agents in the set $N$. Each agent $i \in N$ is equipped with a single-peaked preference relation $R_i$ defined over $\mathbb{R}_+$; this means that there is a number $p(R_i) \in \mathbb{R}_+$, called agent $i$’s peak amount, such that for all $z_i, z'_i \in \mathbb{R}_+$, if $z'_i < z_i \leq p(R_i)$ or $p(R_i) \leq z_i < z'_i$, then $z_i R_i z'_i$. Let $\mathcal{R}$ be the class of all such preference relations, and $\mathcal{R}^N$ be the cross-product of $N$ copies of $\mathcal{R}$. We write $R = (R_i)_{i \in N}$. An economy is a pair $(R, \Omega) \in \mathcal{R}^N \times \mathbb{R}_+$. Let $\mathcal{E}$ be the class of all economies.

A feasible allocation for $e = (R, \Omega) \in \mathcal{E}$ is a vector $x = (x_i)_{i \in N} \in \mathbb{R}_+^N$ such that $\sum x_i = \Omega$. Note that we do not assume free disposability of the commodity. Let $X(e)$ be the set of feasible allocations of $e$. A solution is a correspondence $\varphi: \mathcal{E} \rightarrow \mathbb{R}_+^N$ which associates with each $e \in \mathcal{E}$ a non-empty subset of $X(e)$. The intersection of two solutions $\varphi$ and $\varphi'$ is denoted by $\varphi \varphi'$. Each of the allocations in $\varphi(e)$ is interpreted as a recommendation for $e$. When a solution $\varphi(e)$ is single-valued and $\{x\} = \varphi(e)$, we slightly abuse notation and write $x = \varphi(e)$.

Motivations for the model can be found in Sprumont (1991) and Thomson (forthcoming): rationing in a two-good economy in which prices are in disequilibrium; allocating a task requiring a certain
amount of hours of work among a team of workers, paid an hourly wage and whose disutility of labor is a convex function of labor supplied; etc.

We introduce a solution, which embodies the standard notion of efficiency: an allocation \( x \) is efficient if there is no other feasible allocation such that all agents prefer to \( x \), and at least one agent strictly prefers to \( x \).

**Pareto solution, P:** For all \( e = (R, \Omega) \in \mathcal{E} \), \( x \in P(e) \) if \( x \in X(e) \) and there is no \( x' \in X(e) \) such that for all \( i \in N \), \( x'_i \geq x_i \), and for some \( i \in N \), \( x'_i > x_i \).

For economies with single-peaked preferences, the uniform rule (Benassy, 1982) is the best-known solution.

**Uniform rule, U:** For all \( e = (R, \Omega) \in \mathcal{E} \), \( x = U(e) \) if \( x \in X(e) \) and (i) when \( \Omega \leq \Sigma_N p(R_i) \), \( x_i = \min\{p(R_i), \lambda(e)\} \) for all \( i \in N \), where \( \lambda(e) \) solves \( \Sigma_N \min\{p(R_i), \lambda(e)\} = \Omega \), and (ii) when \( \Omega > \Sigma_N p(R_i) \), \( x_i = \max\{p(R_i), \lambda(e)\} \) for all \( i \in N \), where \( \lambda(e) \) solves \( \Sigma_N \max\{p(R_i), \lambda(e)\} = \Omega \).

3. No-envy and the uniform rule

We now discuss our first distributional requirement, no-envy: every agent should prefer what he receives to what any other agent receives.

**No-envy solution, F:** (Foley, 1967) For all \( e = (R, \Omega) \in \mathcal{E} \), \( x \in F(e) \) if \( x \in X(e) \) and for all \( i, j \in N \), \( x_i R x_j \).

As is well-known from Sprumont (1991) and Thomson (1994a,b), in the context of economies with single-peaked preferences, the set of envy-free allocations is fairly large, and it includes the allocation selected by the uniform rule. Here, we investigate the implications of strengthening no-envy to convex no-envy (Kolm, 1973): every agent should prefer what he receives to any convex combination of what all agents receive. Let \( \lambda \) be the \((|N| - 1)\)-dimensional simplex.

**Convex no-envy solution, C:** (Kolm, 1973) For all \( e = (R, \Omega) \in \mathcal{E} \), \( x \in C(e) \) if \( x \in X(e) \) and for all \( i \in N \) and all \( \lambda \in \Lambda \), \( x_i R \Sigma_N \lambda_j x_j \).

We ask whether there is a subsolution of the convex no-envy and Pareto solution. The uniform rule is such a subsolution. However, we have the following uniqueness result.

**Theorem 1.** The uniform rule is the only subsolution of the convex no-envy and Pareto solution.

**Proof.** It is easy to check that the uniform rule is a subsolution of the convex no-envy and Pareto solution. Now we show that it is the only subsolution. Let \( \varphi \subseteq CP \) and \( e = (R, \Omega) \in \mathcal{E} \). If \( \Omega = \Sigma_N p(R_i) \), then \( \varphi \subseteq P \) implies that, for all \( i \in N \), \( \varphi_i(e) = p(R_i) = U_i(e) \). We assume without loss of generality that \( \Omega < \Sigma_N p(R_i) \). Suppose, by contradiction, that \( \varphi(e) \neq U(e) \). Since \( \varphi \subseteq CP \), there exist \( x \in \varphi(e) \) and \( i, j \in N \) such that \( x_i < U_i(e) \leq U_j(e) < x_j \) and \( x_i < U_i(e) \leq p(R_i) < x_j \). Let \( x^* = \frac{1}{2}[x_i + U_i(e)] \). By construction, \( x^* \in [x_i, x_j] \), and \( x^* \) can be expressed as a convex combination of \( x_i \) and \( x_j \). However, \( x^* \notin \Omega \), in contradiction to \( \varphi \subseteq C \). □

**Remark 1.** As noted by Kolm (1973), on the domain of classical economies, the Walrasian solution from equal division is a subsolution of the convex no-envy and Pareto solution. In fact, if preferences are smooth, it can be shown that the Walrasian solution from equal division is the only subsolution of the convex no-envy and Pareto solution, a result that can be seen as a counterpart of our Theorem 1.
4. Equal division lower bound and the uniform rule

In this section, we start from our second distributional requirement, the equal division lower bound: every agent should prefer what he receives to equal division.

**Equal division lower bound solution, \( B_{ed} \):** For all \( e = (R, \Omega) \in \mathcal{E} \), \( x \in B_{ed}(e) \) if \( x \in X(e) \) and for all \( i \in N, x_i R_i \Omega / |N| \).

It is well-known that the set of allocations satisfying the equal division lower bound is fairly large, and once again it includes the allocation selected by the uniform rule.

We ask the following question: given an economy \( e = (R, \Omega) \in \mathcal{E} \) and a single-valued solution, what is the maximal number \( \beta \in [0,1] \) with the property that for all \( \beta \in [0,1] \) such that \( \beta \leq \beta^e \), each agent prefers what he receives to \( \beta \Omega \), and what is the minimal number \( \beta \in [0,1] \) with the property that for all \( \beta \in [0,1] \) such that \( \beta \geq \beta^e \), each agent prefers what he receives to \( \beta \Omega \). When there is too little of the commodity, agents want to have a large portion of the commodity, and prefer a solution with large \( \beta \). On the other hand, when there is too much of the commodity, agents want to have only a small portion of the commodity, and prefer a solution with small \( \beta \). We will be looking for solutions offering these maximal and minimal bounds.

Now we formally state these definitions. Let \( \varphi \) be a single-valued solution. For all \( e \in \mathcal{E} \), we define the upper bound associated with \( \varphi \), \( \beta^e \), by \( \beta^e = \max\{\beta \in [0,1] \} \) for all \( i \in N \) and all \( \beta \in [0,1] \) such that \( \beta \leq \beta^e \), \( \varphi(e) R_i \beta \Omega \) and the lower bound associated with \( \varphi \), \( \beta^e \), by \( \beta^e = \min\{\beta \in [0,1] \} \) for all \( i \in N \) and all \( \beta \in [0,1] \) such that \( \beta \geq \beta^e \), \( \varphi(e) R_i \beta \Omega \).

**Theorem 2.** When there is too little of the commodity, the uniform rule offers the greatest upper bound, that is, for all \( e \in \mathcal{E} \) and all single-valued solutions \( \varphi \), if \( \Omega \leq \Sigma_N p(R_i) \), then \( \beta^U(e) = \beta^e \). Also, when there is too much of the commodity, the uniform rule offers the least lower bound, that is, for all \( e \in \mathcal{E} \) and all single-valued solutions \( \varphi \), if \( \Omega \geq \Sigma_N p(R_i) \), then \( \beta^U(e) = \beta^e \).

**Proof.** Let \( e = (R, \Omega) \in \mathcal{E} \). If \( \Omega = \Sigma_N p(R_i) \), then for any \( \varphi \subseteq P, \beta^e (e) = 1 = \beta^U(e) \) and \( \beta^e (e) = 0 = \beta^U(e) \).

Now, we consider the case when \( \Omega < \Sigma_N p(R_i) \). From the definition of the uniform rule, for all \( \Omega \in N, U_i(e) = \min\{p(R_i), \lambda(e)\} \), so that \( \beta^U(e) = \max\{\lambda_i \mid 1 \leq i \leq \Omega\} \). If \( \varphi(e) \neq U_i(e) \), there is \( \Omega \in N \) such that \( \varphi(e) < U_i(e) \). Let \( \beta(e) = 1/2[\varphi(e) + U_i(e)] \). Since \( \beta(e) \Omega P_i \varphi(e), \beta^e < \beta^U(e) \), as desired.

The case when \( \Omega > \Sigma_N p(R_i) \) can be handled analogously by using the fact that \( \beta^U(e) = \max\{\lambda_i \mid 1 \leq i \leq \Omega\} \).

**Remark 2.** Another distribution requirement widely discussed in the literature is egalitarian-equivalence, introduced by Pazner and Schmeidler (1978). Given \( e \in \mathcal{E} \), an allocation \( x \in X(e) \) is egalitarian-equivalent for \( e \) if there exists a reference bundle \( r \) such that for all \( i \in N, x_i R_i r \). As shown by Thomson (1994a, footnote 4), the set of egalitarian-equivalent allocations is typically empty in the current context. However, we should note that the greatest upper bound applied to classical economies produces the subsolution of the egalitarian-equivalent solution with the reference bundle proportional to the social endowment. Therefore, the greatest upper bound can be regarded as a weakening of egalitarian-equivalence that is suitable for economies with single-peaked preferences.\(^1\)

\(^1\)This was pointed out to me by an associate editor.
5. Concluding remarks

Schummer and Thomson (1997) show that the uniform rule selects the unique allocation at which the difference between the largest and the smallest amounts received by any two agents is strictly smaller than at any other efficient allocation. Also, they show that it selects the unique allocation at which the variance of the amounts awarded to all the agents is strictly smaller than at any other efficient allocation. On the other hand, de Frutos and Massó (1995) show that the uniform rule selects the only minimizer of the Lorenz ordering over the efficient allocations. Our results, combined with these and other equity properties of the uniform rule, reinforce the distributional merits of this rule in economies with single-peaked preferences.

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