A Beveridge–Nelson smoother

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Abstract

This note defines a Beveridge–Nelson smoother, that is a two-sided signal extraction filter for trends. The smoother is shown to be the optimal estimator of the trend when the ARIMA model can be decomposed into an uncorrelated random walk trend and stationary cycle components. The conditions under which such a decomposition is possible are discussed. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Beveridge and Nelson (1981) showed that any ARIMA\((p,1,q)\) model could be decomposed into a random walk, the trend, and a stationary component, the cycle, driven by the same disturbance. The trend is constructed as a weighted linear combination of current and past observations. This operation is known as the Beveridge–Nelson (BN) filter. This note suggests how to define a corresponding Beveridge–Nelson smoother, that is a trend based on a two-sided weighted average. This smoother is shown to be the optimal estimator of the trend when the ARIMA model can be decomposed into uncorrelated trend and cycle components. The conditions under which such a decomposition is possible are explored.

2. The Beveridge–Nelson decomposition

Consider the ARIMA\((p,1,q)\) model:

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\[ \phi(L) \Delta y_t = \theta(L) \xi_t, \quad \xi_t \sim \text{WN}(0, \sigma^2), \quad t = 2, \ldots, T \]  

where WN(0, \sigma^2) denotes serially uncorrelated disturbances — white noise — with mean zero and variance \( \sigma^2 \), and \( \theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q \) and \( \phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p \) are polynomials in the lag operator, \( L \). The Wold representation is:

\[ \Delta y_t = \psi(L) \xi_t, \quad \sum |\psi_j| < \infty \]

such that \( \psi(L) = \theta(L)/\phi(L) \). The quantity \( \psi(1) = \theta(1)/\phi(1) \) is a measure of persistence.

The BN decomposition allows the observations to be expressed as the sum of trend and cyclical components, that is:

\[ y_t = m_{t, \text{BN}} + c_{t, \text{BN}}, \quad t = 1, \ldots, T \]  

The BN trend, \( m_{t, \text{BN}} \), is defined as the long run prediction of the series (eventual forecast function) at time \( t \), that is \( \lim_{t \to \infty} \tilde{y}_{t+|l|} \). It is formed as:

\[ m_{t, \text{BN}} = w_{\text{BN}}(L) y_t \]  

where the weights in the filter are given by:

\[ w_{\text{BN}}(L) = \psi(1)/\psi(L) = \frac{\theta(1)}{\phi(1)} \frac{\phi(L)}{\psi(1)} \]  

It is easy to see that the weights sum to one, since \( w_{\text{BN}}(1) = 1 \). The Markovian representation of the trend is:

\[ m_{t, \text{BN}} = m_{t-1, \text{BN}} + \psi(1) \xi_t \]  

The representation of the cyclical component, \( c_{t, \text{BN}} \), is:

\[ c_{t, \text{BN}} = y_t - m_{t, \text{BN}} = \frac{\phi(1) \theta(L) - \theta(1) \phi(L)}{\phi(1) \theta(L)} y_t = \frac{\phi(1) \theta(L) - \theta(1) \phi(L)}{\phi(1) \phi(L) \Delta} \xi_t \]

The weights for the extraction of the cycle sum to zero. Since \( \phi(1) \theta(L) - \theta(1) \phi(L) \) must have a unit root, we can write:

\[ \phi(1) \theta(L) - \theta(1) \phi(L) = \Delta \theta(L) \]

and the ARMA representation for this component can be established as:

\[ \phi(L) c_{t, \text{BN}} = \{ \theta(L)/\phi(1) \} \xi_t \]  

Thus \( c_{t, \text{BN}} \sim \text{ARMA}(p, \max(p,q) - 1) \).
3. A Beveridge–Nelson smoother

The proposed BN smoother, \( m_{t|T}^{\text{BNS}} = w^{\text{BNS}}(L)y_t \), is a symmetric two-sided filter which weights the observations according to:

\[
w^{\text{BNS}}(L) = w^{\text{BN}}(L)w^{\text{BN}}(L^{-1}) = \left[ \frac{\theta(1)}{\phi(1)} \right]^2 \frac{\phi(L)\phi(L^{-1})}{\theta(L)\theta(L^{-1})}
\]

(7)

It can be immediately seen that the weights attributed to the observations sum up to unity, that is \( w^{\text{BNS}}(1) = 1 \). Like the BN filter, the BN smoother can always be computed.

The rationalisation for the above expression is given in the next section. It might be thought that a better approach is to regard the BN decomposition (2) as a model and to form the optimal smoother by applying the Wiener–Kolmogorov formula. However, as pointed out in Harvey and Koopman (1999), this just yields the BN filter. (As a general rule, correlated components yield asymmetric signal extraction filters.)

4. Orthogonal decomposition

Suppose an ARIMA(\( p,1,q \)) model (1) can be decomposed into mutually uncorrelated random walk and stationary components, that is:

\[
y_t = \mu_t + c_t
\]

(8)

with:

\[
\Delta \mu_t = \eta_t, \quad c_t = \beta(L)\kappa_t
\]

such that \( \beta(L) = \sum_{j=0}^{\infty} \beta_j L^j \), with \( \sum |\beta_j| < \infty \), and:

\[
\begin{pmatrix} \eta_t \\ \kappa_t \end{pmatrix} \sim \text{WN} \left[ \begin{pmatrix} 0 \\ \sigma^2 \phi \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \phi \end{pmatrix} \right]
\]

Lippi and Reichlin (1992) show that if there is a decomposition into uncorrelated components, then the model cannot be persistent, that is \( \phi(1) \leq 1 \). They go on to show that an orthogonal decomposition is only possible if the spectral generating function is a minimum at the zero frequency.

The trend disturbance variance in (8) is equal to the reduced form disturbance variance times squared persistence, that is:

\[
\sigma^2_{\eta} = \left[ \phi(1)/\theta(1) \right]^2 \sigma^2
\]

(9)

This feature is common to any unobserved components model consisting of a random walk trend and a stationary component. The result is easily shown by equating the spectra of \( \Delta y_t \) and \( \eta_t + \Delta \beta(L)\kappa_t \) and evaluating them at zero frequency. We may therefore write the trend as:
\[ \mu_t = \mu_{t-1} + \{\theta(1)/\phi(1)\} \tilde{\eta}_t = \mu_{t-1} + \psi(1) \tilde{\eta}_t \]

with \( \tilde{\eta}_t \sim \text{WN}(0, \sigma^2) \).

Although (5) and (10) are random walks driven by disturbances with the same variance they differ in that the BN trend is a function of current and past observations whereas (10) is an unobserved component. However, the way in which the observations are weighted in order to extract this unobserved trend component can be found using the Wiener–Kolmogorov filter extended to deal with non-stationary models; see, for example Bell (1984) and Burridge and Wallis (1988). The minimum mean square linear estimator of \( \mu_t \), \( m_{t|T} \), in (18) is obtained as:

\[
m_{t|T} = \sum_{j=-\infty}^{\infty} w_{-j} y_{t-j} = w(L)y_t, \quad w(L) = g_\mu(L)/g_y(L)
\]

where \( g_\mu(L) \) and \( g_y(L) \) are the autocovariance generating functions (ACGFs) of \( \mu_t \) and \( y_t \). Evaluating this expression shows that \( w(L) \) is the BN smoother, \( w_{\text{BNS}}(L) \), of (7).

Now, given an admissible decomposition, the filtered estimator of \( \mu_t \) in (8) is the same as the BN filter. This must be the case because it is equal to the long-run forecast. Note that it is not true, in general, that the filter and the smoother are related as in (7).

As regards the cycle in the BN decomposition (6) its AR coefficients are exactly as in \( c_t^{\text{BN}} \) (and in the ARIMA model) and the order of the MA is the same. This can be shown by first equating the ACGF of \( \Delta y_t \) to that of \( \eta_t + \Delta \beta(L) \kappa_t \) to give:

\[
\sigma_\eta^2 + |1 - L|^2 |\beta(L)|^2 \sigma_\kappa^2 = \{(\theta(L))^2/|\phi(L)|^2\} \sigma^2
\]

where, for example, \( |\theta(L)|^2 \) denotes \( \theta(L)\theta(L^{-1}) \). Substituting from (9) gives:

\[
|1 - L|^2 |\beta(L)|^2 \sigma_\kappa^2 = \alpha^2 \left[ \frac{|\theta(L)|^2}{|\phi(L)|^2} - \frac{\theta(1)^2}{\phi(1)^2} \right] = \phi(1)^2 \theta(L)^2 - \theta(1)^2 |\phi(L)|^2 |\phi(L)|^2 \sigma^2
\]

This shows that the cycle has the ARMA\((p,\max(p,q) - 1)\) representation:

\[
\phi(L)c_t = \{1/\phi(1)\alpha(L)\tilde{\kappa}_t
\]

where \( \alpha(L) \) is an MA polynomial of order \( \max(p,q) - 1 \) such that:

\[
|1 - L|^2 |\alpha(L)|^2 = \phi(1)^2 |\theta(L)|^2 - \theta(1)^2 |\phi(L)|^2
\]

and \( \tilde{\kappa}_t \sim \text{WN}(0, \sigma^2) \). Although the order of the MA polynomial, \( \alpha(L) \), is the same as in the BN cycle, the coefficients will differ from those of \( \beta(L) \). The requirement that \( c_t \) is uncorrelated with the trend component results in restrictions to the parameter space being embedded in (14).
5. Illustrations

The following models illustrate and expand on the points made in the previous sections.

5.1. ARIMA(0,1,1) model

The BN trend in the ARIMA(0,1,1) model:

$$\Delta y_t = (1 + \theta L) \xi_t, \quad \xi_t \sim WN(0, \sigma^2)$$

is

$$m_t^{BN} = w^{BN}(L)y_t = \frac{1 + \theta}{1 - \theta} \sum_{j=0}^{\infty} (-\theta)^j y_{t-j} = (1 + \theta) \sum_{j=0}^{\infty} (-\theta)^j y_{t-j}$$

The corresponding BN smoother for trend extraction is:

$$m_t^{BNS} = (1 + \theta)^2 \frac{1}{1 - \theta} y_t = \frac{1 + \theta}{1 - \theta} \sum_{j=0}^{\infty} (-\theta)^j y_{t+j}$$

The orthogonal decomposition (8) has:

$$\Delta \mu_t = (1 + \theta) \hat{\eta}_t, \quad c_t = \alpha_t \hat{\kappa}_t$$

where $\hat{\eta}_t$ and $\hat{\kappa}_t$ are mutually uncorrelated WN processes with common variance $\sigma^2$, and $\sigma^2 = -\theta$ from (14). Hence, the decomposition is admissible only if $\theta \leq 0$. This amounts to requiring that persistence be no greater than unity. If $\theta > 0$, the BN filter and smoother have alternating negative and positive weights.

5.2. ARIMA(1,1,0)

In the model

$$(1 - \phi L) \Delta y_t = \xi_t, \quad \xi_t \sim WN(0, \sigma^2)$$

the BN filter gives:

$$m_t^{BN} = w^{BN}(L)y_t = \frac{1 - \phi L}{1 - \phi} y_t = (y_t - \phi y_{t-1})/(1 - \phi)$$

The BN smoother is:

$$m_t^{BNS} = \frac{\phi}{(1 - \phi)^2} y_{t-1} + \frac{1 + \phi^2}{(1 - \phi)^2} y_t - \frac{\phi}{(1 - \phi)^2} y_{t+1}$$

The process admits the decomposition into a RW trend plus a stationary AR(1) process:
where $\tilde{\eta}$ and $\tilde{k}$ are mutually uncorrelated WN processes with common variance $\sigma^2$ and $\alpha_0^2 = -\phi$. Thus, the decomposition is admissible only if $\phi \leq 0$. This again amounts to requiring that persistence, $(1 - \phi)^{-1}$, be no greater than unity. Note that with $\phi$ negative all the weights in the smoother are positive.

The smoother in a UC model is not normally truncated since the reduced form has $q > 0$ except in very special cases. If an orthogonal decomposition exists for an ARIMA($p$,1,0) model, the trend signal extraction weights will be truncated at $-p$ and $p$, so that the smoother is a weighted moving average of $2p + 1$ consecutive observations.

5.3. ARIMA(2,1,0)

For mixed and higher order processes the constraints on the AR and MA coefficients can be more restrictive than a simple requirement that persistence be no greater than one. This is illustrated by the ARIMA(2,1,0) model:

$$(1 - \phi_1 L - \phi_2 L^2) \Delta y_t = \xi_t, \quad \xi_t \sim WN(0,\sigma^2)$$

The cycle has the ARMA(2,1) representation:

$$(1 - \phi_1 L - \phi_2 L^2) c_t = \alpha(L) \tilde{k}_t$$

with the coefficients of $\alpha(L) = \alpha_0 + \alpha_1 L$, obtained from (14), satisfying:

$$\begin{align*}
\alpha_0^2 + \alpha_1^2 &= -[\phi_1(1 - \phi_2) + 2\phi_2] \\
\alpha_0\alpha_1 &= -\phi_2
\end{align*}$$

Substituting $\alpha_1^2 = \phi_1^2/\alpha_0^2$ into the first equation gives a quadratic equation for $\alpha_0^2$ with solutions:

$$\alpha_0^2 = -\frac{1}{2} [\phi_1(1 - \phi_2) + 2\phi_2] \pm \frac{1}{2} \sqrt{\phi_1(1 - \phi_2)(\phi_1(1 - \phi_2) + 4\phi_2)}$$

that are real if and only if:

$$\phi_1(1 - \phi_2)[\phi_1(1 - \phi_2) + 4\phi_2] > 0$$

This occurs when:

$$[\phi_1(1 - \phi_2) + 4\phi_2] < 0, \quad \phi_1(1 - \phi_2) < 0$$

since $[\phi_1(1 - \phi_2) + 4\phi_2] > 0$ and $\phi_1(1 - \phi_2) > 0$ implies $[\phi_1(1 - \phi_2) + 2\phi_2] > 0$, which is not admissible in the light of (16). The conditions (18) are exactly those needed to ensure that the spectral generating function of $\Delta y_t$, denoted $g_\Delta(\lambda)$, has a global minimum at the zero frequency. In fact, $[\phi_1(1 - \phi_2) + 4\phi_2] < 0$ is the second-order condition for a minimum at frequency zero, whereas $\phi_1(1 - \phi_2) < 0$ is required to ensure that $g_\Delta(0)$ is actually a global minimum, that is $g_\Delta(0) < g_\Delta(\pi)$, or equivalently, $\phi(1)^2 > \phi(-1)^2$. 

$$\Delta \mu_t = \frac{1}{1 - \phi} \tilde{\eta}_t, \quad c_t = \frac{\alpha_0}{1 - \phi} \tilde{k}_t (1 - \phi L)$$
Fig. 1. Region of the AR parameter space \((\phi_1, \phi_2)\) in which the orthogonal decomposition of the ARIMA(2,1,0) model is admissible (shaded area). The dotted line, \(\phi(1) = 1\), collects all combinations giving unit persistence; non-persistent representations lie below the line. The values such that \(\phi_1(1 - \phi_2) + 4\phi_2 < 0\) lie below the curve \(\phi_2 = -\phi_1/(4 - \phi_1)\).

We also rule out the case \(\phi_1(1 - \phi_2)[\phi_1(1 - \phi_2) + 4\phi_2] = 0\), which corresponds to \(\alpha_0^2 = \alpha_1^2\) and thus to a non-invertible MA polynomial \(a_0 + a_1 L\).

It finally remains to establish which of the roots (17) should be chosen. Invertibility of \(\alpha(L)\) requires \(|\alpha_0| > |\alpha_1|\) which implies \(\alpha_0^2 - \alpha_1^2 > 0\). The only solution is therefore:

\[
\alpha_0^2 = -\frac{1}{2}[\phi_1(1 - \phi_2) + 2\phi_2] + \frac{1}{2}\sqrt{[\phi_1(1 - \phi_2)[\phi_1(1 - \phi_2) + 4\phi_2]}
\]

This example illustrates that persistence less than, or equal to, one is a necessary but not a sufficient condition: the region of the AR parameter space identified by the inequalities in (18) is displayed in Fig. 1 (shaded area) and does not include all combinations for which \(\phi_1 + \phi_2 \leq 0\).

6. Conclusion

We have proposed a two-sided signal extraction filter which corresponds to the BN filter. When the model for the observations can be decomposed into uncorrelated trend and cycle components, the smoothed estimator of the trend is the same as the proposed BN smoother. Persistence less than one is a necessary but not a sufficient condition for an admissible decomposition. If there is no admissible decomposition the appeal of the BN smoother may be lessened, though the same could also be said of the BN filter itself. Indeed in such cases the BN trend may be as variable as the series itself (and sometimes more variable); this is illustrated by the trend arising from the ARIMA(1,1,0) model fitted to US GNP by Watson (1986).
References