A note on distribution in a vote bidding game with general interest and single issue voters

Alex T. Coram*

Department of Infrastructure, 80 Collins Street, Nauru House, Level 23, Melbourne 3000, Australia

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Abstract

This paper explores the strategies of candidates bidding for votes from a population made up of single issue voters and voters who respond in a probabilistic manner. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Social choice theorists have given considerable attention to competition for votes in models in which voters are assumed to be either deterministic or probabilistic. One gap in this literature is that little attention has been paid to the case where there is a mixture of voter types. Apart from some work by Congleton (1991) and Grossman and Helpman (1996), there is not much analysis of candidate strategies in an electorate made up of moderate voters and the type of special interests that might act as single issue voters. Examples of such single issue voters would be farmers, environmentalists, owners and workers in some specific industry such as motor-car manufacture, the elderly, ethnic and nationalistic groups. Congleton suggests that single issue voters might be more important than the literature indicates. Grossman and Helpman (1996, 265) argue that the existence of single issue voters may help explain why the political process does not seem to serve the interests of the median voter. The simplest approach to analysing the impact of single issue voters would be to consider the electoral process as a vote buying competition. In this case, it might be expected that blocks of votes would attract vote buying political parties. Bidding for special interests by offering

E-mail address: alex.coram@doi.vic.gov.au (A.T. Coram)
policies on such issues as expenditure on defence or health subsidies to producers of primary goods seems to be a common aspect of candidate behaviour. What is not clear, however, is the offers it would be best for such parties to make.

This raises the following questions. Do special interest voters get more than a proportional share of the available resources in this type of bidding game? What will happen as the proportion of single issue voters increases? Will the resources available to these voters increase at the same rate?

Mueller (1983, 264) argues, for example, that there is distinction between outcomes in countries with and without strong interest groups. This seems to imply some sort of discontinuity. Is this accounted for by vote bidding, or some other aspect of the political process?

The purpose of this note is to consider a simple model of a multi-stage vote bidding game with a mixture of voter types in order to throw light on these questions. To maximise the transparency of the analysis, the problem to be analysed is set up within the framework of a stripped down Hotelling model.

The model analysed here differs from Grossman and Helpman’s model of interest group payments to influence voter’s opinions and to change the policies offered by parties in that special interests are treated as passive. On the other hand it has more complicated equilibrium properties because it is assumed that what counts for parties is winning. Hence the payoffs switch between all and nothing.

I concentrate on changes in party strategies as the proportion of single issue voters changes. I also analyse the shares of the resources that single issue voters obtain. One interesting result is that, when single issue voters are half, or more than half, the total they get is all the resource.

2. A model of candidate competition in a pure distribution game

Suppose there is a single electorate in which there are two candidates, and a simple majority wins. Some voters vote along single interest lines while others are concerned with a range of public good and general interest issues and behave in a probabilistic manner (Mueller, 1991; 348–369; Carter and Guerette, 1992). Such voters are called moderates. The division of voters in this way captures and adds to some of the theory of voting in Brennan and Lomasky (1997). They argue that voters tend to vote in an expressive fashion. It would be expected, however that groups with an overriding single issue wish to express their support for that issue and to see value in their votes being counted. Voters that are not in this category might be expected to decrease their votes in some manner as the relative support offered by a party declines.

The candidates each have a fixed pool of resources available and bid for votes by offering a distribution of these resources. The fixed pool means that the game is a pure distributional game (Coughlin, 1992, 25). It has often been argued that most elections allocating private goods are precisely elections entailing redistributial issues (Aranson and Ordeshook, 1981). Offers are simultaneous. This represents the notion that candidates have to develop policy positions and cannot change them instantaneously.

The only strategic choice that is of concern is the amount that candidates distribute to the different type of voters. No attempt will be made to analyse the particular coalition of single issue voters that is bought. Cycling problems have been extensively explored elsewhere and are ignored in this model (Ordeshook, 1986; Mueller, 1991). It is assumed that the candidate that makes the greater distribution is able to buy the votes of single issue voters.
These assumptions can be summed up and the model presented as follows:

2.1. The model

The strategy sets for both parties are $S^1 = S^2 = [0, 1]$. A strategy for party one is a distribution to single issue voters $s_1 \in S^1$ and a distribution to the moderates $1 - s_1$. The strategies of party two are $s_2 \in S^2$ and $(1 - s_2)$. A probability distribution of strategies is written $(\sigma_1 s_{11}, \sigma_{12} s_{12}, \ldots)$.

Assume that parties are only concerned with winning. It is also possible that parties are concerned with maximizing their vote. Winning gives a payoff of one unit. $\varphi_1, \varphi_2$ are the payoff functions for parties 1 and 2.

Let $p$ be the probability of getting the votes of moderates and $k$ of getting single issue voters. $p$ and $k$ are density functions. The proportion of single issue voters is $\alpha$ and of moderate voters is $(1 - \alpha)$. This gives

$$
\varphi_i = \varphi_i[p_i(1 - \alpha) + \alpha k_i]
$$

Letting $f_i = p_i(1 - \alpha) + \alpha k_i$

$$
\varphi_i = \begin{cases} 
1 & \text{if } f_1 > f_2 \\
1/2 & \text{if } f_1 = f_2 \\
0 & \text{otherwise.}
\end{cases}
$$

2.2. Assumptions about voters

Moderate voters increase the probability of voting for party $i$ as the amount of resources offered by $i$ increases and decrease the probability of voting for $i$ as the amount of resources offered decreases. Assume that $p_i(s_1, s_2)$ is continuous and differentiable.

$$
\frac{\partial p_i}{\partial s_i} < 0, \frac{\partial p_i}{\partial s_j} > 0, i \neq j.
$$

To make moderate votes worthwhile $p_i(0, 1) = p_2(1, 0) = 1$. In addition the following assumption will be used.

A1. Symmetry. A1 says that moderate voters respond to nothing but the offers made by the parties across the entire range of offers. That is a unit of goodies from party one is always treated in the same way as a unit of goodies from party two. Hence $p_i(s_1, s_2) = p_2(s_2, s_1)$.

Single interest voters support the party that offers them the greatest payoff. Let $k_1$ be the probability of a vote for party 1. Then, $k_1 = 1$ if $s_1 > s_2$, $1/2$ if $s_1 = s_2$, $0$ if $s_1 < s_2$.

Some intuition about this game is given in Fig. 1 for the case where special interest voters are less than half. This sets out the relation between the number of votes and a choice of $s_1 = x \in [0, 1]$ for some given value of $s_2 = y$.

What the figure shows is that $s_1 < y$ gives more moderate votes to party one and the single issue votes to party two. $s_1 = y$ gives $f_1 = f_2 = 1/2$. $s_1 = y + \varepsilon$ for some $\varepsilon$ sufficiently small gives $f_1 = p_1(1 - \alpha) + \alpha > 1/2$. 
Observe that for \( \alpha < 1/2 \) the game is discontinuous at every point \( s_1 = s_2 \).

### 3. Analysis

The game is analysed by taking the cases where the proportion of single issue voters is less than one half and where the proportion is equal to, or more than, one half separately.

#### 3.1. Single issue voters less than a majority: \( 0 < \alpha < 1/2 \)

The analysis of this case shows that there is an equilibrium in mixed strategies. The strategies depend on \( \alpha \) and on whether the parties wish to maximize or minimize the expected returns to single issue voters.

**Proposition 1.** The game has an equilibrium in mixed strategies.

**Proof.** Appendix A.

Proposition 1 tells us that there is an equilibrium in any game with the given structure, but it does not tell us the characteristics of the equilibria. For example, it would be easy to generate cases where the game always gives a higher value for one party. There might be a pure strategy or atomic distributions or a distribution over an interval. To get some information on this we need to give \( p \) some additional structure.

Suppose the game is symmetrical. A1 is sufficient to allow the equilibria to be specified. This is done in corollary one. It is also possible to specify some equilibria under weaker conditions, but these are not as interesting.

**Corollary 1.** The equilibria under condition A1 are given by a probability distribution \((1/2, 1/2)\) over intervals on either side of \( \min\{a^0, b^1\} \) and \( \min\{b^0, a^1\} \) and are symmetrical. These are
Proof. Appendix A.

Fig. 2 illustrates the strategy sets for some specific value of \( 0 < \alpha < 1/2 \) and specified function \( p \) under condition A1. Party one with strategy \( s_1 \) defeats party two in the shaded area.

It is interesting to compare this result with the case where parties wish to maximize their vote. It can be shown that, in this case, the optimum strategy for each party is to pick a probability density function \( h = c \) where \( c \) is constant across the continuous interval \([0,1]\) rather than on discontinuous intervals.

Some political scientists have rejected mixed strategies as a solution to vote bidding games on the grounds that it would not be realistic to expect candidates to employ a randomising device to choose policies (Ordeshook, 1986, 182). This solution is supported here for the following reason. Assume that the game is played repeatedly. What a mixed strategy solution tells us is that there are a number of equilibrium points and that the parties should visit them with the probability indicated. It does not say they employ a randomising device. It only says they would act as if they employed such a device. If the idea that parties shift their policies to buy votes is unacceptable, then the implication must be that competition between parties does not have the form of vote bidding.

3.2. Single issue voters greater than or equal to 1/2 the population.

The result here is that both parties give all the goodies to single issue voters when the proportion of these voters is equal to, or greater than, half the total. This result does not require symmetry.

**Proposition 2.** For \( \alpha \geq 1/2 \), \( s_1 = s_2 = 1 \).

**Proof.** Immediate for \( \alpha \geq 1/2 \) from \( f_1[s_1 = 1, s_2 < 1] \geq 1/2 \).

To confirm that \( s_2 = 0 \) is not a best response when \( \alpha = \frac{1}{2} \) observe that 0 is an element of a discontinuity set. Hence \( s_2 = 0 \) violates Theorem 6 of Dasgupta and Maskin (1986) discussed in the remark to corollary two.

3.3. Distribution to single issue voters with symmetry

The parties may wish to choose the distribution that minimizes or maximizes the expected payoffs to the single issue voters. This introduces a constraint on the objective function and alters the structure of the game. The nature of the payoff to the single issue voters under these choices depends on the response function of the moderate voters.

3.4. The programme that minimizes expected payoffs to single issue voters

The minimum programme is straightforward. In this case each party chooses the strategy \( s^*_1 = (1/2.0, 1/2.a^0) \), \( s^*_2 = (1/2.0, 1/2.b^0) \). Let \( v \) be the value of the game to single issue voters where
this is given as a proportion of the goodies available to distribute. Then it is easy to show that the minimum expected value given strategies $s = \min v$ is

$$E[v|s] = g(\alpha)$$

where $g$ is continuous and monotonically increasing in $\alpha$. $g(0) = 0$ and $g \to 1/2$ as $\alpha \to 1/2$ from below.

The first assertion is established by considering $a^0 = \gamma^{-1}(0) = x$. Note that either of the pure strategies that support the mixed strategy will give the equilibrium payoff against a mixed strategy. Thus, in equilibrium

$$f_i = (1 - \alpha)p_i(x, 0) + \alpha k = 1/2.$$

To maintain equilibrium it must be the case that $df_i/d\alpha = 0$, evaluated at $x = a^0$. Hence

$$-p_i + (1 - \alpha)(\partial p/\partial x)(dx/d\alpha) + k + \alpha k(\partial k/\partial x)/(dx/d\alpha) = 0$$

Since $k = 1$ for $x > 0$, $(\partial k/\partial x) = 0$,

$$dx/d\alpha = (p_i - 1)/(1 - \alpha)(\partial p_i/\partial x)$$

$p$ is continuous and $\partial p/\partial x < 0$. Hence $dx/d\alpha > 0$ and continuous. Similarly for $y = b^0$, $dy/d\alpha > 0$.

To get more information about $g$ take the second derivative of $dx/d\alpha$.

$$d^2x/d\alpha^2 = (\partial p_i/\partial x)^2(dx/d\alpha)(1 - \alpha) - (p_i - 1)[(1 - \alpha)(\partial^2 p/\partial x^2)(dx/d\alpha) - \partial p/\partial x]/[(1 - \alpha)(\partial p/\partial x)]^2$$

The right hand side can be signed except for the term $\partial^2 p/\partial x^2$. If this is positive then $d^2x/d\alpha^2$ is positive. Otherwise it is indeterminate.

To see the second assertion observe that $a^0 = \gamma^{-1}(0) \to 1$ as $\alpha \to 1/2$. Hence $E[v|s]$ approaches $1/2.0 + 1/2.1 = 1/2$.

3.5. The programme that maximizes expected payoffs to single issue voters

The maximum programme is slightly more complicated. In this case each party must choose $s^*_1 = (1/2.m_1, 1/2.x_2 \max)$, $s^*_2 = (1/2.m_2, 1/2.y_2 \max)$ where $m_1 = \min(a^0 - \epsilon, b^1 - \epsilon)$ and $m_2 = \min(b^0 - \epsilon, a^1 - \epsilon)$, and $x_2 \max$ and $y_2 \max$ are given in the proof of corollary 2.

It must be the case that $g = 0$ for $\alpha = 0$ for any programme. Note that as $\alpha \to 1/2$, $a^1 \to 0$. Hence the maximizing programme must use strategies $s^*_1 \to 1/2.0 + 1/2.x_2 \max$ where $x_2 \max > b^0 \to 1$. Hence $g \to 1/2$.

A situation where $g > 1/2$ for $\alpha < 1/2$ is possible under the maximization programme. This has the interesting consequence that, at some point, single issue voters must begin to do worse on average since $g \to 1/2$ for $\alpha \to 1/2$ under all programmes.

What is required for $g > 1/2$ for $\alpha < 1/2$ is that $a^0 = x$ and $1/2 < x < b^1$ or $b^0 = y$ and $1/2 < y < a^1$. In this case an optimum for $s_1$ will be $s^*_1 = 1/2.(a^0 - \epsilon) + 1/2.x_2 \max$, where $x_2 \max > a^0$. For some $\epsilon$, $E(v|s^*_1) > 1/2$. Similarly for $s^*_2$, $E(v|s^*_2) > 1/2$. Hence $g > 1/2$.
This condition can be analysed in terms of the probability functions. Consider $p_i(y)$ for ease of reference in Fig. 2. It is simple to show that a sufficient condition for $y > 1/2$ for $b^o = a^l$ is that $p_i(y)$ is convex. Other conditions may also give the same result. This is demonstrated in Appendix B.

I have illustrated some of these possibilities in Fig. 3. (i) is the path under the minimization programme with $\frac{\partial^2 p}{\partial x^2} > 0$. (iii) is the path with $g > 1/2$ under the maximization programme.

4. Conclusion

The analysis shows that, in a model with single issue and probabilistic voters, parties will distribute all resources to single issue voters when the proportion of these voters is equal to or greater than a
half. This result holds without symmetry. Where the proportion of single issue voters is less than one half there is an equilibrium in mixed strategies. In support of Mueller’s speculation, noted in the introduction, there is a jump at the point where single issue voters are half the population.

An interesting feature of the analysis for the case where single issue voters are less than half is that the distribution partly depends on the intentions of the parties. Both parties can either encourage single issue voters by giving them high expected returns, or discourage single issue voters by giving them a disproportionately low expected return. It is possible, for example, that single issue voters can be held to less than a proportionate share of the resource.

It was shown that parties that are maximizing expected returns to single issue voters may be forced to decrease these returns as the proportion of single issue voters approaches half the total. This is somewhat paradoxical.

It was also noted that more generally, according to the Dasgupta-Maskin theorem, the equilibrium for this type of game is always in mixed strategies. For those who argue that mixed strategies are not appropriate in party competition, it is not clear what this result tells us. Since heterogeneity of voters seems sensible, the only conclusion can be that parties do not bid for votes. If so, we need some other specification of party behaviour.

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Appendix A. Proof of Proposition 1

The proof applies the following theorem (Dasgupta and Maskin, 1986). Let $S$ be a closed interval of $R$. Suppose that $\varphi_i$ is continuous except on a subset of $S^{**}$ of $S^*$, that $\Sigma \varphi_i$ is upper semi-continuous and $\varphi_i$ is bounded and weakly lower semi-continuous in $s_i$. Then the game has a mixed strategy equilibrium. (i) $S_j \subseteq [0, 1]$ is a closed interval on $R$. (ii) $\Sigma \varphi_i = 1$ and hence continuous. (iii). Weak lower semi-continuity in $\varphi_i$ requires that there exists $\mu \in [0, 1]$ such that $\mu(\lim\inf \varphi_i(s_i, s_2)) + (1 - \mu)(\lim\inf \varphi_i(s_i', s_2)) \geq \varphi_i(s_i, s_2)$ where the first limit is taken as $s_i'$ approaches $s_i$ from below and the second as $s_i'$ approaches $s_i$ from above. Set $\mu = 0$ or 1 as required and the condition is met.

Definition.

$$\omega(s_i) = s_i \quad p_1(1 - \alpha) = p_2(1 - \alpha) + \alpha$$
$$\gamma(s_i) = s_i' \quad p_2(1 - \alpha) = p_1(1 - \alpha) + \alpha$$

Corollary 1. The proof illustrates the following result (Dasgupta and Maskin, 1986, lemma 7). Let the game satisfy condition A1. Then the equilibrium mixed strategies are symmetrical.

The Nash equilibria are $s_1^* = \{1/2, x_1 \in [0, \min a^0, b^1), 1/2, x_2\; y \in \gamma^{-1}(y = x_1)\}$, $s_2^* = \{y_1 \in [1/2[0, \min b^-], a^1), 1/2, y_2; \omega^{-1}(x = y_2)\}$. The proof is easy but tedious.

(i) Symmetry is obvious. (ii) Sufficiency. Suppose $s_1^* \neq s_2^*$. Then $x_1 \in [0, \min a^0, b^1)$, defeats one
of \(y_1\), \(y_2\), and loses to the strategy it does not defeat. If it is defeated by \(y_1\) for example it must intercept \(y_1\) in a region \(f_2 > f_1\). Hence \(x_1 < y_1 < \omega(x_1) < \omega(x = y_1) = y_2\). From the last inequality \(x_1\) defeats \(y_2\). Similarly \(y_1 < a^1 = b^1 < x_2 = \gamma^{-1}(y = x_1) < \gamma^{-1}(y_1) = x' = y_2 = \omega(x = y_1)\) from A1 (think of the symmetry around the diagonal). Hence \(x_2\) defeats \(y_1\) from \(y_1 < x_2 = \gamma^{-1}(y = x_1) < \gamma^{-1}(y_1)\). Since \(b^1 < x_2 < y_2\), \(y_2\) defeats \(x_2\). If \(s^*_1 = s^*_2\) the strategies draw. (iii). Necessity. Suppose \(s^*_2 \neq [y_1 \in 1/2[0, \min b^0, a^1)]\), \(1/2, y_2: \omega(x = y_1)]\). Then there exists an \(x\) such that \(f_2(x, s_2) = 0\) from A1. In equilibrium \(f_1 = f_2\) and \(v_1 = v_2 = 1/2\). Let \(s_2\) have the same \(y\) values as above and there be some probability mix \(\sigma \neq (1/2, 1/2)\) which gives \(E[s_2^*|\sigma(s_2)] > 1/2\). But \(E[s_1^*|\sigma(s_2)] = \sigma 1/2 + (1 - \sigma) 1/2 = 1/2\). Contradiction.

**Remark.** The corollary gives a distribution on discontinuous intervals. It is also possible to pick a Nash equilibrium that consists of two points that satisfy the condition required for these intervals. Theorem 6 (Dasgupta and Maskin) says that equilibria are also atomless on the discontinuity sets for symmetric games. The existence of an equilibrium that puts positive weight on two points does not violate this theorem. The reason is that the theorem requires the following condition. For any \(s_1 = x\) and \(x\) a point of discontinuity \(\mu(\lim \inf \varphi_i(s_1', s_2)) + (1 - \mu)(\lim \inf \varphi_i(s_1', s_2)) > \varphi_i(s_1, s_2)\) where the first limit is taken as \(s_1'\) approaches \(s_1\) from below and the second as \(s_1'\) approaches \(s_1\) from above. Since the equilibria is determined across two intervals in the game the strict inequality does not hold. This is because the gains for deviating from \(s_1\) in one interval in the equilibrium set are lost in the other interval.

Theorem 6 does hold for the case of the vote maximizing parties discussed immediately after corollary one in the text. In this case the equilibrium is a distribution across a continuous interval.

**Appendix B. Proof that \(y > 1/2\) for non convex \(p\)**

\(b^0 = y: 2p_1(0, y)(1 - \alpha) - 1 = \alpha\). \(a^1 = y: 1 - 2p_1(1, y)(1 - \alpha) = \alpha\). Hence \(h(y) = p_1(0, y) + p_1(1, y) = 1/(1 - \alpha) = q\). We have \(h: [0, 1] \rightarrow [1/2, 3/2]\) and \(q': [0, 1/2] \rightarrow [1, 2]\). Therefore for an equilibrium \(h > 1\). It is easy to show that for \(h > 1\), \(y > 1/2\) for \(p_1(y)\) convex.

**References**