Solution and control of linear rational expectations models with structural effects from future instruments

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Received 22 March 1999; accepted 2 December 1999

Abstract

A solution and control method is developed for rational expectations models where future instruments enter the model directly, such as when expected interest rates feature in an inflation control problem. Such models are intrinsically time-inconsistent without additional jump variables. © 2000 Elsevier Science S.A. All rights reserved.

Keywords: Expected instruments; Optimal control

JEL classification: C61; E60

1. Motivation

We describe the solution and optimal control of linear rational expectations models which have direct effects from future instrument values in the structural model form, such as that proposed by Svensson (1998). These models have the property that they are intrinsically time-inconsistent with respect to optimal policy formulation. There are additional control problems. The usual Lagrange multiplier approach for optimal control cannot be applied. Dynamic programming (used to obtain time-consistent equilibria) cannot be used without some modification, such as the one proposed by Svensson (1998) or Amman et al. (1995).

We develop a method for solution that is a generalization of the Blanchard and Kahn (1980) (henceforth BK) approach and a method for optimal time-inconsistent control with a quadratic objective function that is based on the Theil (1964) stacking procedure. The time-inconsistent control solution is a vital part of the assessment of any policy regime: It indicates the value of using commitment mechanisms such as delegation or legislation.

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Our procedure is necessarily restricted to a finite horizon control problem. However, we develop expressions for the impact of the instruments past the end of the solution period to deal with steady-state aspects which allows us to approximate an infinite horizon problem. We also note how to deal with models which are not saddlepath stable without control.

2. Model and rational solution

We write:

\[
\begin{bmatrix}
    z_{t+1} \\
    E_t x_{t+1}
\end{bmatrix} = A \begin{bmatrix}
    z_t \\
    x_t
\end{bmatrix} + \sum_{i=0}^{k} B_i E_i u_{t+i}
\]

with \( n - q \) predetermined variables \( z_t \), \( q \) non-predetermined variables \( x_t \), \( m \) exogenous variables \( u_t \), and \( E_t \) denotes the expectation at time \( t \). \( A \) is a \( n \times n \) matrix and \( B_i \) a sequence of \( k + 1 \) \( n \times m \) matrices. The difference from the standard BK formulation is the structural forward effects of the expected instruments appearing in the model independently from the saddlepath condition outlined below. For \( k = 0 \) this reduces to the standard model.

We characterize the solution by considering the eigenvalue problem \( MA = \Lambda M \) where \( \Lambda \) is a diagonal matrix of eigenvalues in increasing absolute value and \( M \) is the non-singular matrix of left eigenvectors. These matrices, obtained by standard methods, allow us to diagonalize \( A \) and write \( A = M^{-1} \Lambda M \).

Premultiplying (1) by \( M \) gives:

\[
Ms_{t+1} = \Lambda Ms_t + M \sum_{i=0}^{k} B_i E_i u_{t+i}
\]

where \( s_t = [z_t' \ x_t']' \) and the lead is similarly defined. BK argue that uniqueness will require that there are as many unstable eigenvalues as there are jump variables. Define \( \xi_t = Ms_t \). Partition \( M \) conformably with the number of jump variables as:

\[
\begin{bmatrix}
    \xi^s_t \\
    \xi^u_t
\end{bmatrix} = \begin{bmatrix}
    M_{11} & M_{12} \\
    M_{21} & M_{22}
\end{bmatrix} \begin{bmatrix}
    z_t \\
    x_t
\end{bmatrix}
\]

and \( A \) similarly. Write (2) as:

\[
\begin{bmatrix}
    \xi^s_{t+1} \\
    \xi^u_{t+1}
\end{bmatrix} = \begin{bmatrix}
    A_s & 0 \\
    0 & A_u
\end{bmatrix} \begin{bmatrix}
    \xi^s_t \\
    \xi^u_t
\end{bmatrix} + \sum_{i=0}^{k} B_i E_i u_{t+i}
\]

where we have partitioned the eigenvalues into the stable, \( A_s \), and unstable, \( A_u \), and let \( M_i = [M_{1i} \ M_{2i}] \) with \( M_2 \) similarly defined.

Saddlepath stability of the transformed system can only be achieved if \( \xi^u_t \) is ‘solved forward’, i.e.:

\[
\xi^u_t = A_u^{-1} \xi^u_{t+1} - A_u^{-1} M_2 \sum_{i=0}^{k} B_i E_i u_{t+i} = -\sum_{j=0}^{\infty} A_u^{-j-1} M_2 \left( \sum_{i=0}^{k} B_i E_i u_{t+i+j} \right)
\]
It will prove convenient to eliminate the double summation on the right-hand side of (5) in what follows. If we expand the right-hand side we can collect terms straightforwardly to give:

\[ \xi_t^u = - \sum_{j=0}^{k-1} A_u^{-j-1} \Gamma_j E_t u_{t+j} - \sum_{i=k}^{\infty} A_u^{-i-1} \Gamma_i E_t u_{t+i} \]  \hspace{2cm} (6) 

where we define \( \Gamma_p = \sum_{j=0}^{p} A_u^j M_2 B_i \).

From the definition of \( \xi \) in (3) and the dynamics implied in (4) it is clear that the jump variables are only on the saddlepath if the relation \( \xi_t^u = M_{21} z_t + M_{22} x_t \) holds. Together with (5) this implies a reaction function for \( x_t \) of:

\[ x_t = - N z_t - \sum_{j=0}^{k-1} M_{22}^{-1} A_u^{-j-1} \Gamma_j E_t u_{t+j} - \sum_{i=k+1}^{\infty} M_{22}^{-1} A_u^{-i-1} \Gamma_i E_t u_{t+i} \]  \hspace{2cm} (7) 

where \( N = M_{22}^{-1} M_{21} \). Substituting (7) into the first line of (1) gives:

\[ z_{t+1} = (A_{11} - A_{12} N) z_t + \sum_{j=0}^{k} (B_{1,j} - A_{12} M_{22}^{-1} A_u^{-j-1} \Gamma_j) E_t u_{t+j} - \sum_{i=k+1}^{\infty} A_{12} M_{22}^{-1} A_u^{-i-1} \Gamma_i E_t u_{t+i} \]  \hspace{2cm} (8) 

where we have partitioned \( B_i = \begin{bmatrix} B_{1,i} \\ B_{2,i} \end{bmatrix} \).

Formulated in this way there are \( k \) different \( \Gamma \)'s to be calculated for the reaction function (7). The only variation in the effect from future instruments comes from the inverse powers of \( L \). This is achieved by including positive powers of \( A_u \) in the definition of \( \Gamma \). In (8) we need to add in the \( k+1 \) direct effects on the instruments on the predetermined states. As expected, setting \( k = 0 \) reduces the whole system to the familiar BK one, but if there are no jump variables \( (q=0) \) the system does not reduce to the usual causal one because of the direct impact of future instruments.

When we consider the infinite horizon control problem it is useful to derive an expression for the impact of a constant instrument vector past some horizon. If there is some \( T > k \) after which \( u_{t+l} = u_T = \bar{u} \) for \( l > 0 \) then we can modify Eq. (6) to:

\[ \xi_t^u = - \sum_{j=0}^{k-1} A_u^{-j-1} \Gamma_j E_t u_{t+j} - \sum_{i=k}^{T-1} A_u^{-i-1} \Gamma_i E_t u_{t+i} - (I - A_u^{-1})^{-1} A_u^{-T-1} \Gamma_T \bar{u} \]  \hspace{2cm} (9) 

The solutions for \( x_t \) and \( z_t \) given by (7) and (8) can then be modified by using (9) instead of (6).

The modification to the BK method also gives some insight into the interpretation of the model parameters. The lead effects by the \( B_i \) are very similar to the forward terms generated by the BK algorithm. As such the model can be viewed as a reduced form where some rational expectations variables have been solved out, leaving additional jump variables which rely on terms further out.

3. Finite horizon optimal control

We initially derive the linear-quadratic optimal control under two separate assumptions. First, that the control problem is defined up to a finite horizon, \( T \). Second, that the instrument values are set to
zero for all periods past this. Neither of these are particularly sensible assumptions for any rational expectations control problem; equilibrium values can easily be away from zero for arbitrary control problems. We describe how to work around these restrictions below.

Define the control problem as:

$$W_0 = \frac{1}{2} \sum_{i=0}^{T} \rho^i (s'_i Q s_i + 2u'_i U s_i + u'_i R u_i)$$

(10)

where $Q$ and $R$ are positive semi-definite weighting matrices, $U$ an additional weighting matrix and $0 < \rho \leq 1$ a discount factor. We can introduce linear costs, as well as modify the control model to include constants or extra exogenous variables without any difficulty in what follows.

We will redefine the control problem as a stacked system in the form associated with Theil (1964). We need to establish some more notation. Rewrite (7) as:

$$x_t = -Nz_t - \sum_{i=0}^{\infty} \Phi_i E_t u_{t+i}$$

(11)

so that $\Phi_0 = M_{22}^{-1} A_u^{-1} I_0$ and so on. Define

$$\Psi_0 = \begin{bmatrix} B_{1,0} \\ 0 \end{bmatrix}, \quad \Psi_1 = \begin{bmatrix} B_{1,1} \\ -\phi_0 \end{bmatrix}, \quad \Psi_2 = \begin{bmatrix} B_{1,2} \\ -\phi_1 \end{bmatrix} \quad \text{and} \quad \Psi_k = \begin{bmatrix} B_{1,k} \\ -\phi_{k-1} \end{bmatrix}$$

and for $i > k$

$$\Psi_i = \begin{bmatrix} 0 \\ -\phi_{i-1} \end{bmatrix}$$

Additionally, to take account of initial conditions define

$$\Phi_i = \begin{bmatrix} 0 \\ \phi_i \end{bmatrix}$$

Finally let

$$\tilde{N} = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{A} = \begin{bmatrix} A_{11} \\ A_{12} \\ 0 \\ 0 \end{bmatrix}$$

Assuming that the instrument values are set to zero past the control horizon $T$ (an assumption we relax below) the model can be rewritten as:

$$\begin{bmatrix} \tilde{N} \\ 0 \\ -\tilde{A} \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{N} \\ 0 \\ -\tilde{A} \end{bmatrix}$$

or

$$\begin{bmatrix} \tilde{N} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(12)
\[ \hat{A}s = \hat{B}u + \hat{b} \]  

(13)

Define \( G = \hat{A}^{-1}\hat{B} \) and \( b = \hat{A}^{-1}\hat{b} \) so that \( s = Gu + b \). The matrix \( \hat{B} \) is made up from row shifting operations (except for the initial conditions) and is easily implemented. Forming \( G \) this way avoids the need for calculating the recursive solution for the model as the inverse generates it. Note that \( G \) is not lower triangular because \( \hat{B} \) is not.

The objective function needs to be similarly ‘stacked’. Define new weight matrices as:

\[
Q = \begin{bmatrix}
Q & 0 & \cdots & 0 \\
0 & \rho Q & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \rho^T Q
\end{bmatrix}
\]

with \( U \) and \( R \) similarly defined. The control problem can be compactly written as:

\[
W_0 = \frac{1}{2} s'Qs + u'Us + \frac{1}{2} u'Ru
\]

\[
= \frac{1}{2} (b' + u'G')Q(Gu + b) + u'U(Gu + b) + \frac{1}{2} u'Ru
\]

(14)

From (14) the optimal control can be found very simply. First-order conditions are:

\[
\frac{\partial W_0}{\partial u} = (G'QG + UG + R)u^* + (G'Q + U)b = 0
\]

(15)

so that the optimal instrument values are:

\[ u^* = -(G'QG + UG + R)^{-1}(G'Q + U)b \]

(16)

implying optimal trajectories for the endogenous variables under control of:

\[ s^* = (I - G(G'QG + UG + R)^{-1}(G'Q + U))b \]

4. Intrinsic time-inconsistency, infinite horizons and computation

We are able to show from the above control solution that this class of models is intrinsically time-inconsistent, even without additional jump variables. The stacked model matrix \( G \) is not lower triangular both because of the rational expectations terms generated by the \( \Phi \) terms but also the direct \( B_t \) terms. This accords with our interpretation of the ‘structural’ model incorporating at least a partial rational expectations solution. We conclude that it is therefore very important to evaluate the degree to which time-inconsistency impinges on optimal policy formulation before being able to assess alternative policy regimes.

The control algorithm above assumes the instruments past the terminal time are set to zero. If we wish to compare the solution to that derived by Svensson (1998) we need to approximate an infinite horizon problem. To do so we need to solve the problem over a long enough horizon to put the
endogenous variables and instruments onto their equilibrium paths and treat the instruments past the terminal date properly.

We can use (9) to calculate the rational expectation past a given terminal date $T$ for a constant instrument. Thus we modify the last column of $\hat{B}$ in (12) to include the relevant constant which depends on both $T - t$ where for east $s$, and whether $t$ is less than $k$ periods from the final period. The first of these effects the constant at the end of (9) and the latter indicates how we need to treat the cumulation of the $B_t$ values for the impact effect of the constant instrument past the end. The modifications are straightforward and essentially reduce to varying the power term in the constant and having $k$ special reaction functions at the end rather like the $k$ at the beginning.

If we assume that the instruments and the endogenous variables are on their equilibrium paths at time $T$ then we can modify (10) to an infinite horizon problem by modifying the bottom right-hand corner of $Q$ to $(I - Q)^{-1}Q^T$ and $U$ and $R$ similarly. We can then approximate the true infinite horizon problem by solving this ‘infinite horizon’ formulation for an arbitrary ‘equilibrium’ horizon $T$, which is then increased until the solution does not alter.

A final computational note is that the approach depends on the model being saddlepath stable open-loop to calculate the reaction functions. If this is not the case, such as the Svensson (1998) model, incorporating an arbitrary simple feedback rule which has no effect on the resulting optimal control can be used to ensure saddlepath stability. This may require forecasts of relevant target variables to be generated for $k > 1$.

**Acknowledgements**

I would like to thank Tibor Hledik for stimulating discussions about the problem addressed in this paper. This research was funded by the ESRC Macroeconomic Modelling Consortium.

**References**