On the end-point issue in unit root tests in the presence of a structural break

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Abstract

This paper shows that the spurious rejection problem illustrated by Leybourne et al. (1998) [Leybourne, S.J., Mills, T., Newbold, P., 1998. Spurious rejections by Dickey–Fuller tests in the presence of a break under the null. Journal of Econometrics 87, 191–203] is restricted to the DF type test, which is based on the conditional likelihood function discarding the first observation. © 2000 Elsevier Science S.A. All rights reserved.

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JEL classification: C12; C15; C22

1. Introduction

It is now well known from the pioneering work of Perron (1989) that standard unit root tests lose power when the stationary alternative is true and an existing structural break is ignored. A converse question is: what will happen to the standard unit root tests if a time series contains a unit root and a structural break is ignored under the null? Amsler and Lee (1995) initially examined this question and showed that, unlike under the alternative, the usual unit root tests such as Dickey–Fuller (1979, DF hereafter) and Schmidt–Phillips (1992, SP hereafter) tests are not affected asymptotically by ignoring the break if the null is true. The asymptotic distributions of these tests remain unchanged when the data generating process involves a break under the null and there will be no size distortion, even if the break is ignored in the testing procedure. A recent work by Leybourne et al. (1998), however, shows that using the standard Dickey–Fuller test can lead to spurious rejections of the null if a structural break occurs early in the series for finite samples. Thus, the result advocated by Amsler and Lee does

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This paper shows that the spurious rejection problem under the null is peculiar to the DF type test, which is based on the conditional distribution of the data discarding the first observation. The spurious rejection problem does not occur in finite samples of even moderate size for the LM unit root test of Schmidt and Phillips (1992) and in other unit root tests that utilize the full unconditional distribution of the data. When these tests use the first observation in deriving test statistics, they are free of spurious rejections even if the break occurs early in the series.¹

2. Test statistics

We consider a data-generating model,

\[ Y_t = \gamma Z_t + X_t, X_t = \beta X_{t-1} + u_t, \quad t = 2, \ldots, T, \]  

where the \( u_t \) are iid random variables with mean 0 and variance \( \sigma^2 \), and \( Z_t \) is a vector of exogenous variables. We are interested in testing \( \beta = 1 \). When \( Z_t = 1 \), we can rewrite (1) as

\[ Y_t = \gamma(1 - \beta) + \beta Y_{t-1} + u_t. \]  

The standard Dickey–Fuller test statistic is given as the \( t \)-ratio for \( \beta = 1 \) in the above equation. We assume that the first observation \( Y_1 \) is independent of \( u_t, t > 1 \). We can construct different tests using likelihood procedures assuming that \( u_t \) are normally distributed. Depending on what we assume about the distribution of \( Y_t \), we can obtain different test statistics. When \( Y_t \) is assumed fixed, maximizing the likelihood function is equivalent to minimizing

\[ Q_c = (n - 1) \log \sigma^2 + \sigma^{-2} \sum_{t=2}^{T} [Y_t - \gamma(1 - \beta) - \beta Y_{t-1}]]^2, \]

where \( Y_t \) is excluded. The conditional ML estimator of \( \beta \), given \( Y_t \), is the same as the OLS estimator of the regression where \( Y_t \) is regressed on \( 1 \) and \( Y_{t-1} \). The DF statistic is obtained as a \( t \)-statistic from this OLS estimator. On the other hand, when \( Y_t \) is assumed to follow a normal distribution with mean \( \gamma \) and variance \( \sigma^2 \), maximizing the unconditional likelihood function is equivalent to minimizing

\[ Q_u = n \log \sigma^2 + \sigma^{-2}(Y_1 - \gamma)^2 + \sigma^{-2} \sum_{t=2}^{T} [Y_t - \gamma(1 - \beta) - \beta Y_{t-1}]]^2. \]

Note that the first observation is used in (4). Gonzalez-Farias (1992) obtained the unconditional ML estimators as

¹A recent study of Huh and Dickey (1999) shows that the unit root test based on a symmetric estimator is also free of the spurious rejection problem under the null. We note that the symmetric estimator is also based on the full unconditional distribution of the data.
\[
\gamma = \frac{Y_1 + (1 - \beta) \sum_{t=2}^{T} (Y_t - \hat{\beta} Y_{t-1})}{1 + (T - 1)(1 - \hat{\beta})^2}, \quad \hat{\beta} = \frac{\sum_{t=2}^{T} (Y_t - \hat{\gamma})(Y_{t-1} - \hat{\gamma})}{\sum_{t=2}^{T} (Y_{t-1} - \hat{\gamma})^2}
\]

\[
\hat{\sigma}^2 = T^{-1}(Y_1 - \hat{\gamma})^2 + T^{-1} \sum_{t=2}^{T} (Y_t - \hat{\gamma}(1 - \hat{\beta}) - \hat{\beta} Y_{t-1})^2.
\] (5)

Pantula et al. (1994) suggest obtaining the above ML estimators by an iterative procedure. Starting with an OLS estimator of \(\beta\), say \(\hat{\beta}^{(0)}\), as an initial value, one can compute iteratively \(\hat{\beta}^{(i)}\) and \(\hat{\gamma}^{(i)}\) using (5) until they converge. Pantula et al. use 10 iterations in their simulation study. The unit root test statistics using the ML estimators are also suggested by Gonzalez-Farias (1992) as

\[
\hat{\tau}_{ML} = (\hat{\beta} - 1) \left[ \hat{\sigma}^{-2} \sum_{t=2}^{T} (Y_{t-1} - \hat{\gamma})^2 \right]^{1/2},
\]

\[
\Phi_{ML} = T(s^2 - \hat{\sigma}^2)/s^2,
\] (6)

where \(s^2 = \sum_{t=2}^{T} (Y_t - Y_{t-1})^2/(T - 2)\). The uniformly most powerful (UMP) test of Elliott and Stock (1994) is also given from the unconditional likelihood function. The UMP test considers testing \(\beta = 1\) against the alternative \(\beta = \beta_\gamma\), where \(\beta_\gamma\) is given as \(\beta_\gamma = 1 - 7T^{-1}\). For a unit root hypothesis, Elliott and Stock suggest

\[
\hat{\Phi}_{ES} = T(\hat{\sigma}^2 s^2 - s^2)/s^2 + 7
\] (7)

where \(\hat{\sigma}^2 s^2\) is given as \(\hat{\sigma}^2\) in (5), except that \(\hat{\beta}\) is replaced with \(\beta_\gamma = 1 - 7T^{-1}\) and \(\hat{\gamma}\) is evaluated with \(\beta_\gamma\). Pantula et al. (1994) provide a summary of the asymptotic distributions of these tests and their corresponding critical values. Lastly, the LM unit root test statistic of SP can be obtained according to the LM (score) principle from the restricted score vector and the hessian of the unconditional likelihood function. SP provide a convenient method of obtaining the test statistic using the following regression

\[
\Delta Y_t = \delta' \Delta Z_t + \phi \hat{S}_{t-1} + u_t,
\] (8)

where \(\hat{S}_t = Y_t - \hat{\psi}_t - Z_t \hat{\delta}, t = 2, \ldots, T\), \(\hat{\delta}\) are coefficients in the regression of \(\Delta Y_t\) on \(\Delta Z_t\), and \(\hat{\psi}_t\) is the restricted MLE of \(\psi_t(= \psi + X_0)\) given by \(Y_1 - Z_1 \hat{\delta}\). The LM statistic is the t-statistic for \(\phi = 0\) and is denoted as \(\hat{\tau}_{LM}\).

3. Simulation results

In this section, we examine the effect of a structural break under the null on the tests discussed in the previous section. In particular, we are interested in whether spurious rejections can be found from these tests when a structural break occurs early in the series. Among the test statistics we consider, only the DF statistic is obtained from the estimator based on the conditional likelihood function and
denoted as $\hat{T}_{DF}$. Other test statistics are based on the unconditional likelihood function utilizing the information of the first observation. They include the ML based tests ($\hat{T}_{ML}$ and $\hat{F}_{ML}$) of Gonzalez-Farias, the UMP test ($\hat{F}_{ES}$) of Elliott and Stock, and the LM statistic ($\hat{T}_{LM}$) of SP.

The data generating process (DGP) in the simulation includes a structural break under the null as

$$Y_t = d \cdot B_t + Y_{t-1} + u_t,$$  \hspace{1cm} (9)

where $B_t = 1$ for $t = T_B + 1$, and zero otherwise, and $T_B$ is the time period when a break occurs. This DGP is equivalent to that in Eq. (1) of Leybourne et al. (1998), p. 193. We treat $Y_t$ as a random variable with mean 0 and variance $\sigma^2$. In the simulations, we generate pseudo-iid $N(0,1)$ random numbers using the Gauss (version 3.2.12) RNDNS procedure. All simulations are performed using 5000 replications. We examine various break locations $\lambda = T_B/T$. The finite sample performance is examined with a sample of size $T = 100$. We also examine the asymptotic behavior with a sample of size $T = 500$. 5% rejection rates are reported in all simulations.

The first panel in Table 1 provides results with $T = 100$ and $d = 5$. Results indicate no serious spurious rejections in all tests based on the unconditional likelihood function. The DF test is subject to spurious rejections for $\lambda \leq 0.12$. Instead, most other tests exhibit a small negative size distortion with somewhat fewer rejections. The second panel in Table 1 provides results for a larger break size of $d = 10$. Clearly, the spurious rejection problem of the DF test becomes more serious as $d$ increases. This result is consistent with that of Leybourne et al. (1998). Fewer rejections become more evident in all other tests as $d$ increases.

We next examine the large sample case in Table 2 when $T = 500$. With a larger sample size all tests improve. Even the DF test makes a big improvement in size as the spurious rejection problem is significantly mitigated. We can expect that spurious rejections will disappear as $T$ increases for a fixed $d$. This finding supports the notion in Amsler and Lee (1995, Theorem 2) that the usual unit root tests are not affected asymptotically by the presence of a structural break under the null. Most of the tests based on the unconditional likelihood function give rejection rates close to their 5% nominal size. In particular, the LM test appears the most accurate; even when $d = 10$, there is almost no size distortion.

### Table 1
Rejection Rates under the null in the presence of a structural break ($T = 100$)

| Test Position of break ($\lambda$) | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 | 0.12 | 0.14 | 0.16 | 0.18 | 0.20 | 0.50 | 0.80 | 0.90 | 0.95 | 0.99 |
|-----------------------------------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $\hat{\tau}_{DF}$ $d = 5$       | 0.223| 0.201| 0.184| 0.161| 0.143| 0.125| 0.107| 0.096| 0.085| 0.074| 0.063| 0.052| 0.048| 0.043| 0.038| 0.038| 0.041| 0.039| 0.040 |
| $\hat{\tau}_{ML}$ $d = 5$       | 0.045| 0.042| 0.042| 0.042| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.045| 0.047| 0.047| 0.048 |
| $\hat{D}_{ML}$ $d = 5$          | 0.032| 0.033| 0.035| 0.037| 0.040| 0.039| 0.040| 0.042| 0.042| 0.043| 0.043| 0.043| 0.045| 0.045| 0.044| 0.044| 0.047| 0.046| 0.047 |
| $\hat{D}_{ES}$ $d = 5$          | 0.032| 0.034| 0.036| 0.038| 0.039| 0.042| 0.043| 0.044| 0.044| 0.045| 0.045| 0.045| 0.046| 0.046| 0.047| 0.049| 0.051| 0.050| 0.049 |
| $\hat{T}_{ES}$ $d = 5$          | 0.049| 0.047| 0.047| 0.047| 0.047| 0.047| 0.047| 0.044| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043| 0.043 |
| $\hat{\tau}_{DF}$ $d = 10$     | 0.550| 0.519| 0.487| 0.444| 0.396| 0.348| 0.309| 0.288| 0.231| 0.185| 0.145| 0.087| 0.077| 0.059| 0.036| 0.026| 0.022| 0.019| 0.015 |
| $\hat{\tau}_{ML}$ $d = 10$     | 0.072| 0.038| 0.027| 0.023| 0.024| 0.025| 0.024| 0.026| 0.024| 0.027| 0.027| 0.030| 0.030| 0.030| 0.032| 0.033| 0.031| 0.031| 0.036 |
| $\hat{D}_{ML}$ $d = 10$        | 0.007| 0.008| 0.009| 0.011| 0.013| 0.014| 0.017| 0.017| 0.017| 0.021| 0.023| 0.023| 0.025| 0.026| 0.026| 0.028| 0.036| 0.036| 0.045 |
| $\hat{D}_{ES}$ $d = 10$        | 0.005| 0.007| 0.009| 0.011| 0.012| 0.013| 0.017| 0.017| 0.017| 0.021| 0.023| 0.023| 0.025| 0.026| 0.026| 0.028| 0.036| 0.036| 0.045 |
| $\hat{\tau}_{ES}$ $d = 10$     | 0.030| 0.031| 0.030| 0.030| 0.027| 0.031| 0.030| 0.029| 0.028| 0.030| 0.029| 0.029| 0.030| 0.030| 0.030| 0.031| 0.032| 0.036| 0.033 |
Table 2

Rejection Rates under the null in the presence of a structural break ($T = 500$)

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<th>Test Position of break ($d$)</th>
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<th>0.03</th>
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4. Concluding remarks

In this paper, we examine a variety of unit root tests to see the effect of a structural break under the null. We find that the spurious rejection problem illustrated in Leybourne et al. (1998) is restricted to the DF type test based on the conditional likelihood function that discards the first observation. All other unit root tests considered in this paper are mostly free of this spurious rejection problem, as they are based on the unconditional likelihood function that includes the first observation. In this regard, our finding is in line with results of Park and Mitchell (1980) that the Cochrane–Orcutt procedure using $T−1$ observations after discarding the first observation is less efficient than the estimator based on the ML estimator using all $T$ observations.

References