Caution in macroeconomic policy: uncertainty and the relative intensity of policy

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Abstract

In a one-state two-control Quadratic Linear Problem, we use the Riccati equations to examine the effect of an increase in uncertainty in a future time period on the optimal use of the control variables in the first time period. We find that caution or more intensity in the outcome depends on the ratio of the weight on each control variable to the variance of the parameter which multiplies that control. When these ratios are not equal, the control with a smaller weighted variance will always be used more intensely. When these ratios are equal, both control variables will always be used more intensely. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Recently, there has been a return of interest in the effects of parameter uncertainty on macroeconomic policy, viz. Ghosh and Masson (1994), Wieland (1998), Mercado and Kendrick (1999), and Amman and Kendrick (2000). This work has mainly consisted of empirical or experimental studies. Our focus in this article is theoretical.

Working with a one-state two-control static model Johansen (1978) found that when the relative magnitude of the variance on one of the control parameter increases w.r.t. the variance of the other control parameter, the control associated with it will be used more cautiously, while the other control will be used more intensely. Working with a one-state one-control dynamic model Craine (1979) found that when there is an increase in the future variance of the control parameter, it will be optimal...
to use the first-period control more intensely. In this article we move forward these results to dynamic models with a single state and two control variables. Using the Riccati equations, we show that if there is an increase in the uncertainty of a parameter associated with a future control there will be an increased use of both, or at least one, of the first-period controls, depending on the relative magnitude of their first-period weighted variances.

2. The problem

We will focus on an optimal policy problem defined as a Quadratic Linear Problem. The state parameter and the control parameters are assumed to be uncertain with no covariance among them. The model parameters and the weights on the state and control variables can be time varying or time invariant. In formal terms, the problem is expressed as one of finding the controls \((u_k)_{k=0}^{N-1}\) to minimize a quadratic criterion function \(J\) of the form:

\[
J = E \left\{ \frac{1}{2} x_k^T w x_k + \frac{1}{2} \sum_{k=0}^{N-1} (x_k' A x_k + u_k' A u_k) \right\}
\]

subject to:

\[
x_{k+1} = A x_k + B u_k + \epsilon_k
\]

where \(E\) = expectation operator; \(x\) = scalar state variable; \(u\) = control variables; \(A\) = diagonal matrix of positive weights on the control variables; \(w\) = positive scalar weight on the state variable; \(\epsilon\) = random disturbance and:

\[
A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad A = a = \text{scalar}, \quad B = b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\]

Desired paths for the state and the controls are zero. All means and parameter variances are assumed to be known, and \(a\), \(b\), \(w\) and \(A\) can be either fixed or time varying (we have omitted the corresponding time subscripts to simplify the exposition). The solution to the optimization problem is the feedback rule (see Kendrick, 1981, pp. 48–49):

\[
u_k = G_k x_k
\]

where, for our one-state two-control model, the feedback gain coefficients are:

\[\text{Even though in this case we have more instruments than targets, the case is not trivial. As it was pointed out in Brainard’s (1969) seminal paper, in the presence of parameter uncertainty it will be optimal, in general, to use all available instruments even if there is only one target variable.}\]

\[\text{It is standard procedure to focus on the first-period behavior of the control variables. The qualitative behavior of the controls may well change beyond the first period because their optimal values after the first period will be computed recursively, using the model equation to determine the value of the state variable for period “}k+1\text{” (see Kendrick, 1981, Chapter 6).}\]

\[\text{This is the usual case for log-linearized models or models with variables expressed in percentage changes with respect to a base case.}\]
\[ G_k = \begin{bmatrix} G_{1k} \\ G_{2k} \end{bmatrix} = -\frac{1}{\text{det}} \begin{bmatrix} ab_1 \left( A_{22}k_{k+1} + k_{k+1}^2\sigma_{b_2}^2 \right) \\ ab_2 \left( A_{11}k_{k+1} + k_{k+1}^2\sigma_{b_1}^2 \right) \end{bmatrix} \]  

(2.4)

where \( k \) is the Riccati matrix (in this case, a scalar) and where:

\[
\text{det} = A_{11}A_{22} + A_{11}k_{k+1}b_1^2 + A_{22}k_{k+1}b_2^2 + A_{11}k_{k+1}\sigma_{b_2}^2 + A_{22}k_{k+1}\sigma_{b_1}^2 + k_{k+1}^2\sigma_{b_2}^2 + k_{k+1}^2\sigma_{b_1}^2 + k_{k+1}\sigma_{b_1}\sigma_{b_2}^2 
\]  

(2.5)

Consider now the effects on the intensity of the first-period response in the policy variables \( u \) (or, equivalently, on \( G \)) when there is a change in the future uncertainty of one of the parameters which are multiplied by the control variables.\(^4\) We will analyze the case of an increase in the uncertainty associated with parameter \( b_1 \), and thus \( \sigma_{b_1(T)}^2 \), where \( T \) can take any value between \( I \) and \((N - 1)\). Equivalent results can be derived for the case of \( \sigma_{b_2(T)}^2 \). First notice that the Riccati matrix will always be positive. Indeed, for a model like ours the Riccati equations are \( k = w \) for the terminal period and:

\[
k_k = w + a^2k_{k+1}\left(1 - \frac{A_{11}k_{k+1}b_2^2 + A_{22}k_{k+1}b_1^2 + k_{k+1}^2\sigma_{b_1}^2 + k_{k+1}^2\sigma_{b_2}^2}{\text{det}} \right) 
\]  

(2.6)

for any other period (see Kendrick, 1981, pp. 48–49). Eq. (2.6) is solved by backward integration starting from period \( “N - 1”. \) Thus, the value of \( k_{N-1} \) will be positive, given that \( k_N = w \) is positive, that the weights on the controls were assumed to be positive, and that the numerator in the term between parentheses is always smaller than \( \text{det} \), making the term between parentheses always positive. Given that \( k_{N-1} \) is positive, by the same reasoning \( k_{N-2} \) will be positive, and so on. Thus, we can conclude that \( k \) will always be positive.

Given that \( k \) is positive, given that the weights on the controls were assumed to be positive, and given that the absolute value of quadratic terms is their own value, from (2.4) we can write:

\[
|G_1| = \frac{|ab_1|\left( A_{22}k_{(1)} + k_{(1)}^2\sigma_{b_2}^2 \right)}{\text{det}} 
\]  

(2.7)

and

\[
|G_2| = \frac{|ab_2|\left( A_{11}k_{(1)} + k_{(1)}^2\sigma_{b_1}^2 \right)}{\text{det}} 
\]  

(2.8)

Notice that the multiperiod link between future uncertainty \( \sigma_{b_1(T)}^2 \) and first-period feedback gain coefficients \( G \) is made by the successive Riccati matrices \( k \). Thus, from Eqs. (2.7) and (2.8) we obtain:

\(^4\)To simplify notation, from now on time subscripts corresponding to periods other than zero (that is, the first period) will be between parentheses. Thus, variables and parameters without subscripts between parentheses will correspond to time zero, that is, the first period.
\[
\frac{\partial |G|}{\partial \sigma_{b_i(T)}^2} = \frac{\partial |G|}{\partial k_{(1)}} \frac{\partial k_{(1)}}{\partial k_{(2)}} \cdot \frac{\partial k_{(T-1)}}{\partial k_{(T)}} \frac{\partial k_{(T)}}{\partial \sigma_{b_i(T)}^2}
\]  
\tag{2.9}

where \(i = 1, 2\). From Eq. (2.6) we can obtain:

\[
\frac{\partial k_{(T)}}{\partial \sigma_{b_i(T)}^2} > 0 \quad \text{and} \quad \frac{\partial k_{(1)}}{\partial k_{(2)}} > 0, \ldots, \frac{\partial k_{(T-1)}}{\partial k_{(T)}} > 0
\]  
\tag{2.10}

and from (2.7) and (2.8) we obtain, respectively:

\[
\frac{\partial |G|}{\partial k_{(1)}} = \left| a b_1 \left\{ A_{11} A_{22} + 2 A_{11} A_{22} k_{(1)} \sigma_{b_2}^2 + A_{11} k_{(1)}^2 \sigma_{b_2}^4 + \left[ k_{(1)} b^2 (A_{11} \sigma_{b_2}^2 - A_{22} \sigma_{b_1}^2) \right] \right\} \right| \frac{\det^2}{\text{det}^2}
\]  
\tag{2.11}

and

\[
\frac{\partial |G|}{\partial k_{(1)}} = \left| a b_2 \left\{ A_{11} A_{22} + 2 A_{11} A_{22} k_{(1)} \sigma_{b_1}^2 + A_{11} k_{(1)}^2 \sigma_{b_1}^4 + \left[ k_{(1)} b^2 (A_{22} \sigma_{b_1}^2 - A_{11} \sigma_{b_2}^2) \right] \right\} \right| \frac{\det^2}{\text{det}^2}
\]  
\tag{2.12}

The signs of these derivatives, and thus the signs of (2.9) depend on the sign of \((A_{11} \sigma_{b_2}^2 - A_{22} \sigma_{b_1}^2)\) — or, equivalently, on the sign of \((A_{22} \sigma_{b_1}^2 - A_{11} \sigma_{b_2}^2)\). Notice that:

\[
\text{sign} \left( A_{11} \sigma_{b_2}^2 - A_{22} \sigma_{b_1}^2 \right) = \text{sign} \left[ \left( \frac{\sigma_{b_2}^2}{A_{22}} \right) - \left( \frac{\sigma_{b_1}^2}{A_{11}} \right) \right]
\]  
\tag{2.13}

Thus, the sign of the derivatives of \(|G_1|\) and \(|G_2|\) w.r.t. \(\sigma_{b_i(T)}^2\), and the corresponding responses of the control variables \(u_1\) and \(u_2\), will be determined by the relative magnitude of the first-period weighted variances of \(b_1\) (that is, \((\sigma_{b_1}^2 / A_{11})\)) and \(b_2\) (that is \((\sigma_{b_2}^2 / A_{22})\)). There are three possible cases.

If \((\sigma_{b_2}^2 / A_{22})\) and \((\sigma_{b_1}^2 / A_{11})\) are equal then the derivatives of \(|G_1|\) and \(|G_2|\) w.r.t. \(\sigma_{b_i(T)}^2\) will both be positive: an increase in future uncertainty associated with \(b_1\) will always induce a more intense first-period response from both control variables. Facing an increase in future uncertainty, it makes sense to make a more intense use of the first-period controls, since they will now have, in relative terms, a more precise impact on the state variable.

If \((\sigma_{b_2}^2 / A_{22})\) is greater than \((\sigma_{b_1}^2 / A_{11})\) then the derivative of \(|G_1|\) will be positive: an increase in future uncertainty will always induce a more intense first-period response from \(u_1\). If \((\sigma_{b_2}^2 / A_{22})\) is smaller than \((\sigma_{b_1}^2 / A_{11})\) then the derivative of \(|G_2|\) will be positive: an increase in future uncertainty will always induce a more intense first-period response from “the other” control variable, that is \(u_2\). Facing an increase in future uncertainty, it also makes sense to rely more on the first-period control with a more certain (in relative terms within the first period) impact on the state variable.

\(^5\)Notice that the response of \(|G_1|\) will be ambiguous, since the sign of \((\partial |G_1| / \partial b_{(1)} )\) will depend, as can be seen in Eq. (2.12), on the relative magnitude of the first three terms in the numerator sum versus the term between brackets.

\(^6\)Notice that in this case the response of \(|G_1|\) will be ambiguous.
3. Conclusions

For a one-state two-control model, we use the Riccati equations to examine the effect of an increase in uncertainty in a future time period on the optimal use of the control variables in the first time period. Here we find that caution or more intensity in the outcome depends on the ratio of the weight on each control variable to the variance of the parameter which multiplies that control. When these ratios are not equal, the control with a smaller weighted variance will always be used more intensely. When these ratios are equal, both control variables will always be used more intensely.

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