A note on mutually absolutely continuous belief systems

Kin Chung Lo*

Department of Economics, York University, 4700 Keele Street, North York, Toronto, Ontario, Canada M3J 1P3

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Abstract

This paper shows that in a strategic game, if players’ beliefs are mutually absolutely continuous, then epistemic results that are established using the notion of belief can be reformulated using the standard notion of knowledge. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In a very interesting paper, Aumann and Brandenburger (1995) provide epistemic conditions for Nash equilibrium. The model that they use to discuss epistemic matters is called an interactive belief system (or simply belief system). To guarantee that Nash equilibrium arises in games with \( n > 2 \) players, they assume that players’ beliefs come from a common prior. Stuart (1997) also uses the framework of belief system to provide epistemic conditions for defection in every period of the finitely repeated prisoner’s dilemma. Instead of assuming that players’ beliefs come from a common prior, he assumes that the belief system is mutually absolutely continuous. Roughly speaking, this assumption says that if a player believes that a given state is possible, then all players believe that the state is possible. As Stuart points out in his paper, if the belief system has a common prior, then it is mutually absolutely continuous.

Both of the above papers adopt the notion of belief. That is, a player believes an event \( E \) if he assigns probability one to \( E \). In this paper, we show that if the belief system is mutually absolutely continuous, then epistemic results that are established using the notion of belief can be reformulated
using the standard notion of knowledge. An advantage of such a reformulation is that standard intuition about knowledge can be applied to make the results more transparent. At the end of the paper, we demonstrate this point by reformulating the results of Aumann and Brandenburger (1995) and Stuart (1997).

2. Belief system and partitional structure

Fix a strategic game form $A = A_1 \times \cdots \times A_n$, where $A_i$ is a finite set of strategies for player $i$. Throughout, the indices $i$, $j$ and $k$ vary over distinct players in $\{1, 2, \ldots, n\}$. Unless specified otherwise or emphasis is necessary, the quantifier ‘for all such $i$, $j$, and $k$’ is to be understood. Define $A_{-i} = \times_{j \neq i} A_j$. Elements in $A_i$ and $A_{-i}$ are denoted by $a_i$ and $a_{-i}$, respectively. A game is defined as $u = (u_1, \ldots, u_n)$, where $u_i : A \to \mathbb{R}$ is $i$’s payoff function.

2.1. Belief system

A belief system for the game form $A$ is denoted as $\{T_i, \hat{p}_i, \hat{u}_i, \hat{a}_i\}$. Each of its components is explained as follows.

- Player $i$ has a finite set $T_i$ of types. Define $T = \times_{i=1}^n T_i$ and $T_{-i} = \times_{j \neq i} T_j$. Any subset of $T$ is called an event and any element $t \in T$ is called a state of the world.
- For each type $t_i \in T_i$ of player $i$, $i$’s beliefs over $T$ are represented by a probability measure $\hat{p}_i(\cdot; t_i)$ with the property that $\hat{p}_i(t; t_i) = 0$ if $t \notin t_i \times T_{-i}$.
- For each $t_i \in T_i$, $i$’s payoff function is $\hat{u}_i(t_i) : A \to \mathbb{R}$.
- For each $t_i \in T_i$, $i$ chooses the strategy $\hat{a}_i(t_i) \in A_i$.

A conjecture $\phi_i$ of player $i$ is a probability measure on $A_{-i}$. The marginal probability measure of $\phi_i$ on $A_j$ is denoted by $\text{marg}_{A_j} \phi_i$. If player $i$’s type is $t_i$, then $i$’s conjecture $\phi_i(t_i)$ is given by

$$\phi_i(t_i)(a_{-i}) = \hat{p}_i([\tilde{t} \in T : \hat{a}_{-i}(\tilde{t}_{-i}) = a_{-i}] ; t_i), \quad \forall a_{-i} \in A_{-i},$$

where $\hat{a}_{-i}(\tilde{t}_{-i}) = (\hat{a}_{i}(\tilde{t}_i), \ldots, \hat{a}_{i-1}(\tilde{t}_{i-1}), \hat{a}_{i+1}(\tilde{t}_{i+1}), \ldots, \hat{a}_{n}(\tilde{t}_n))$. An $n$-tuple $(\phi_i(t_1), \ldots, \phi_i(t_n))$ of conjectures is denoted by $\phi(t)$. Similarly, an $n$-tuple $(\hat{u}_i(t_1), \ldots, \hat{u}_i(t_n))$ of payoff functions is denoted by $\hat{u}(t)$.

Say that player $i$ of type $t_i$ is rational if

$$\hat{a}_i(t_i) \in \text{arg max}_{a_i \in A_i, a_{-i} \in A_{-i}} \sum_{a_i \in A_i, a_{-i} \in A_{-i}} \hat{u}_i(t_i)(a_i, a_{-i}) \phi_i(t_i)(a_{-i}).$$

That is, player $i$ of type $t_i$ is rational if his strategy $\hat{a}_i(t_i)$ maximizes expected payoff when the payoff function is $\hat{u}_i(t_i)$ and the conjecture is $\phi_i(t_i)$.

1A number of papers have been written on examining the equivalence between belief and knowledge (see, for instance, Brandenburger and Dekel, 1987).
Consider a mutually absolutely continuous belief system description of the N-period repeated prisoner’s dilemma. Proposition 2 below is due to Stuart (1997, p. 139).

Proposition 2. Consider a game \( u \), an \( n \)-tuple \( \phi \) of conjectures, and a belief system with a common prior \( \hat{p} \). Suppose there is a state \( t \) such that \( \hat{p}(t) > 0 \), \( \{\hat{t} \in T: \text{every player } i \text{ of type } \hat{i} \text{ is rational} \} \) and \( \{\hat{t} \in T: \hat{u}(\hat{t}) = u \} \) are mutually believed at \( t \), and \( \{\hat{t} \in T: \phi(\hat{t}) = \phi \} \) is commonly believed at \( t \). Then there is an \( n \)-tuple \( \sigma = (\sigma_1, \ldots, \sigma_n) \), where \( \sigma_i \) is a probability measure on \( A_i \), such that \( \phi_i = \times_{j \neq i} \sigma_j \) and \( \sigma \) is a Nash equilibrium of \( u \).

Say that a belief system is mutually absolutely continuous if for any player \( i \), \( \hat{p}_i(t; t_i) > 0 \) implies that for all players \( j \), \( \hat{p}_j(t; t_i) > 0 \) (Stuart, 1997, p. 136). It is immediate that if the belief system has a common prior, then it is mutually absolutely continuous. Suppose \( n = 2 \) and the game \( (\hat{u}_1(t_1), \hat{u}_2(t_2)) \) at every state \( t \in T \) is the \( N \)-period repeated prisoner’s dilemma (where \( N \) is any positive integer). Proposition 2 below is due to Stuart (1997, p. 139).

Proposition 2. Consider a mutually absolutely continuous belief system description of the \( N \)-period repeated prisoner’s dilemma. Suppose there is a state \( t \) such that \( \hat{p}_i(t; t_i) > 0 \) and the event \( \{\hat{t} \in T: \text{every player } i \text{ of type } \hat{i} \text{ is rational} \} \) is commonly believed at \( t \). Then at state \( t \), both players defect in every period of the game.

2.2. Partitional structure

A partitional structure for the game form \( A \) is denoted as \( \{\Omega, H, p, u, a\} \). Each of its components is explained as follows.

- All the players are facing a finite set \( \Omega \) of states of the world. Any subset of \( \Omega \) is called an event.
- Player \( i \)'s information structure is represented by a partition \( H_i \) of \( \Omega \). For every state \( \omega \in \Omega \), \( H_i(\omega) \subseteq H_i \) is the partitional element which contains \( \omega \).
- At each \( \omega \in \Omega \), \( i \)'s beliefs over \( \Omega \) are represented by a probability measure \( p_i(\cdot; \omega) \) with the property that \( p_i(\omega'; \omega) = 0 \) if \( \omega' \notin H_i(\omega) \).
- At each \( \omega \in \Omega \), \( i \)'s payoff function is \( u_i(\omega): A \to \mathbb{R} \).
- At each \( \omega \in \Omega \), \( i \) chooses the strategy \( a_i(\omega) \in A_i \).

As we pointed out above, we assume the set \( T \) of types to be finite. Aumann and Brandenburger allow \( T \) to be infinite.
Note that \( p, u, \) and \( a \) are functions on \( \Omega \) and they are assumed to be measurable with respect to \( H_i \).

At every state \( \omega \), \( i \)'s conjecture \( \phi_i(\omega) \) is given by

\[
\phi_i(\omega)(a_{-i}) = p_i([a_{-i}]|\omega) \quad \forall a_{-i} \in A_{-i},
\]

(1)

where \([a_{-i}] = \{ \tilde{\omega} \in \Omega: a_{-i}(\tilde{\omega}) = a_{-i} \}\). An \( n \)-tuple \((\phi_1(\omega), \ldots, \phi_n(\omega))\) of conjectures is denoted by \( \phi(\omega) \). Similarly, an \( n \)-tuple \((u_1(\omega), \ldots, u_n(\omega))\) of payoff functions is denoted by \( u(\omega) \).

Say that player \( i \) is rational at \( \omega \) if

\[
a_i(\omega) \in \arg \max_{u_i \in A_i, a_{-i} \in A_{-i}} u_i(\omega)(a_i, a_{-i}) \phi_i(\omega)(a_{-i}).
\]

That is, \( i \) is rational at \( \omega \) if his strategy \( a_i(\omega) \) maximizes expected payoff when the payoff function is \( u_i(\omega) \) and the conjecture is \( \phi_i(\omega) \).

Given any event \( E \), say that player \( i \) knows \( E \) at \( \omega \) if \( H_i(\omega) \subseteq E \). Define \( K_i(E) \) as the set of all those \( \omega \) at which \( i \) knows \( E \). Set \( K(\omega) = K_1(\omega) \cap \cdots \cap K_n(\omega) \). That is, \( K(\omega) \) is the event that all players know \( E \). If \( \omega \in K(\omega) \), say that \( E \) is mutually known at \( \omega \). For any positive integer \( m \), define \( K^{n+1}(\omega) = K^1(K^n(\omega)) \). Define \( CK(\omega) = \cap_{m=1}^{\infty} K^m(\omega) \). If \( \omega \in CK(\omega) \), say that \( E \) is commonly known at \( \omega \).

For any event \( E \), say that \( E \) is self-evident to player \( i \) if \( K_i(E) = E \) (that is, if \( E \) is the union of some of \( i \)'s partition cells). An event that is simultaneously self-evident to all the players is called a public event. Aumann (1976) provides the following convenient characterization of commonly known events.

**Proposition 3.** An event \( E \) is commonly known at \( \omega \) if and only if there exists a public event \( E \) such that \( \omega \in E \subseteq F \).

A probability measure \( p \) on \( \Omega \) is called a common prior of the partitional structure if for all \( i \) and all \( \omega \), \( p(H_i(\omega)) > 0 \) and

\[
p_i(\tilde{\omega}, \omega) = \frac{p(\tilde{\omega})}{p(H_i(\omega))} \quad \forall \tilde{\omega} \in H_i(\omega).
\]

Note that if the partitional structure has a common prior \( p \), then \( i \)'s conjecture \( \phi_i(\omega) \) defined in (1) will be the conditional probability measure which is derived from \( p \) as follows:

\[
\phi_i(\omega)(a_{-i}) = \frac{p([a_{-i}] \cap H_i(\omega))}{p(H_i(\omega))} \quad \forall a_{-i} \in A_{-i}.
\]

(2)

It is well known that conditional probabilities satisfy the sure-thing principle. That is, given any two disjoint events \( F \) and \( G \), if the probability of an event is equal to \( \alpha \) conditional on \( F \), and similarly, if the probability of the event is also \( \alpha \) conditional on \( G \), then the probability of the event conditional on \( F \cup G \) is also \( \alpha \). Therefore, for any nonempty event \( E \) that is self-evident to \( i \), if \( E \subseteq \{ \omega \in \Omega: \phi_i(\omega) = \phi_j \} \), then (2) and the sure-thing principle imply that

\(^3\)See, for instance, Aumann (1976) for an application of this result.
\[
\phi_i(a_{-i}) = \frac{p([a_i] \cap E)}{p(E)} \quad \forall a_{-i} \in A_{-i}.
\]

(3)

Recall that the function \(a_i\) is measurable with respect to \(H_i\). This implies that for all \(a_i \in A_i\), the event \([a_i] = \{\omega \in \Omega : a_i(\omega) = a_i\}\) is self-evident to \(i\). Clearly, the intersection of two self-evident events is a self-evident event. Therefore, for any \(a_i \in A_i\) such that \([a_i] \cap E \neq \emptyset\), (3) can be written as

\[
\phi_i(a_{-i}) = \frac{p([a_i] \cap [a_i] \cap E)}{p([a_i] \cap E)} \quad \forall a_{-i} \in A_{-i}.
\]

(4)

Note that (3) and (4) are also valid for \(i\)’s conjecture over any subset of his opponents. That is, \([a_{-i}]\) in the two equations can be replaced by \(\cap_{j \notin i} [a_j]\), where \(J\) is any subset of \([1, \ldots, i-1, i+1, \ldots, n]\). The equivalence between (3) and (4) is a key to the proof of Proposition 5 in the sequel.

Finally, say that a partitional structure has full support if for all \(i\) and all \(\omega\), \(p(\omega; \omega) > 0\).

3. Decision-theoretic transformation

Fix a mutually absolutely continuous belief system \(\{T_i, \hat{p}_i, \hat{u}_i, \hat{a}_i\}\). Define

\[S = \{t \in T : \hat{p}(t; t_i) > 0\}\]

Since the belief system is mutually absolutely continuous, the definition of \(S\) does not depend on \(i\). Say that a partitional structure \(\{\Omega, H, \hat{p}, \hat{u}, \hat{a}\}\) is a decision-theoretic transformation of \(\{T_i, \hat{p}_i, \hat{u}_i, \hat{a}_i\}\) if

\[\Omega = S\]

and for all \(t \in S\),

\[
\begin{align*}
H_i(t) &= \{\hat{t} \in S : \hat{t}_i = t_i\}, \\
p_i(\hat{t}; t) &= \hat{p}_i(\hat{t}; t_i) \quad \forall \hat{t} \in S, \\
u_i(t) &= \hat{u}_i(t_i), \\
a_i(t) &= \hat{a}_i(t_i);
\end{align*}
\]

in addition, if the belief system has a common prior \(\hat{p}\), then the partitional structure also has a common prior \(p\) such that for all \(t \in S\), \(p(t) = \hat{p}(t)\).

Proposition 4 below provides some useful relationship between a mutually absolutely continuous belief system and its decision-theoretic transformation. Its main message is that belief of rationality and conjectures in a belief system can be restated as knowledge of rationality and conjectures in its decision-theoretic transformation.

**Proposition 4.** Fix a mutually absolutely continuous belief system \(\{T_i, \hat{p}_i, \hat{u}_i, \hat{a}_i\}\) and its decision-theoretic transformation \(\{\Omega, H, \hat{p}, \hat{u}, \hat{a}\}\). Then

(a) \(\{\Omega, H, \hat{p}, \hat{u}, \hat{a}\}\) has full support;
(b) for any game \( u \), \( \{ t \in T : \tilde{u}(t) = u \} \cap S = \{ \omega \in \Omega : u(\omega) = u \} \);
(c) for any \( n \)-tuple \( \phi \) of conjectures, \( \{ t \in T : \phi(t) = \phi \} \cap S = \{ \omega \in \Omega : \phi(\omega) = \phi \} \);
(d) \( \{ t \in T : \) every player \( i \) of type \( t_i \) is rational \( \} \cap S = \{ \omega \in \Omega : \) every player \( i \) is rational at \( \omega \) \);
(e) for all \( E \subseteq T \) and for all positive integers \( m \), \( B^m(E) \cap S = K^m(E \cap S) \);
(f) for all \( E \subseteq T \), \( CB(E) \cap S = CK(E \cap S) \).

**Proof.** Parts (a), (b), (c) and (d) follow from the definition of decision-theoretic transformation, and (f) follows from (e). To establish (e), note that for all \( t \in T \) and for all \( E \subseteq T \),

\[
t \in B_i(E) \cap S \iff \{ \tilde{t} \in T : \phi_i(t_i) > 0 \} \subseteq E \quad \text{and} \quad t \in S \\
\iff H_i(t) \subseteq E \cap S \\
\iff t \in K_i(E \cap S)
\]

Therefore,

\[
B_i(E) \cap S = K_i(E \cap S). \tag{5}
\]

This implies that \( \bigcap_{i=1}^n B_i(E) \cap S = \bigcap_{i=1}^n K_i(E \cap S) \). That is, \( B^1(E) \cap S = K^1(E \cap S) \). Since (5) holds for all \( E \subseteq T \), we have \( B(B(E)) \cap S = K(B(E) \cap S) \) and, therefore, \( \bigcap_{i=1}^n B_i(B(E)) \cap S = \bigcap_{i=1}^n K_i(B(E) \cap S) \). That is, \( B^2(E) \cap S = K^2(E \cap S) \). Clearly, the argument can be repeated as many times as desired to conclude that for any positive integer \( m \), \( B^m(E) \cap S = K^m(E \cap S) \). \( \square \)

Proposition 4 implies that Proposition 1 can be reformulated as Proposition 5 below. We also provide a proof of Proposition 5 which is relatively more transparent than the proof of Theorem B in Aumann and Brandenburger (1995, pp. 1168–1169).

**Proposition 5.** Consider a game \( u \), an \( n \)-tuple \( \phi \) of conjectures, and a full support partitional structure with a common prior \( p \). Suppose there is a state \( \omega \) such that \( \{ \tilde{\omega} \in \Omega : \) every player \( i \) is rational at \( \tilde{\omega} \} \cap \{ \tilde{\omega} \in \Omega : u(\tilde{\omega}) = u \} \) are mutually known at \( \omega \) and \( \{ \tilde{\omega} \in \Omega : \phi(\tilde{\omega}) = \phi \} \) is commonly known at \( \omega \). Then there is an \( n \)-tuple \( \sigma = (\sigma_1, \ldots, \sigma_n) \), where \( \sigma_i \) is a probability measure on \( A_i \), such that \( \phi_i = \times \sigma_{i \neq j} \sigma_j \) and \( \sigma \) is a Nash equilibrium of \( u \).

**Proof.** The hypothesis that \( \{ \tilde{\omega} \in \Omega : \phi(\tilde{\omega}) = \phi \} \) is commonly known at \( \omega \) implies agreement of marginal conjectures: \( \text{marg}_{\phi_i}(\phi_i) = \text{marg}_{\phi_i}(\phi_i) \) (Aumann, 1976, p. 1237). This enables us to define, independent of \( j \), \( \sigma_i = \text{marg}_{\phi_i}(\phi_i) \). According to (3), there exists a public event \( E \subseteq \{ \tilde{\omega} \in \Omega : \phi(\tilde{\omega}) = \phi \} \) such that

\[
\sigma_i(a_i) = \frac{p([a_i] \cap E)}{p(E)} \forall a_i \in A_i. \tag{6}
\]

Without loss of generality, prove \( \phi_i = \times \sigma_{i \neq j} \sigma_j \). Given (3) and (6), it suffices to establish that for all \( a_{i-1} \in A_{i-1} \),

\[
\frac{p([a_2] \cap \cdots \cap [a_n] \cap E)}{p(E)} = \frac{p([a_2] \cap E)}{p(E)} \cdot \frac{p([a_3] \cap E)}{p(E)} \cdot \cdots \cdot \frac{p([a_{n-1}] \cap E)}{p(E)} \frac{p([a_n] \cap E)}{p(E)}. \tag{7}
\]
If there exists \( j \neq 1 \) such that \([a_i] \cap E = \emptyset\), then it is obvious that both sides of (7) are zero. Suppose \([a_i] \cap E \neq \emptyset\) for all \( j \neq 1 \). According to the argument for the equivalence between (3) and (4), it is legitimate to write

\[
\frac{p([a_i] \cap E)}{p(E)} = \frac{p([a_i] \cap [a_i] \cap E)}{p([a_i] \cap E)},
\]

(8)

implying that the right hand side of (7) can be simplified to

\[
\frac{p([a_i] \cap [a_i] \cap E)}{p(E)} \cdot \frac{p([a_i] \cap E)}{p(E)} \cdot \ldots \cdot \frac{p([a_i] \cap E)}{p(E)}.
\]

Simplifying (9) and its subsequent expressions using the same argument as many times as desired, the right hand side of (7) will eventually be transformed to its left hand side.

All the knowledge hypotheses in the proposition imply that for every \( i \) and for every \( j \neq i \),

\[
H_i(\omega) \subseteq \{\omega \in \Omega: \text{player } i \text{ is rational at } \omega\} \cap \{\omega \in \Omega: u_i(\omega) = u_i\} \cap \{\omega \in \Omega: \phi_i(\omega) = \phi_i\}.
\]

That is, every \( a_i(\omega):\omega \in H_i(\omega) \} \) maximizes \( i \)'s expected payoff when the payoff function is \( u_i \) and the conjecture is \( \phi_i \), where \( \phi_i \) has been shown to be equal to \( \times_{j \neq i} \sigma_j \). Since the partitional structure has full support and \( \sigma_i \) is \( j \)'s marginal conjecture about \( i \) at \( \omega \), the support of \( \sigma_i \) is actually\( a_i(\omega):\omega \in H_i(\omega) \}. \) Therefore, \( \sigma \) is a Nash equilibrium of \( u \).

Proposition 4 also implies that Proposition 2 can be reformulated as Proposition 6 below.

**Proposition 6.** Consider a full support partitional structure description of the \( N \)-period repeated prisoner’s dilemma. Suppose there is a state \( \omega \) such that \( \{\omega \in \Omega: \text{every player } i \text{ is rational at } \omega\} \) is commonly known at \( \omega \). Then at state \( \omega \), both players defect in every period of the game.

**Proof.** According to Proposition 3, there exists a public event \( E \) such that \( \omega \in E \subseteq \{\omega \in \Omega: \text{every player } i \text{ is rational at } \omega\} \). Since the partitional structure has full support, \( p_i(\omega; \omega) > 0 \) for all \( \omega \in E \). These two facts imply that for every \( \omega \in E \), the hypotheses stated in Claim (a) below are satisfied. The claim establishes that at every \( \omega \in E \), both players defect in the last period of the game. Given Claim (a), Claim (b) below can be applied repeatedly to conclude that at every \( \omega \in E \), both players defect in every period of the game. We provide a proof of Claim (b), which can be easily simplified to prove Claim (a).

**Claim a.** Suppose that at a state \( \omega \), \( i \) is rational and \( p_i(\omega; \omega) > 0 \). Then at \( \omega \), \( i \) defects in the last period of the game.

**Claim b.** Suppose that at a state \( \omega \), \( i \) is rational and \( p_i(\omega; \omega) > 0 \). Suppose that at every state in \( H_i(\omega) \), \( j \) defects in the last \( M \) periods of the game (where \( 1 \leq M \leq N - 1 \)). Then at \( \omega \), \( i \) defects in the last \( M + 1 \) periods of the game.

**Proof of Claim b.** Suppose, to the contrary, there exists \( K \geq M + 1 \) such that \( i \) does not defect in period \( K \) at \( \omega \). By hypothesis, at every state in \( H_i(\omega) \), \( j \) defects in the last \( M \) periods. This guarantees
that if $i$ follows $\mathbf{a}_i(\tilde{\omega})$ only in the first $N - (M + 1)$ periods and then switches to defection in the last $M + 1$ periods, his payoff at every state in $H_i(\tilde{\omega})$ will not decrease and that at $\tilde{\omega}$ will strictly increase. This and $\mathbf{p}_i(\tilde{\omega};\omega) > 0$ imply that his expected payoff at $\tilde{\omega}$ will also strictly increase, contradicting the hypothesis that $i$ is rational at $\tilde{\omega}$. □

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