Strong Condorcet efficiency of scoring rules

Dominique Lepelley\textsuperscript{a,}\textsuperscript{*}, William V. Gehrlein\textsuperscript{b}

\textsuperscript{a}GEMMA-CREME, University of Caen, 14032 Caen Cedex, France
\textsuperscript{b}University of Delaware, Delaware, DE, USA

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Abstract

In an election, an alternative is said to be a strong Condorcet winner when more than 50\% of the voters rank this alternative first in their preference orders. The strong Condorcet efficiency of a voting rule is defined as the probability of electing the strong Condorcet winner, given that such an alternative exists. In this paper, we provide some analytical representations for the strong Condorcet efficiency of some specific scoring rules in three-alternative elections.

Keywords: Voting theory; Condorcet efficiency; Borda rule

JEL classification: D7

1. Introduction

In an election, an alternative $A$ is a Condorcet winner when every other alternative is beaten by $A$ in a series of pairwise comparisons by majority rule; and $A$ is said to be a strong Condorcet winner when more than 50\% of the voters rank $A$ first in their preference orders. The Condorcet criterion, which requires the selection of the Condorcet winner when such an alternative exists, certainly is the most commonly used principle for evaluating alternative voting rules.

A well known result in social choice theory establishes that every scoring rule violates the Condorcet criterion. Scoring rules constitute an important class of voting rules (of which plurality rule and Borda rule are the best known examples), where the winner is determined by computing a score that depends on the rank of the alternatives in the voters’ preference orders. Numerous studies have considered the Condorcet efficiency of various scoring rules, i.e. the conditional probability that these rules will elect the Condorcet winner, given that a Condorcet winner exists.
A probably less known result is the following: every scoring rule but one, the plurality rule, can fail to select the strong Condorcet winner when such an alternative exists (see Lepelley and Merlin, 1998). Lepelley and Gehrlein (1999) showed that the probability of having a strong Condorcet winner is far from being negligible in three-alternative elections when the Impartial Anonymous Culture condition is assumed. It is therefore of interest to compute, for alternative scoring rules, a measure of their propensity to elect the strong Condorcet winner. We define the strong Condorcet efficiency of a voting rule as the conditional probability of electing the strong Condorcet winner, given that such an alternative exists. The aim of this paper is to provide some analytical representations for the strong Condorcet efficiency of several scoring rules (including the Borda rule) in three-alternative elections. We obtain representations for the situation in which individual preferences are unrestricted in Section 2, and for the case of single-peaked preferences in Section 3.

2. Representations for the strong Condorcet efficiency of some scoring rules

With three alternatives \{A,B,C\} there are six linear preference orders that voters might have on these alternatives (without precluding any pattern of individual opinions):

1. \(ABC\); 2. \(ACB\); 3. \(BAC\);
4. \(CAB\); 5. \(BCA\); 6. \(CBA\).

Let \(n_i\) denote the number of voters having the linear order number \(i\). A voting situation is a vector \(s = (n_1, \ldots, n_6)\). The Impartial Anonymous Culture condition (IAC), that we shall use throughout the paper, assumes that every possible voting situation is equally likely to occur.

Consider a three-alternative election with \(n\) voters. We know from Gehrlein and Fishburn (1976) that the number of elements in the set, \(N(n)\), of possible voting situations is given by \(|N(n)| = \frac{\prod_{i=1}^{5}(n+i)}{120}\). Lepelley and Gehrlein (1999) showed that the cardinality of the set \(SC(n)\) of voting situations for which a strong Condorcet winner exists is (for \(n\) odd)

\[|SC(n)| = \frac{(n+1)(n+3)(n+5)(n+7)(3n+7)}{640}\]

and the probability that a strong Condorcet winner exists under IAC for \(n\) odd is given by \(P_{SC}(n)\), with

\[P_{SC}(n) = \frac{3(n+7)(3n+7)}{16(n+2)(n+4)}\].

Let \(SCE(R,n)\) be the strong Condorcet efficiency of rule \(R\) when voters preferences are unrestricted and IAC is assumed. The first scoring rule we consider is the negative plurality rule. This rule requires voters to vote for their two most preferred alternatives and the winner is the alternative receiving the most votes.

**Result 1.** If \(n\) is an odd multiple of three and \(n - 1\) a multiple of four, then
\[ \text{SCE(Negative plurality, } n) = \frac{197n^3 + 1145n^2 + 1623n - 405}{108(3n + 7)(n + 5)(n + 1)} \]

**Proof.** Let \( NP^A(n) \) denote the set of voting situations with \( n \) voters for which \( A \) is both the strong Condorcet winner and the negative plurality winner. The conditions for a voting situation to belong to \( NP^A(n) \) are given by:

\[
\begin{align*}
    n_1 + n_2 &> n/2, \\
    n_2 + n_4 &> n_5 + n_6, \text{ and } n_1 + n_3 > n_5 + n_6.
\end{align*}
\]

Assuming that \( n \) is odd and taking into account the fact that the \( n_i \)s sum up to \( n \), these inequalities are equivalent to the following ones:

\[
\begin{align*}
    0 &\leq n_{56} \leq \frac{n - 3}{3}, \\
    n_{56} + 1 &\leq n_{13} \leq n - 1 - 2n_{56}, \\
    0 &\leq n_3 \leq \text{Min} \left[ \frac{n - 1}{2} - n_{56}, 0 \leq n_4 \leq \text{Min} \left[ n - n_{13} - n_{56}, \frac{n - 1}{2} - n_{56} - n_3 \right] \right].
\end{align*}
\]

Here, \( \text{Min}[x,y] \) is the minimum of \( x \) and \( y \), and \( n_{ij} = n_i + n_j \). In order to eliminate the \( \text{Min} \) arguments, we partition the set \( NP^A(n) \) into five subsets defined as follows (it is assumed that \( n \) is an odd multiple of three and \( n - 1 \) a multiple of four):

\[
\begin{align*}
    s &\in NP^A_1(n) \iff 0 \leq n_{56} \leq \frac{n - 5}{4}, n_{56} + 1 \leq n_{13} \leq n - 1 - 2n_{56}, 0 \leq n_3 \leq n_{13}, 0 \leq n_4 \\
    &\leq \frac{n - 1}{2} - n_{56} - n_3; \\
    s &\in NP^A_2(n) \iff 1 \leq n_{56} \leq \frac{n - 5}{4}, \frac{n + 1}{2} - n_{56} \leq n_{13} \leq \frac{n - 1}{2}, 0 \leq n_3 \leq \frac{n - 1}{2} - n_{56}, 0 \leq n_4 \\
    &\leq \frac{n - 1}{2} - n_{56} - n_3; \\
    s &\in NP^A_3(n) \iff 0 \leq n_{56} \leq \frac{n - 5}{4}, \frac{n + 1}{2} \leq n_{13} \leq n - 1 - 2n_{56}, 0 \leq n_3 \leq n_{13} - \frac{n + 1}{2}, 0 \leq n_4 \\
    &\leq n - n_{56} - n_{13}; \\
    s &\in NP^A_4(n) \iff 0 \leq n_{56} \leq \frac{n - 5}{4}, \frac{n + 1}{2} \leq n_{13} \leq n - 1 - 2n_{56}, n_{13} - \frac{n + 1}{2} \leq n_3 \\
    &\leq \frac{n - 1}{2} - n_{56}, 0 \leq n_4 \leq \frac{n - 1}{2} - n_{56} - n_3; \\
    s &\in NP^A_5(n) \iff \frac{n - 1}{4} \leq n_{56} \leq \frac{n - 3}{3}, n_{56} + 1 \leq n_{13} \leq n - 1 - 2n_{56}, 0 \leq n_3 \leq \frac{n - 1}{2} - n_{56}, \\
    0 &\leq n_4 \leq \frac{n - 1}{2} - n_{56} - n_3.
\end{align*}
\]

By using known relations for sums of powers of integers, we can obtain an algebraic representation for the cardinality of each of these five subsets. We then obtain the cardinality of \( NP^A(n) \) as:
Thus, given the symmetry of IAC with respect to $A, B$ and $C$, the joint probability that an alternative $A$ is both the strong Condorcet winner and the negative plurality winner is $\frac{3|NP^A|}{|N(n)|}$. This probability is then divided by $P_{SC}(n)$ to obtain the strong Condorcet efficiency of the negative plurality rule. □

We now consider the strong Condorcet efficiency of the well-known Borda rule, which requires voters to rank order the three alternatives. Each voter’s first ranked alternative is given 2 points, the second ranked alternative is given 1 point, and the third ranked alternative is given 0 point. The winner is the alternative receiving the most total points.

**Result 2.** If $n$ is an odd multiple of three and $n - 1$ a multiple of four, then

$$SCE(\text{Borda}, n) = \frac{2(39n^5 + 690n^4 + 4370n^3 + 12420n^2 + 17991n + 13770)}{27(3n + 7)(n + 7)(n + 5)(n + 3)(n + 1)}$$

**Proof.** Let $BO^A(n)$ be the set of voting situations for which alternative $A$ is both the strong Condorcet winner and the winner by Borda rule, and let $BO^{B > A}(n)$ denote the set of voting situations for which $A$ is the strong Condorcet winner and for which $B$ beats $A$ by Borda rule. It is easily checked that a strong Condorcet winner cannot obtain the lowest score under the Borda rule. The symmetry of IAC with respect to $B$ and $C$ then implies that

$$|BO^A(n)| = |SC(n)| - 2|BO^{B > A}(n)|.$$

Since we know the cardinality of $SC(n)$, we only need a representation for the cardinality of $BO^{B > A}(n)$. A voting situation belongs to $BO^{B > A}(n)$ if and only if

$$n_1 + n_2 > n/2 \text{ and } 2(n_3 + n_5) + n_1 + n_6 > 2(n_1 + n_2) + n_3 + n_4.$$

As the $n_i$s sum up to $n$, it follows that (with $n$ odd):

$$0 \leq n_{56} \leq \frac{n - 1}{2}, \ 0 \leq n_5 \leq n_{56}, \ \text{Max}(0, \frac{n - n_5 - 2n_{56}}{2}) \leq n_3 \leq \frac{n - 1}{2} - n_{56},$$

$$0 \leq n_4 \leq \frac{n - 1}{2} - n_{56} - n_3, \ 0 \leq n_2 \leq 2n_3 + n_5 + 2n_{56} - n.$$

These inequalities are quite similar to those obtained by Gehrlein and Lepelley (1998) when evaluating the Condorcet efficiency of Borda rule: the only difference concerns the upper limit of $n_4$ which is $(n - 1/2) - n_{56} - n_3$ instead of $(n - 1/2) - n_{56}$. So we have just to adapt the partitioning used by Gehrlein and Lepelley (1998) to finally obtain:
The cardinality of $BO^A(n)$, then the probability that the strong Condorcet winner and the Borda winner coincide, and finally the strong Condorcet efficiency of Borda rule follow from this representation.

The third scoring rule we consider is the Coombs rule, which is (in three-alternative elections) a two-stage procedure: the voters vote for their two most preferred alternatives, and the alternative receiving the fewest votes is eliminated. The remaining two candidates are then carried to a second stage, where the winner is determined by majority rule.

**Result 3.** If $n$ is an odd multiple of 3, then

$$SCE(Coombs, n) = \frac{79n^2 + 546n + 675}{27(3n + 7)(n + 5)}$$

**Proof.** Alternative $A$ is the strong Condorcet winner and is not the unique Coombs winner if and only if $n_{12} > n/2$, $n_{56} \geq n_{13}$ and $n_{56} \geq n_{24}$. We deduce from these inequalities the following restrictions:

$$\frac{n}{3} \leq n_{56} \leq \frac{n - 1}{2}, n - 2n_{56} \leq n_{13} \leq n_{56},$$

$$0 \leq n_{3} \leq \frac{n - 1}{2} - n_{56}, 0 \leq n_{4} \leq \frac{n - 1}{2} - n_{56} - n_{3},$$

where $n$ is assumed to be an odd multiple of three. From these inequalities, a representation for the corresponding number of voting situations can be directly obtained, and the desired probability easily follows.

Table 1 shows computed values of the strong Condorcet efficiency of negative plurality, Borda and Coombs rules using the representations from above.

From these values, we see that the strong Condorcet efficiency of the three scoring rules under consideration increases as the number of voters increases. Moreover, our results indicate a rather good performance of both the Borda and the Coombs rules, whereas negative plurality gives rise to the selection of an alternative different from the strong Condorcet winner in about $[1 - (197/324)](9/16) = 22.05\%$ of the possible voting situations when the number of voters is large (this proportion is $2.08\%$ for the Borda rule and $1.39\%$ for the Coombs rule).

It is of interest to compare the above results to those obtained for the Condorcet efficiency of scoring rules under IAC. Denoting by $CE(R,n)$ the IAC Condorcet efficiency of rule $R$ in three-alternative elections, we have in the limit of the voters (see, e.g., Gehrlein and Lepelley, 1998): $CE(\text{Negative plurality}, \infty) = 197/324$, $CE(\text{Borda}, \infty) = 78/81$ and $CE(\text{Coombs}, \infty) = 79/81$. Put together with our results, these figures show that, when a Condorcet winner is not elected with a rule $R$ in an election with a large electorate, then the probability that this Condorcet winner is a strong
Table 1
Strong Condorcet efficiency (three alternatives): unrestricted preferences

<table>
<thead>
<tr>
<th>n</th>
<th>Negative Plurality</th>
<th>Borda</th>
<th>Coombs</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.48739</td>
<td>0.90756</td>
<td>0.93277</td>
</tr>
<tr>
<td>21</td>
<td>0.54656</td>
<td>0.93549</td>
<td>0.95604</td>
</tr>
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<td>33</td>
<td>0.56654</td>
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</tr>
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<td>45</td>
<td>0.57673</td>
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<td>0.96620</td>
</tr>
<tr>
<td>57</td>
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<td>0.95233</td>
<td>0.96810</td>
</tr>
<tr>
<td>69</td>
<td>0.58703</td>
<td>0.95414</td>
<td>0.96935</td>
</tr>
<tr>
<td>81</td>
<td>0.58999</td>
<td>0.95543</td>
<td>0.97023</td>
</tr>
<tr>
<td>93</td>
<td>0.59222</td>
<td>0.95639</td>
<td>0.97089</td>
</tr>
<tr>
<td>105</td>
<td>0.59396</td>
<td>0.95713</td>
<td>0.97139</td>
</tr>
<tr>
<td>117</td>
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<td>0.95773</td>
<td>0.97179</td>
</tr>
<tr>
<td>129</td>
<td>0.59650</td>
<td>0.95821</td>
<td>0.97212</td>
</tr>
<tr>
<td>141</td>
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<td>0.95861</td>
<td>0.97239</td>
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<td>153</td>
<td>0.59826</td>
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<td>0.97262</td>
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<td></td>
<td></td>
<td></td>
<td>0.97531</td>
</tr>
</tbody>
</table>

Condorcet winner is 63.5% if $R$ is the negative plurality\(^1\), 25% if $R$ is the Borda rule, and 50% if $R$ is the Coombs rule.

3. The case with single-peaked preferences

An interesting question is to investigate what the above results become when single-peakedness is assumed, i.e. when some degree of consensus exists among the voters, so that some patterns of individual preferences are precluded. Black (1958) noticed that Coombs rule always selects the Condorcet winner, and hence the strong Condorcet winner, when voters’ preferences are single-peaked. However, single-peakedness is not sufficient to secure the election of the strong Condorcet winner when negative plurality or Borda is used.

When only three alternatives are in contention, single-peakedness simply means that one alternative is never ranked in last position by the voters. We shall assume that this alternative is $B$. Thus, the four admissible linear preference orders are\(^2\):

1. $ABC$; 2. $BAC$;
3. $BCA$; 4. $CBA$.

\(^1\)This result is obtained as follows: $[(1 - 197/324)9/16]/[(1 - 68/108)15/16] = 127/200 = 0.635$, where $15/16$ is the IAC probability of having a Condorcet winner in the limit of the voters.

\(^2\)Another way to deal with the single-peakedness assumption in our model is to consider that the alternative which is never ranked last is not given but chosen at random. The two ways of doing are not equivalent for small values of $n$ (the number of voters) but lead to the same results when $n$ is large.
We know from Lepelley and Gehrlein (1999) that the number of elements in the set $N^{sp}(n)$ of single-peaked voting situations is given as:

$$|N^{sp}(n)| = \frac{(n + 1)(n + 2)(n + 3)}{6}.$$

Moreover, the cardinality of the set $SC^{sp}(n)$ of voting situations in which a strong Condorcet winner exists is (Gehrlein and Lepelley, 1998):

$$|SC^{sp}(n)| = \frac{(n + 1)(n + 3)^2}{8}$$

and

$$P_{SC}^{sp}(n) = \frac{3(n + 3)}{4(n + 2)}$$

where $P_{SC}^{sp}(n)$ is the proportion of single-peaked voting situations with a strong Condorcet winner and $n$ is supposed to be odd. Let $SCE^{sp}(R,n)$ denote the strong Condorcet efficiency of rule $R$ when voters’ preferences are single-peaked and when it is assumed that every single-peaked voting situation is equally likely to occur.

**Result 4.** If $n$ is odd, then

$$SCE^{sp}(\text{Negative plurality}, n) = \frac{2(n - 1)(n + 6)}{3(n + 3)^2}$$

**Proof.** When preferences are single-peaked in three-alternative elections, the median alternative $B$ cannot be beaten under negative plurality. Thus, the set of voting situations in which the strong Condorcet winner exists and is not the only winner by negative plurality is constituted of three subsets of voting situations: (i) the subset with $A$ the strong Condorcet winner, (ii) the subset with $C$ the strong Condorcet winner, (iii) the subset in which $B$ is the strong Condorcet winner and is not the only negative plurality winner. It is easily shown, with $n$ odd, that the cardinality of each of the two first subsets is $[(n + 1)(n^2 + 8n + 15)/12]$, and that the third subset contains $n + 1$ elements. The strong Condorcet efficiency of negative plurality is derived from these observations. □

**Result 5.** If $n$ is an odd multiple of three, then

$$SCE^{sp}(\text{Borda}, n) = \frac{2(4n^2 + 15n + 15)}{9(n + 3)(n + 1)}$$

**Proof.** Under single-peakedness, the Borda scores of the alternatives are the following: $s(A) = 2n_1 + n_2$, $s(B) = 2(n_2 + n_3) + n_1 + n_4$ and $s(C) = 2n_4 + n_3$. It can be shown that, if the strong Condorcet winner is not the Borda winner, then the strong Condorcet winner is either $A$ or $C$. Suppose that the strong Condorcet winner is $A$; if an alternative obtains a better Borda score than $A$, then this alternative necessarily is $B$. So, $A$ will be the strong Condorcet winner and will not be the unique Borda winner if and only if $n_1 > n/2$ and $2(n_2 + n_3) + n_1 + n_4 \geq 2n_1 + n_2$, and this implies ($n$ being an odd multiple of three):

$$\frac{n + 1}{2} \leq n_1 \leq \frac{2n}{3}, 0 \leq n_2 \leq 2n - 3n_1, 2n_1 - n \leq n_3 \leq n - n_1 - n_2.$$
Table 2
Strong Condorcet efficiency (three alternatives): single-peaked preferences

<table>
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<tr>
<th>n</th>
<th>Negative plurality</th>
<th>Borda</th>
</tr>
</thead>
<tbody>
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<td>9</td>
<td>0.55556</td>
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<td>0.66165</td>
<td>0.88739</td>
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<tr>
<td>153</td>
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</tr>
<tr>
<td>∞</td>
<td>0.66667</td>
<td>0.88889</td>
</tr>
</tbody>
</table>

The number of voting situations corresponding to these inequalities is \( (n^3 + 9n^2 + 15n - 9)/144 \). Hence, given the symmetry of A and C, the total number of voting situations in which the strong Condorcet winner is the unique Borda winner is

\[
\frac{(n + 1)(n + 3)^2}{8} - 2 \frac{n^3 + 9n^2 + 15n - 9}{144} = \frac{4n^3 + 27n^2 + 60n + 45}{36}
\]

and the strong Condorcet efficiency of Borda rule follows from this representation.

Computed values of \( SCE^{sp}(\text{Negative plurality}, n) \) and \( SCE^{sp}(\text{Borda}, n) \) are given in Table 2. These values indicate that single-peakedness (slightly) improves the ability of the negative plurality rule to elect the strong Condorcet winner, a rather expected result. On the other hand, it is surprising to find that the strong Condorcet efficiency of the Borda rule decreases when single-peakedness is assumed: with this assumption, the Borda rule selects an alternative different from the strong Condorcet winner in about \( (1 - \frac{8}{9})^{\frac{3}{4}} = 8.33\% \) of the voting situations when the number of voters is large.

References