An axiomatic characterization of Yitzhaki’s index of individual deprivation

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Abstract

This note presents and discusses a number of properties a measure of individual deprivation should satisfy. The axioms we propose are necessary and sufficient to characterize Yitzhaki’s index of deprivation making the structure of the index more transparent. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction\textsuperscript{1}

Twenty years ago Yitzhaki (1979) provided a quantification of the concept of individual deprivation suggested earlier by Runciman (1966). This concept supposes that an individual compares herself with some reference group, namely with all persons having a commodity which she would also like to have. Yitzhaki made this concept more precise by considering income as the relevant variable. He assumed that an individual feels deprived if there is anybody with a higher income. Her deprivation with respect to this person is measured by the corresponding income gap; her total deprivation is a normalized sum of these income gaps (summed over all individuals belonging to the reference group of relatively richer people). This sum defines an individual index of deprivation. Furthermore Yitzhaki

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demonstrated that the aggregation of the individual indices over society results in the corresponding absolute Gini coefficient.

Thus measurement of social, or overall, deprivation involves a two-stage process. At first a deprivation profile is defined which consists of the list of individual deprivations felt by each individual in society. In a second step these individual indices are aggregated into overall deprivation. So far, all indices proposed in the literature (Yitzhaki, 1979; Chakravarty and Chakraborty, 1984; Berrebi and Silber, 1985; Paul, 1991; Chakravarty and Mukherjee, 1999) have been derived by means of this approach. Up to now most papers focused on the aggregation stage taking for granted that individual $k$’s deprivation is suitably measured by

$$D_k(X) = \frac{1}{n} \sum_{i=k+1}^{n} (X_i - X_k)$$

where $X$ denotes the vector of incomes and incomes are ordered increasingly.

This note will present and discuss a number of properties a measure of individual deprivation should satisfy. The axioms proposed are sufficient to characterize Yitzhaki’s index by means of a simple straightforward proof. They describe the reference group, introduce a type of additive decomposability, postulate anonymity, establish the way different incomes are taken into account and normalize the index, respectively. The preceding makes the structure and the fundamental properties of the index of deprivation transparent. No regularity condition such as differentiability or continuity needs to be employed.

The note is organized as follows. In Section 2 the fundamental axioms are proposed and discussed. Section 3 provides the characterization of Yitzhaki’s index and discusses further properties the index fulfills and their relationship to the axioms. Finally Section 4 offers some conclusions. In particular possibilities of aggregating the individual indices into an index of social deprivation are discussed.

2. Framework and properties

We consider a fixed homogeneous population $N\{1, \ldots, n\}$ of $n(n \geq 2)$ individuals. They are identical, but generally differ in income. A feasible income distribution or situation $X$ is given by an income vector $(X_1, \ldots, X_n) \in \mathbb{R}^n$ where $X_i$ is individual $i$’s income, $i = 1, \ldots, n$. Let $P(N)$ be the power set of $N$ and $P_k(N) = \{S \in P(N) | k \in S \text{ and } S \neq \emptyset\}$ be the set of all subsets of $N$ not containing $k$, for $k \in N$. For any income distribution $X$ and any nonempty subset $S$ of $N$, we denote the income vector of all individuals belonging to $S$ by $X(S)$. It is defined by $X(S) = (X_{i_1}, \ldots, X_{i_m})$ if $m = |S|$ is the number of individuals and if $S$ can be written as $\{i_1, \ldots, i_m\}$ where $i_1 < i_2 < \ldots < i_m$.

In this section we examine the deprivation of an arbitrary individual $k \in N$. We propose.

**Definition.** $d_k(X, S)$ denotes an index of individual $k$’s deprivation with respect to subgroup (or reference group) $S$, given the income distribution $X$, for any $X \in \mathbb{R}^n$ and $S \in P_k(N)$.

This definition is a bit more general than the usual one since it does not only apply to $N$ (the total population), but also to arbitrary subgroups not containing individual $k$. Of course, up to now, the meaning of $d_k(X, S)$ is rather vague. In the following we introduce a number of axioms or properties
an index of individual deprivation should satisfy. The choice of axioms is always based on (subjective) value judgements.

The first axiom restricts the class of indices considerably:

2.1. Axiom INDependence of irrelevant individuals

For all \( X, Y \in \mathbb{R}^n \) and \( i \in N \neq k \):

\[
X_i = Y_i \text{ and } X_j = Y_j \text{ imply } d_k(X, \{i\}) = d_k(Y, \{i\}).
\]

Here the deprivation of individual \( k \) in two different situations is compared. The choice of the income distributions \( X \) and \( Y \) is restricted: individual \( k \) and individual \( i \), respectively, receive the same income. Then individual \( k \)’s deprivation with respect to \( i \) has also to be the same. As a consequence, the income distributions \( X(N - \{k, i\}) \) and \( Y(N - \{k, i\}) \) can be arbitrary \((N - S)\) denotes the complement of \( S \) in \( N \)) without affecting \( d_k(x, \{i\}) \) or \( d_k(Y, \{i\}) \). Only the incomes of individual \( k \) and \( i \) are relevant. In particular the rank of \( k \)’s or \( i \)’s income in \( X \) does not play any role. Thus this axiom cannot be fulfilled by any index \( d_k(X, S) \) using positional information of incomes with respect to \( X \) or employing any income from \( X(N - (S \cup \{k\})) \).

Similarly, the incomes of individuals who are poorer than \( k \) are not to be taken into account:

2.2. Axiom FOCus

If \( X \not\leq X \) then \( d_k(X, S) = d_k(X, S - \{i\}) \) for all \( i \in S, S \in P_k(N) \) and \( X \in \mathbb{R}^n \).

Property FOC reflects the basic idea that individual \( k \) compares her income situation only to that of individuals who are (relatively) richer. So any income \( X_i \leq X_k \) for \( i \in S \) has no impact on individual \( k \)’s deprivation.

Up to now the naming of individuals might be of concern. This is excluded by

2.3. Axiom ANONymity

Consider any permutation \( \pi \) of \( N \) such that \( \pi(k) = k \). Then \( d_k(X, S) = d_k(X^{\pi}, S^{\pi}) \) for all \( S \in P_k(N) \) and \( X \in \mathbb{R}^n \) where \( X^{\pi} = (X_{\pi(1)}, \ldots, X_{\pi(n)}) \) and \( S^{\pi} = \{\pi(i) \mid i \in S\} \).

The numbering of individuals different from \( k \) is not relevant for individual deprivation. This property is a kind of anonymity. As a consequence any information dependent on the numbering must not be used: all individuals are treated equally.

The next property is related to the choice of the reference group \( S \):

2.4. Axiom Additive DEComposition

\[
d_k(X, S) = d_k(X, S_1) + d_k(X, S_2) \text{ if } S = S_1 \cup S_2, S_1 \cap S_2 = \emptyset \text{ for all } S, S_1, S_2 \in P_k(N) \text{ and } X \in \mathbb{R}^n.
\]

It essentially postulates that only individuals count: the size of the respective subgroup \( S \) or any relation between members of \( S \) are not (directly) relevant. Deprivation is an ‘additive’ concept. Repeated application of DEC demonstrates that overall individual deprivation is equal to the sum of the deprivation individual \( k \) feels with respect to each single individual belonging to the reference group \( S \).

The particular way individual \( k \) compares her income with others is described by
3. Axiom INVariance

\[ d_k(X + \alpha 1, S) = d_k(X, S) \] for all \( X \in \mathbb{R}^n \), \( \alpha \in \mathbb{R} \) and \( S \in P_k(N) \), where \( 1 = (1, \ldots ,1) \in \mathbb{R}^n \).

The index of individual deprivation does not change whenever all incomes are changed by the same amount. Therefore deprivation depends on the absolute difference between \( X_k \) and higher incomes. INV is comparable to the invariance property of absolute inequality measures (cf. e.g. Blackorby and Donaldson (1980), Ebert (1988)).

In contrast to relative inequality measures an index of individual deprivation is influenced by the scale of incomes.

4. Axiom LINear homogeneity

\[ d_k(\lambda, X, S) = \lambda d_k(X, S) \] for all \( X \in \mathbb{R}^n \), \( \lambda \in \mathbb{R}^+ \) and \( S \in P_k(N) \).

A proportional change of all incomes increases or decreases deprivation by the same factor. Formally the index is sensitive to the unit of income, or more precisely, differences in living standards as measured by income gaps are reflected by the index of individual deprivation: If for example all incomes double, society is twice as rich and deprivation doubles too.

Finally, the index is normalized by

4.1. Axiom NORMalization

\[ d_k((0, \ldots ,0,1),(n)) = \frac{1}{n} \] for \( k < n \).

Of course, the choice of the normalization rule is arbitrary. Here the deprivation individual \( k \) feels with respect to the richest individual, individual \( n \) (the difference of income is one unit) is normalized to \( 1/n \), i.e. the income gap is divided by the number of individuals in \( N \). NORM will imply that \( d_k(x, N) \) can be interpreted as the average income gap (see below).

5. Characterization and further properties

The axioms proposed above restrict an index of individual deprivation. They are consistent with one another (i.e. they are not contradictory) and sufficient to characterize exactly one index. We are able to establish the following proposition:

**Proposition.** An index of individual deprivation satisfies the axioms IND, FOC, ANON, DEC, INV, LIN, and NORM if and only if it is identical to

\[ d^*_k(X, S) \frac{1}{n} \sum_{i \in S \setminus S_k} (X_i - X_k) \] for all \( S \in \mathbb{R}^n \), \( S \in P_k(N) \).

\( d^*_k(X, S) \) measures the ‘average’ income gap between individual \( k \) and all individuals belonging to \( S \) and being richer than \( k \), normalized by the population size \( n \). \( d^*_k(X, N - \{k\}) \) is nothing other than the relative deprivation function of person \( k \) proposed by Yitzhaki (1979) as a quantification of
Runciman’s (1966) concept. As will be discussed in Section 4 \( d_k^*(x, N - \{k\}) \) is only one possible index. Whenever we drop one of the above axioms other possibilities arise. It is important to stress that no regularity condition is required for the characterization of \( d_k^*(X, S) \). Continuity is implied implicitly.

The proof of the proposition is straightforward.

**Proof.** Assume that \( d_k(X, S) \) satisfies the properties listed in the proposition. FOC and repeated application of DEC imply

\[
d_k(X, S) = \sum_{i \in S, x_i > x_k} d_k(x, \{i\}).
\]

IND implies that there is an elementary function \( f_i: \{x, y\} | x, y \in \mathbb{R}, x < y \} \rightarrow \mathbb{R} \) such that

\[
d_k(X, \{i\}) = f_i(x_k, x_i) \text{ for } i \in N, i \neq k
\]

since incomes \( X_j > X_k \) for \( j \neq i, j \neq k \) must not play any role (cf. the discussion of IND in Section 2). Because of ANON \( f_i = f_j = f \) for \( i, j \in N \).

INV guarantees that

\[
f(X_k + \alpha, X_i + \alpha) = f(X_k, X_i) \text{ for all } X \in \mathbb{R}^n, \alpha \in \mathbb{R}, i \neq k.
\]

Set \( \alpha := -X_k \) and obtain

\[
d_k(X, \{i\}) = f(0, X_i - X_k).
\]

Because of LIN we get

\[
f(0, t) = ct
\]

and continuity of \( d_k(X, S) \).

NORM implies that \( c_k = 1/n \). This proves that \( d_k(X, S) \) has to be identical to \( d_k^*(X, S) \). The converse is obvious. \( \square \)

The axioms used are independent. The Appendix demonstrates that, for each axiom, an index exists satisfying the remaining ones, which differs from the solution characterized.

In addition to the properties mentioned the index of individual deprivation \( d^*(X, S) \) possesses further properties\(^2\) which might be useful for the reader. Some are given here. The respective axioms implying them are listed in square brackets in order to relate the properties to the basic axioms. We obtain

\[
d_k(X, \emptyset) = 0 \quad [\text{DEC}, \text{set } S_2 := \emptyset]
\]

If individual \( k \) compares herself with nobody then there is no deprivation.

\[
\text{If } X_i \leq X_k \text{ then } d_k(X, \{i\}) = 0. \quad [\text{FOG}, \text{DEC}]
\]

\(^2\)(ii), (iv), and (ix) are also discussed by Paul (1991), (iv), (ix), (x), (xi) by Chakravarty and Chakraborty (1984).
Individuals, poorer than \( k \), do not contribute to her feeling of deprivation. The proof is immediate: Suppose that \( i \in S \) and \( X_i \leq X_k \) then by DEC
\[
d_i(X, S) = d_i(X, S - \{i\}) + d_i(X, \{i\})
\]
Now apply FOC to get \( d_i(X, \{i\}) = 0 \).
\[
d_i(X, S) = 0 \text{ if } X_i \geq X_i \text{ for } i \in S. \quad \text{[IND, FOC, DEC]} \tag{iii}
\]
The richest individual does not feel deprived.
\[
d_i(X, S) \text{ is decreasing in } X_i \text{ for all } X \in \mathbb{R}^n \text{ and } S \in P_i(N). \quad \text{[all axioms except FOC]} \tag{iv}
\]
If individual \( k \)'s situation improves, i.e. if her income increases, her deprivation decreases correspondingly.
\[
d_i(X, S) \geq 0 \quad \text{[all axioms]} \tag{v}
\]
The index is nonnegative. The essential property in proving this claim is monotonicity (iv).
\[
\text{If } X_k \leq X_i \text{ then } d_i(X, S) \geq d_i(X, S). \quad \text{[all axioms]} \tag{vi}
\]
Whenever the reference group used is the same, the deprivation of an individual richer than \( k \) is not greater than \( k \)'s deprivation.
\[
d_i(X, S) \text{ is independent of a rank-preserving increase in income } X_i \text{ if } X_i < X_k. \quad \text{[IND, FOC, DEC]} \tag{vii}
\]
As long as somebody stays poorer than individual \( k \), an increase in her income does not matter.
\[
d_i(X, S) \text{ is independent of a progressive transfer between two individuals being poorer than individual } k. \quad \text{[IND, FOC, DEC]} \tag{viii}
\]
Such individuals are not taken into account by \( d_i^p(X, S) \).
\[
\text{If } X_k < X_i \text{ and } i \in S, \text{ an increase of } X_i \text{ increases } d_i(X, S). \quad \text{[all axioms except FOC]} \tag{ix}
\]
Such a change obviously increases deprivation.
\[
\text{A progressive transfer among individuals } i \text{ and } j, \text{ being richer than } k, \text{ leaves } d_i(X, S) \text{ unchanged if-}\quad i, j \in S. \quad \text{[IND, FOC, ANON, DEC, INV, LIN]} \tag{x}
\]
Since the index is linear and is determined by the sum of the income gaps, it is not affected by this kind of progressive transfer.
\[
\text{A rank-preserving progressive transfer from any individual } i \text{ richer than } k \text{ to anybody poorer than } k \text{ decreases } d_i(X, S) \text{ if } i \in S. \quad \text{[all axioms]} \tag{xi}
\]
In this case individual \( k \)'s aggregate income gap is diminished.
If the population is replicated, an individual having the same income $X_k$ in both populations feels the same deprivation with respect to the rest of the respective society. (xii)

This property means that the index $d_k(x, N - \{k\})$ satisfies the so-called principle of population initially proposed by Dalton (1920) for inequality measures. Then the degree of individual deprivation depends on the distribution of income (in the statistical sense). It is independent of the actual size of the population. This property is essentially an implication of NORM.

6. Conclusion

The proposition proved above shows that the axioms introduced are consistent: there is (exactly) one index satisfying all of them. It is easy to see by constructing simple examples that they are independent (cf. the Appendix), i.e. it is impossible to dispense with any of them. On the other hand several authors have proposed other indices of individual deprivation (Berrebi and Silber, 1985; Paul, 1991; Chakravarty et al., 1995). For instance, Paul’s index is defined on $\mathbb{R}_{++}$ and does not satisfy INV, REL and NORM, but satisfies the remaining axioms. Therefore the latter could be supplemented by others to obtain an appropriate characterization of this measure. Insofar this note also lays the foundations for an axiomatization of further indices.

Yitzhaki (1979) and Chakravarty and Mukherjee (1999) demonstrate the close relationship between the index of individual deprivation and an index reflecting satisfaction which could be called the dual. The same is true, of course, for the properties of either index. Thus the discussion of Section 2 and 3 can be translated to an examination of satisfaction indices (and conversely).

Finally we briefly comment on aggregate or social indices of deprivation. They measure the total deprivation of society. Since $d_k^n(X, N - \{k\})$ is equal to the average income gap of individual $k$ it is an obvious idea to define an aggregate index by averaging and to use a symmetric mean. Chakravarty and Chakraborty (1984), Paul (1991) and Chakravarty et al. (1995) follow up this idea. In our framework some additional axioms have to be employed. The details need not be discussed here since Diezert (1993) provides a survey on and several characterizations of symmetric means. The interested reader is referred to this excellent source.

Appendix A

The following indices $d_k^h$ satisfy all properties IND, FOC, DEC, ANON, INV, LIN, and NORM but one.

Let $\pi$ be a permutation such that $X_{(i)} = (X_{(1)}, \ldots, X_{(n)}) = (X_{\pi(1)}, \ldots, X_{\pi(n)})$ where $X_{\pi(i)} \leq X_{\pi(i+1)}$ for $1 \leq i \leq n - 1$ and given $X$. Then define

$$d_k^h(X, S) = \frac{1}{n} \sum_{X_{(i)} > X_k} (n - i + 1)(X_{(i)} - X_k)$$

(not IND)
\[
d_2^i(X, S) = \frac{1}{n} \sum_{i \in S} (X_i - X_k) \quad \text{(not FOC)}
\]
\[
d_3^i(X, S) = \sqrt{\frac{1}{n^2} \sum_{X_i > X_k} (X_i - X_k)^2} \quad \text{(not DEC)}
\]
\[
d_4^i(X, S) = \frac{1}{n} \sum_{X_i \leq X_k} (n - i + 1)(X_i - X_k) \quad \text{(not ANON)}
\]
\[
d_5^i(X, S) = \frac{1}{n} \sum_{X_i \leq X_k} \sqrt{\left(X_i^2 - X_k^2\right)} \quad \text{(not INV)}
\]
\[
d_6^i(X, S) = \frac{1}{n} \sum_{X_i > X_k} (X_i - X_k)^2 \quad \text{(not LIN)}
\]
\[
d_7^i(X, S) = \sum_{X_i > X_k} (X_i - X_k) \quad \text{(not NORM)}
\]

References


