On resource monotonicity in the fair division problem

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Abstract

In the context of the classical fair division problem, we show that Efficiency and Resource Monotonicity are incompatible with the following “Conditional Equal Split” condition: If equal split of the collective endowment is efficient, it should be among the recommended allocations. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

In the so-called fair division problem, a group of agents must choose how to split a bundle of private goods to which they are collectively entitled. Since no one has any individual property right, sharing the collective endowment equally is probably fair. It is, however, typically inefficient. This opens the problem of defining an equitable efficient sharing rule.

The axiomatic approach to this problem starts by defining properties that one might wish to require from such a rule. Resource Monotonicity is one such property. It requires that no agent be worse off when the collective endowment grows. While this condition is compatible with efficiency, it turns out to be quite demanding. Moulin and Thomson (1988) have shown that Efficiency and Resource Monotonicity are incompatible with either the so-called Equal Split Lower Bound axiom, which demands that every agent find what he receives at least as good as the average bundle, or the No Domination principle, which requires that no agent receive less of every good than any other agent. When preferences are monotonic, No Domination is weaker than the familiar No Envy condition demanding that no agent ever prefer somebody else’s bundle to his own.
The purpose of this note is to point out another difficulty. We consider the following “Conditional Equal Split” condition, introduced in the literature by Thomson (1988) under the name “Condition α”: Whenever equal split of the collective endowment is efficient, it should be among the recommended allocations. If the crux of the fair division problem is really to reconcile equality with efficiency, this axiom seems rather natural. We show that Efficiency, Resource Monotonicity, and Conditional Equal Split are incompatible.

2. A formal expression of the result

Our notation for vector inequalities is $\succ$, $>$, $\geq$. There is a finite set $L$ containing $l \geq 2$ goods and a set $N$ containing $n \geq 2$ agents. For each $i \in N$, we denote agent $i$’s preference by $R_i$ and we write $I_i$ and $P_i$ for the indifference and strict preference relations associated with $R_i$. Preferences belong to some subset $\mathcal{R}$ of the set of classical preferences. By a classical preference we mean a complete, transitive, continuous, monotone and convex preference relation on $\mathbb{R}_+^N$. Monotonicity is understood in the strict sense: $R_i$ is monotonic if $x_iP_iy_i$ whenever $x_i > y_i$. A (division) problem (or economy) is a list $e = (R, \omega)$, where $R \in \mathcal{R}^N$ is the agents’ preference profile and $\omega \in \mathbb{R}_+^L$ is the collective endowment to be split among them. We let $E = \mathcal{R}^N \times \mathbb{R}_+^L$. An allocation for the problem $e \in E$ is a vector $x \in \mathbb{R}_+^{Nn}$ satisfying the feasibility constraint $\sum_{i \in N} x_i \leq \omega$, where, for each $i \in N$, $x_i = (x_{i1}, \ldots, x_{in})$ denotes the bundle allocated to agent $i$. We denote by $Z(e)$ the set of allocations for $e \in E$. A rule $F$ assigns to each $e \in E$ a nonempty set $F(e) \subseteq Z(e)$.

Efficiency. For every $e \in E$, $F(e) \subseteq P(e)$.

We now turn to Resource Monotonicity. Because we allow for multi-valued rules, the general idea of resource monotonicity can be formulated in many different ways. The requirement we use is the strongest version; for weaker versions, see Geanakoplos and Nalebuff (1988) and Moulin (1991).

Resource Monotonicity. For every $e = (R, \omega) \in E$ and $e' = (R, \omega') \in E$ such that $\omega' \succeq \omega$, $x_i^IR_i x_i$ for all $x \in F(e)$, all $x' \in F(e')$, and all $i \in N$.

Finally, we give the formal definition of the conditional equal split property.

Conditional Equal Split. For every $e \in E$ such that $(\frac{m_i}{n}, \ldots, \frac{m_n}{n}) \in P(e)$, $(\frac{m_i}{n}, \ldots, \frac{m_n}{n}) \in F(e)$.

We are now ready to show that the three conditions just stated are generally incompatible. Of course, this requires some richness assumption on the preference domain: The conditions would be compatible if $\mathcal{R}$ were, say, the set of Cobb–Douglas preferences, as the familiar Walrasian rule from equal split shows.

Theorem. If the preference domain $\mathcal{R}$ contains the homothetic classical preferences, no rule satisfies Efficiency, Resource Monotonicity, and Conditional Equal Split.
**Proof.** Suppose \( L = N = \{1,2\} \). Let \( e = ((R_1,R_2),\omega) \in E \) satisfy the following conditions. The classical preference \( R_1 \) is homothetic and has marginal rate of substitution of \(-6\) along the ray \( x^2 = \frac{2}{3}x^1 \), \(-1\) along the ray \( x^2 = 2x^1 \), and \(-\frac{1}{6}\) along the ray \( x^2 = \frac{3}{2}x^1 \). Furthermore, \( (2,4)P_1(\frac{15}{2},3) \). The preference \( R_2 \) is symmetric to \( R_1 \), that is, \((x^1,x^2)\sim_{R_2}(y^1,y^2)\) if and only if \((x^1,x^2)\sim_{R_1}(y^1,y^2)\). It is easy to see that such preferences exist, as illustrated in Fig. 1. Finally, \( \omega = (6,6) \).

Observe that \((2,4),(4,2)\) \( P(e) \). Let \( x = (x_1,x_2) \in F(e) \). By Efficiency, \( x_1R_1(2,4) \) or \( x_2R_2(4,2) \). If the former statement is true, let \( \omega' = (15,6) \) and \( x' \in F(e') \). By Resource Monotonicity, \( x'_1R_1x_1 \), hence \( x'_1P_1(\frac{15}{2},3) = \omega' \). This contradicts Conditional Equal Split since \((\frac{15}{2},3)\sim_{P(e)}(\frac{15}{2},3)\). By a symmetrical argument, supposing \( x_2R_2(4,2) \) leads to a similar contradiction. The incompatibility is easily extended to more than two goods and two agents. \( \square \)

We conclude with two remarks. (1) If all preferences in \( \mathcal{R} \) are strictly convex, Equal Split Lower Bound implies Conditional Equal Split. On the other hand, it is easy to construct efficient envy-free rules that violate Conditional Equal Split. (2) In any two-agent classical economy \( e \) whose efficient allocations are ordered (in the sense that \( x_1 \equiv y_1 \) or \( x_1 \geq y_1 \) whenever \( x,y \in P(e) \)), it follows from either the combination of Efficiency and No Domination, or Equal Split Lower Bound alone, that equal split must be among the recommended allocations whenever it is efficient. Since efficient allocations are ordered in two-agent economies with strictly convex homothetic classical preferences, Conditional Equal Split may be replaced with either No Domination or the Equal Split Lower Bound in the statement of our theorem.

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References