The first-order stochastic dominance ordering of the Singh–Maddala distribution

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Abstract

Given two distributions from the Singh–Maddala family, this paper investigates how to determine whether one distribution first-order stochastically dominates the other. The resulting criteria are also applied to the Dagum type I family of distributions. © 2000 Elsevier Science S.A. All rights reserved.

Keywords: Ranking income distributions; Singh–Maddala distribution; Inequality; Social welfare

JEL classification: C49; D31; D63

1. Introduction

The family of distributions proposed by Singh and Maddala (1976) has been a popular model for describing the distribution of income or consumption expenditure (see McDonald, 1984; Brachmann et al., 1996). The cumulative distribution function (cdf) is given by

\[ F(x; a, b, q) = 1 - \left[ 1 + \left(\frac{x}{b}\right)^{a} \right]^{-q}. \]  

In empirical applications, the parameters \( b, a \) and \( q \) are estimated to facilitate intertemporal or international comparisons of income distributions with a view to drawing conclusions about inequality and social welfare. Where inequality is concerned, comparisons are usually made using the Lorenz-ordering. For the Singh–Maddala family, Wilfling and Krämer (1993) have derived necessary and sufficient conditions to determine the Lorenz-ordering of two distributions in terms of the parameters \( a \) and \( q \). Since it is mean-free, however, the Lorenz-ordering does not provide an answer to the question which one of two distributions implies higher social welfare. If we focus on the subclass of

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additively separable welfare functions satisfying monotonicity in each individual’s income, then a comparison with respect to social welfare will be equivalent to a comparison of the associated cdf’s with respect to first-order stochastic dominance (FSD). Moreover, FSD implies higher order stochastic dominance as well as generalized Lorenz-dominance (see Cowell, forthcoming) which is a central concept for comparing income distributions in many fields of economics.\footnote{See Lambert (1989) for an application to public finance.}

Distribution $G$ is said to (weakly) first-order stochastically dominate $H$ if, and only if, $G(x) \leq H(x)$ for all $x$ (see Cowell, forthcoming). Although some distribution-free tests for FSD have been developed (see Schmid and Trede, 1996, for a recent example), virtually no attempts have been made to address this issue in the context of parametric distributions. This I shall attempt to do for the Singh–Maddala (SM) family.

As in the case of the Lorenz-ordering, the FSD ordering of distributions from the Singh–Maddala family is not complete, i.e. in many cases the cdf’s will intersect. Moreover, while for the Lorenz-ordering the scale parameter $b$ plays no role, the FSD ordering will depend on all three parameters, rendering it impossible to give a set of closed form ‘if, and only if’ conditions.

2. Necessary and sufficient conditions for first-order stochastic dominance

Theorem 1 gives necessary conditions for first-order stochastic dominance.

**Theorem 1.** Let $F_1$ and $F_2$ be SM distribution functions, with parameters $a_i$, $b_i$ and $q_i$ ($i = 1, 2$), respectively. If $F_1$ first-order stochastically dominates $F_2$, then

(a) $a_1 \geq a_2$ and

(b) $a_1 q_1 \leq a_2 q_2$.

**Proof.** Define the family of strictly increasing functions $u_i(x) = x^{t_i}/t, t \neq 0$ and the corresponding family of additively separable social welfare functions $W_i(F) = \int u_i(x) dF(x) = \mu_i(F)/t$, where $\mu_i(F)$ denotes the $r$th moment associated with $F$. We further need the following representation of the $r$th moment of the SM family obtained by McDonald (1984):

$$\mu_i(F) = b^r \Gamma(1 + t/a) \Gamma(q - t/a)/\Gamma(q)$$

where $\Gamma(\cdot)$ denotes the complete gamma function.

(a) Assume that $a_1 < a_2$ and let $t$ approach $-a_1$ from above. In this case, $(1 + t/a_1)$ will approach zero. Inspection of (2) and recalling that $\lim_{z \to 0} \Gamma(z) = \infty$ reveals that this implies that $W_i(F_1)$ will approach minus infinity, while $W_i(F_2)$ will approach a finite negative number. Thus for some $t' > -a_1$ we have $W_i(F_1) < W_i(F_2)$. Since $F_1(x) \leq F_2(x)$ for all $x$ implies $W_i(F_1) \geq W_i(F_2)$ for all $t$ (see Saposnik, 1981), $W_i(F_1) < W_i(F_2)$ contradicts $F_1 \leq F_2$.

(b) Assume that $a_1 q_1 > a_2 q_2$ and let $t$ approach $a_2 q_2$ from below. Now the term $\Gamma(q_2 - t/a_2)$ and thus $W_i(F_2)$ will approach plus infinity, while $W_i(F_1)$ will approach a finite positive number. Thus for some $t^* < a_2 q_2$ we have $W_i(F_1) < W_i(F_2)$, which contradicts $F_1 \leq F_2$. QED
It is interesting to note that the necessary conditions given in Theorem 1 are in direct contrast to those for Lorenz-dominance obtained by Wilfling and Krämer (1993) who showed that \( F_1 \) Lorenz-dominates \( F_2 \) if, and only if, \( a_i \geq a_2 \) and \( a_i q_1 \geq a_2 q_2 \). Thus, if two distributions can be Lorenz-ordered, there will be no ordering according to FSD and vice versa. This is a serious drawback of the SM family since, in general, a distribution \( G \) can first-order and Lorenz-dominate a distribution \( H \) at the same time.

We state an inequality for sums introduced by Pringsheim (1902a,b; see also Hardy et al., 1952, Theorem 19) that will be needed for the proof of Theorem 2:

**Lemma 1.** For positive \( r, p \) and \( c_k \), \( k = 1, \ldots, n \), \( (\Sigma_{k=1}^n c_k^r)^{1/r} \leq (\Sigma_{k=1}^n c_k^p)^{1/p} \) if, and only if, \( r \geq p \).

A set of sufficient conditions for first-order stochastic dominance is set out in:

**Theorem 2.** If \( a_1 \geq a_2 \), \( a_1 q_1 \leq a_2 q_2 \) and \( b_1 \geq b_2 \), then \( F_1 \) first-order stochastically dominates \( F_2 \).

**Proof.** For the SM family \( F_1 \leq F_2 \) is equivalent to

\[
(1 + (x/b_1)^{a_1})^{q_1} \leq (1 + (x/b_2)^{a_2})^{q_2}.
\]

Since \( (1 + (x/b)^{a})^{q} \) is decreasing in \( b \) and increasing in \( q \) and, by hypothesis, \( b_1 \geq b_2 \) and \( a_1 q_1 \leq a_2 q_2 \), we have \( (1 + (x/b_1)^{a_1})^{q_1} \leq (1 + (x/b_2)^{a_2})^{q_2} \). It thus suffices to show that \( (1 + (x/b_2)^{a_2})^{q_2} \leq (1 + (x/b_2)^{a_2})^{q_2} \) which, after a change of variable, is equivalent to \( (1 + z^{a_2})^{q_2} \leq (1 + z^{a_2})^{q_2} \) for all positive \( z \). That this holds for all \( 0 < a_2 \leq a_1 \) follows at once from Lemma 1. QED

The next theorem covers the special cases where either (a) or (b) of Theorem 1 holds with equality and, as a third case, \( b_1 \geq b_2 \):

**Theorem 3.**

(a) Suppose \( a_1 = a_2 = a \). Then \( F_1 \) first-order stochastically dominates \( F_2 \) if, and only if, \( q_1 \leq q_2 \) and \( b_1 / b_2 = (q_1/q_2)^{1/a} \).

(b) Suppose \( a_1 q_1 = a_2 q_2 \). Then \( F_1 \) first-order stochastically dominates \( F_2 \) if, and only if, \( a_1 \geq a_2 \) and \( b_1 \geq b_2 \).

(c) Suppose \( b_1 \geq b_2 \). Then \( F_1 \) first-order stochastically dominates \( F_2 \) if, and only if, \( a_1 \geq a_2 \) and \( a_1 q_1 \leq a_2 q_2 \).

**Proof.** (a) In this case, after a change of variable, \( F_1 \leq F_2 \) is equivalent to \( (1 + z)^{a_1/q_2} \leq (1 + b_1/b_2)^{a_2} \) for all positive \( z \). Clearly, this inequality will hold for all \( z \) if, and only if, \( q_1/q_2 \) is not bigger than unity and, for \( z = 0 \), the derivative of the left hand side is not bigger than the derivative of the right hand side. This latter condition is equivalent to \( b_1 / b_2 \geq (q_1/q_2)^{1/a} \).

(b) Necessity of \( a_1 \geq a_2 \) follows from Theorem 1. For the necessity of \( b_1 \geq b_2 \), rewrite \( F_1 \leq F_2 \) as \( b_2 / b_1 \leq (1 + (b_2 x)^{a_2})^{1/a_2} / (1 + (b_1 x)^{a_1})^{1/a_1} \). The right hand side of this inequality approaches unity for large \( x \). Thus \( b_2 / b_1 \geq 1 \) contradicts \( F_1 \leq F_2 \). Sufficiency follows from Theorem 2.

(c) Necessity follows from Theorem 1, sufficiency from Theorem 2. QED

In many cases, Theorems 1, 2 and 3 will not suffice to determine whether the cdf’s of two
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distributions under comparison intersect or not, namely when conditions (a) and (b) of Theorem 1 both hold with strict inequality together with \( b_2 > b_1 \). Therefore I provide a table that, together with Theorem 1, fills this gap. First note that \( F_1 \leq F_2 \) is equivalent to
\[
(b_2/b_1)^{z^2} \leq z/(1 + z^{(a_1/a_2)q_1/q_2} - 1)
\] (4)
for all positive \( z \). Further, the necessary conditions of Theorem 1 can be written as
\[
1 \leq a_1/a_2 \leq q_2/q_1.
\] (5)
Table 1 therefore reports the minimum of the r.h.s. of (4) with respect to \( z, v(a_1/a_2, q_2/q_1) \), for pairs of \( a_1/a_2 \) and \( q_2/q_1 \) over a range that is sufficient for most applications and that satisfies (5). Thus, \( F_1 \) first-order stochastically dominates \( F_2 \) if, and only if, (a) and (b) of Theorem 1 hold and \( (b_2/b_1)^{z^2} \leq v(a_1/a_2, q_2/q_1) \).

3. The Dagum type I family

The cdf of the Dagum type I model (Dagum, 1977) is given by
\[
G(x; \tilde{a}, \tilde{b}, \tilde{q}) = (1 + (\tilde{b}/x)^{\tilde{a}})^{-\tilde{q}}.
\]
A comparison of this with the SM family can be found in Kleiber (1996), who also obtained necessary and sufficient conditions for Lorenz-dominance.

Rearranging (3) together with a change of variable yields the following lemma.

**Lemma 2.** Let \( G_1 \) and \( G_2 \) be Dagum type I distribution functions with parameters \( \tilde{a}_i, \tilde{b}_i, \tilde{q}_i \) \( (i=1,2) \), respectively, and define \( F(x; \tilde{a}, \tilde{b}, \tilde{q}) \) as in (1). Then \( G_1 \) first-order stochastically dominates \( G_2 \) if, and only if, \( F(x; \tilde{a}_2, 1/\tilde{b}_2, \tilde{q}_2) \) first-order stochastically dominates \( F(x; \tilde{a}_1, 1/\tilde{b}_1, \tilde{q}_1) \).

Thus all results obtained for the SM family in Section 2 can be applied to the Dagum type I family if we replace \( a_1, b_1 \) and \( q_1 \) by \( \tilde{a}_2, 1/\tilde{b}_2 \) and \( \tilde{q}_2 \), respectively, and \( a_2, b_2 \) and \( q_2 \) correspondingly.

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