The law of demand implies limits to chaos

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Abstract

This article shows that there is a regular progression of increasingly strong form of the law of demand, both in compensated and uncompensated versions, which increasingly restrict the dimensionality of the attractor that occurs under tatonnement. In their weakest forms, these conditions impose very little structure on aggregate demand, whereas in their strongest forms, they yield global stability of equilibrium. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

There is a large gap between the essentially arbitrary tatonnement dynamics permitted by the standard framework of general equilibrium theory and conditions sufficient for the global stability of equilibrium suggested by the invisible hand tradition. The trouble, of course, is with the possible distribution of individual income effects, since individual substitution effects are as well-behaved as one could want, obeying a generalized law of compensated demand for all price changes. However, since recent empirical work suggests that actual distributions of consumers are far from arbitrary,\footnote{See for instance, Hildenbrand (1994). Note, however, that empirical work has primarily concerned so-called distribution economies, and that for exchange economies an endowment effect must also be accounted for.} one may expect the aggregate law of demand to survive in the uncompensated form for some if not all price changes. This raises the question of whether this buys one anything dynamically, that is whether...
aggregate demand that is well-behaved for some but not all price changes has dynamic phenomena any less complex than that permitted by completely arbitrary demands. We show that, indeed, there is a regular progression of increasingly strong forms of the generalized law of demand, in either compensated or uncompensated form, which imply increasingly severe restrictions on the size of the limiting behavior possible for tatonnement dynamics. The strongest of these conditions yield the standard results concerning the global stability of equilibrium. The weakest of these conditions, on the other hand, are quite mild forms of the law of demand, so that together this string of results contributes toward closing the aforementioned gap between arbitrary dynamics and those that assure the stability of a single equilibrium.

2. The dimensionality of attractors

We work with the tatonnement process

$$\dot{p} = z(p) \quad p \in R^{n+1}$$

(1)

where the excess demand $z(p)$ has the standard properties of a regular exchange economy – smoothness, homogeneity of degree zero, Walras’ Law, and a boundary condition for extreme prices. Since Walras’ Law assures that the Euclidean distance $\|\phi(p_0,t)\|$ remains constant for any forward solution $\phi(p_0,t)$ of the above differential equation, the dynamics can be considered to be a vectorfield living on $S^{n+} = \{ p \in R^{n+1}^+ \mid \|p\| = 1 \}$, the positive portion of the unit sphere. The boundary condition for such an economy assures that forward trajectories exist throughout time, and so, we can think in terms of the action of the dynamics, $\phi_t: S^{n+} \to S^{n+}$, over the entire price domain, for any given time. Since our interest is only in the long-run behavior of the dynamics, we concentrate on describing the system’s global attractor,

$$A = \bigcap_{t>0} \phi_t(S^{n+})$$

(2)

which necessarily forms a compact, invariant set (i.e. $\phi(A) = A$).

Now, the size of any compact set $A$ can be expressed in terms of its Hausdorff dimension. Since this definition is somewhat complicated, we merely record it and make some observations; the reader may wish to consult Falconer (1997) or Leonov et al. (1996) for a more detailed discussion. Given $A$, for any two numbers $d$ and $\epsilon$, cover $A$ by balls of radius $r_i \leq \epsilon$, and put

$$\mu(A,d,\epsilon) = \inf \sum r_i^d$$

(3)

where the lower bound is taken over all finite coverings of $A$. The function $\epsilon \to \mu(A,d,\epsilon)$ is decreasing and therefore has a limit as $\epsilon \to 0$; so, let the $d$-dimensional Hausdorff measure of $A$ be

$$\mu(A,d) = \lim_{\epsilon \to 0} \mu(A,d,\epsilon)$$

(4)
It can be shown that there exists a positive real number $d_0$ such that $\mu(A,d) = \infty$ for $d < d_0$ and $\mu(A,d) = 0$ for $d > d_0$. This number $d_0(A) \equiv d_0$ is then simply called the Hausdorff dimension of $A$. Note that, while for a smooth $m$-dimensional manifold $A$, the Hausdorff dimension will of course be $m$, the attractors of smooth dynamics can themselves be quite non-smooth, and may, indeed, have fractional Hausdorff dimensions (being so-called fractal sets.)

Consider the Jacobian $Dz(p)$ representing uncompensated price effects and symmetrize it to form $J(p) = Dz(p) + Dz(p)^T$, whose eigenvalues we rank in decreasing size as $\lambda_1(p), \ldots, \lambda_{n+1}(p)$. We say that $z(p)$ obeys the law of demand of order $i$ if $\lambda_1 + \cdots + \lambda_{i-1} < 0$. Note that in the extreme case of order $n$, all eigenvalues but one are negative, which is the greatest possible, since by homogeneity, $J(p)$ can be at most negative semidefinite, never negative definite. We refer to this condition as the strong generalized law of demand. The other extreme is order 1, where all that is required is that the sum of the eigenvalues, which coincides with (twice) the sum of the diagonal own effects of $Dz(p)$, should be negative in total. Here all that is required is that negative own effects should dominate any positive ones; this mild condition may be referred to as the weak generalized law of demand. We have the following result.

**Theorem 1.** If aggregate demand $z(p)$ obeys the law of demand of order $i$, then the dimension of the attractor $A$ must be less than $n + 1 - i$. Economics obeying the strong law of demand (i.e., of order $n$) have a globally stable equilibrium.

**Proof.** The above falls out of a more general result in $n + 1$ dimensions saying that if for some $d \in [0,n]$ and $s \in [0,1]$, $\lambda_1(p) + \cdots + \lambda_d(p) + s\lambda_{d+1}(p) < 0$ for all $p \in A$, then $\dim A < d + s$. In an abstract form, suitable for infinite dimensions, this result is first developed in Douady and Oesterle (1980.) The finite-dimensional version for differential equations is presented in Smith (1986.) Notice that this more general result allows for “laws of demand” of fractional orders. Translating the result to our context of $(n + 1)$ price dimensions immediately yields only $\dim A < n + 2 - i$, but since the attractor in $R^{(n+1)+}$ must be but a radial extension of the one lying in $S^n$, we can subtract another dimension in restricting our attention to the dynamics there. Notice that for order $n$, we only immediately know that the attractor is of dimension less than 1. This, however, requires the attractor to consist of components of dimension 0, since all trajectories other than equilibria are of dimension 1. At any such equilibrium, $p^T J(p) = 0^T$, as well as $J(p)p = 0$, so this requires the one nonnegative eigenvalue to in fact be zero, and thus for each of the equilibria to be locally stable. This, in turn, implies a unique, hence globally stable equilibrium, by the usual argument of Dierker (1972). □

**Remarks.** The above result for the strong law of demand is actually stronger than the usual result for monotone demands, since while a negative semidefinite Jacobian $Dz(p)$ of rank $n$ certainly implies the two largest eigenvalues of $J(p)$ are negative in sum, the converse does not follow, away from equilibrium.

The result for the weak law of demand is just Liouville’s theorem, the classical result that dynamics where the Jacobian has a negative trace necessarily shrink volumes, so that the limiting set must be of zero measure. When there are but three goods, a negative trace implies global stability (see Keenan and Rader, 1985), so even the weak law of demand is sufficient for stability.
An aggregate compensated price change is one such that \( z(p) \cdot dp = 0 \), that is \( dp \in T_{z(p)} = \{x \in R^{n+1}| x \cdot z(p) = 0\} \). Thus, aggregate compensated price effects concern \( Dz(p)|_{T_{z(p)}} \). Given that we must then restrict \( Dz(p) \) to a hyperplane anyway, unlike the uncompensated case, we will drop the redundant dimension arising from homogeneity and work only with \( Dz(p)|_{T_{p}} \). Thus, the compensated behavior will be described in terms of the eigenvalues \( \mu_{1}(p), \ldots , \mu_{n-1}(p) \) of \( J(p) = (Dz(p) + Dz(p)^{T})|_{T_{p} \cap T_{z(p)}} \). We say that demand obeys the compensated law of demand of order \( i \) if \( \mu_{1} + \cdots + \mu_{n-i} < 0 \). In the extreme case of \( i = n-1 \), all eigenvalues are negative and we have the differential form of WARP (the weak axiom of revealed preference.) In the other extreme case, \( i = 1 \), it is only required that the sum of the eigenvalues, which is (twice) the sum of the diagonal elements of \( Dz(p)|_{T_{p} \cap T_{z(p)}} \), or indeed of the compensated matrix \( Dz(p)|_{T_{z(p)}} \), be negative in total. This is, of course, a quite weak form of the law of compensated demand.

**Theorem 2.** If demand obeys the compensated law of demand of order \( i \), then the dimension of the attractor \( A \) must be less than \( n+1-i \). In case of order \( n-1 \) (WARP), then there is a globally stable equilibrium.

**Proof.** The previously discussed result in Smith (1986) has been adapted by Leonov et al. (1996) for the case of the Jacobian orthogonal to the direction of the vectorfield. This result in \( n \) dimensions says that if there is a \( d \in [0,n-1] \) and \( s \in [0,1] \) such that \( \mu_{i}(p) + \cdots + \mu_{d+1}(p) + s\mu_{d}(p) < 0 \) for all \( p \in A \) (where \( \mu_{i} \) is the eigenvalue of \( (Dz(p) + Dz(p)^{T})|_{T_{z(p)}} \)) then \( dimA < d + s \). If \( d = 2 \) and \( s = 0 \), then \( dimA \leq 1 \). One way to apply this to our case is by changing the coordinates \( (p_{1}, \ldots , p_{n+1}) \) into the polar coordinates \( (\theta_{1}, \ldots , \theta_{n}, r) \), an orthogonal transformation at each point, and then dropping the degenerate radial dimension \( r \). The Jacobian of the remaining system, \( \theta = \tilde{z}(\theta) \), acting as a quadratic form, coincides with \( Dz(p)|_{T_{p}} \) in the original coordinates, and hence \( D\tilde{z}(\theta) + D\tilde{z}(\theta)^{T} \) corresponds with \( (Dz(p) + Dz(p)^{T})|_{T_{p}} \).

In the case \( d = 2, s = 0 \), (i.e. WARP,) the attractor is composed of elements of at most 1 dimension. It is easy to show that the order \( n-1 \) condition implies that each critical element – closed orbit or equilibrium – is itself stable. While there could potentially be one-dimensional sets of the global attractor other than critical elements, they would have to consist of trajectories approaching the equilibria, and this cannot happen, given their local stability. Since the disjoint, open basins of attraction of the critical elements, which then constitute the global attractor, must cover the entire price domain, there can be just one such element, which given the existence of equilibrium, is necessarily a globally stable equilibrium. \( \square \)

**Remark.** While the proof of global stability given WARP is rather convoluted, considering the classical nature of the result, it is still useful to have an argument following from our general dimensionality considerations.

**References**