Optimal insurance design with random initial wealth

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Abstract

An optimal insurance policy is designed when the policyholder faces both insurable and uninsurable risks. Under a realistic behavioral assumption, it may display a disappearing deductible if the uninsurable loss becomes riskier, according to any degree of stochastic dominance, as the insurable loss increases. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

An important result on optimal insurance is due to Raviv (1979) who proves that, when the risk-averse insurer faces increasing and convex administrative costs, the optimal insurance policy displays coinsurance above a deductible. However, this result holds under the implicit assumption that the policyholder faces only one source of risk.

Raviv’s result is re-examined when the insured agent is also exposed to an uninsurable risk that affects his initial wealth. When this background risk is independent of the insurable risk, the optimal insurance contract is shown to still display coinsurance above a deductible. If an increase in the insurable loss makes riskier the policyholder’s random initial wealth according to any degree of stochastic dominance and if the derivatives of his utility function alternate in sign, the optimal insurance contract may display a disappearing deductible. Under the additional assumption that the premium is proportional to the expected indemnity, a disappearing deductible always provides an optimal risk sharing.

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2. The model

A risk-averse agent, with increasing and concave utility function \( u \), is endowed with an initial random wealth \( \tilde{w}_0 = w_0 + \tilde{e} \), where \( w_0 \) is nonrandom and \( \tilde{e} \) is a zero-mean background risk. He faces a risk of loss \( \tilde{x} \). He has the opportunity to purchase an insurance policy against the random loss \( \tilde{x} \) but the uncertainty affecting his initial wealth cannot be insured.

The insurance contract is described by a couple \([I(\cdot), P]\) where \( I(\cdot) \) is the indemnity schedule and \( P \) the premium. Hence, \( I(x) \) is the indemnity paid by the insurance company if the realized loss is \( x \). A feasible coverage function satisfies:

\[
0 \leq I(x) \leq x \text{ for all } x. \tag{1}
\]

Following Raviv (1979), the insurance company with utility function \( v \) is assumed to be risk averse, \( v' > 0 \) and \( v'' < 0 \), and it faces administrative costs which are increasing and convex with the indemnity payment, \( c'(I) \geq 0 \) and \( c''(I) \geq 0 \) for all \( I \).

An optimal insurance contract is the couple \([I(\cdot), P]\) that maximizes the insured’s expected utility of final wealth subject to the constraint that the insurer’s expected utility is constant:

\[
\max_{I(\cdot), P} \int_0^{x_{\max}} \int_a^b u(w_0 + \varepsilon - x + I(x) - P) \, dG(\varepsilon/x) \, dF(x) \tag{2}
\]

subject to conditions (1) and

\[
\int_0^{x_{\max}} v(\pi_0 + P - I(x) - c[I(x)]) \, dF(x) \geq v(\pi_0) \tag{3}
\]

where \( \pi_0 \) is the insurer’s nonrandom initial wealth, \( F(x) \) is the cumulative distribution function (CDF) of the random variable \( \tilde{x} \) defined over the support \([0, x_{\max}]\) with \( x_{\max} \geq 0 \), and \( G(\varepsilon/x) \) is the CDF of the random variable \( \tilde{e} \) conditional on \( \tilde{x} = x \). It is defined over the support \([a, b]\), with \( a < 0 < b \), which is assumed to be independent of the realization of \( \tilde{x} \).

The purpose of this paper is to examine the design of an optimal insurance policy against a random loss \( \tilde{x} \) when the insurance purchaser’s initial wealth \( w_0 \) is affected by an uninsurable background risk \( \tilde{e} \) with \( E\tilde{e} = 0 \), \( E \) being the expectation operator. Such a contract is designed when both sources of risk are independent and when they are dependent according to a criterion of stochastic dominance.

3. Independent random initial wealth

The risk of loss \( \tilde{x} \) and the background risk \( \tilde{e} \) are assumed to be independent. Defining the indirect utility function \( \hat{u} \) by \( \hat{u}(w) = Eu(w + \tilde{e}) \) for all \( w \), problem (2) turns out to be the same as the one studied by Raviv (1979) where the insured’s utility function \( u \) is replaced by \( \hat{u} \). He proves that, under the optimal insurance policy, indemnity payments are made when the realized loss \( x \) is higher than a deductible \( D \geq 0 \). The marginal coverage satisfies for all \( x > D \):
\[ I(x) = \frac{R_{a}(w)}{R_{a}(w) + R_{b}(\pi)(1 + c') + \frac{c''}{(1 + c')}} \]  

where \( w = w_0 - x + I^*(x) - P, \pi = \pi_0 + P - I^*(x) - c[I^*(x)] \), \( R(.) \) denotes the index of absolute risk aversion, \( c' \) and \( c'' \) are evaluated at \( I^*(x) \). Since \( I'^*(x) < 1 \) for all \( x > D \), the optimal insurance policy displays coinsurance above a deductible.

**Proposition 1.** Suppose that the policyholder’s initial wealth is affected by an independent zero-mean background risk. The optimal insurance contract displays coinsurance above a deductible.

Therefore, the presence of an independent background risk does not alter the form of the optimal insurance contract derived when the policyholder’s initial wealth is nonrandom. Nevertheless, this background risk should affect the level of coinsurance above the deductible. This effect is usually ambiguous because the presence of the independent background risk affects not only the policyholder’s index of absolute risk aversion through his indirect utility function but also his wealth, the insurer’s one and his cost function through a change in \( I^*(x) \). However, this ambiguity can be solved if a marginal change in \( x \) is considered such that \( I^*(x) \) remains unchanged. As a consequence, the independent background risk only affects the policyholder’s risk aversion. This effect is analyzed by using the concept of “risk vulnerability” defined by Gollier and Pratt (1996). An individual is risk vulnerable if any unfair independent background risk induces him to behave in a more risk-averse way. All utility functions belonging to the well-known class of Hyperbolic Absolute Risk Aversion (HARA) functions, including constant and relative absolute risk aversion, exhibit risk vulnerability. Under risk vulnerability, \( R_{a}(w) \) is thus higher than \( R_{a}(w) \) for all \( w \). The marginal coverage \( I'^*(x) \) increases with \( x \) above the deductible but remains lower than unity. The presence of an independent background risk entails the transfer of a larger fraction of the insurable loss above the deductible from the risk-vulnerable policyholder to the risk-averse insurer.

If the insurer is risk neutral (\( R_{a}(.) = 0 \) and his cost function is linear (\( c''(.) = 0 \)), i.e. the premium is proportional to the expected indemnity, the optimal insurance contract displays full insurance above the deductible. In comparison with Arrow’s (1971) theorem, this schedule is not altered by the introduction of an independent background risk, as noticed by Gollier (1996).

### 4. Dependent random initial wealth

The background risk \( \tilde{e} \) is now assumed to be correlated with the insurable random loss \( \tilde{x} \). Doherty and Schlesinger (1983) show that the optimal insurance contract could take any form depending on the stochastic dependency between both sources of risk. Consequently, I assume that an increase in \( x \) induces a riskier conditional distribution of \( \tilde{e} \) by \( n^{th} \)-degree stochastic dominance. This assumption is expressed by the following two conditions:

\[ T_{a}(e/x) \geq 0 \text{ for all } e \in [a, b], \]

\[ T_{i}(b/x) \geq 0 \text{ for all } i = 1, 2, \ldots, n, \]
where

\[ T_1(\epsilon/x) = G_1(\epsilon/x) = \frac{\partial G(\epsilon/x)}{\partial x} \text{ and } T_n(\epsilon/x) = \int_a^c T_{n-1}(s/x) \, ds \text{ for } n \geq 2. \]

When \( n \) equals 1 (resp. 2, 3), an increase in \( x \) makes \( G(x) \) riskier by the first- (resp. second-, third-) degree stochastic dominance, which are well-known rules for ordering random variables (see, for example, Hadar and Russell, 1969; Whitmore, 1970).

Furthermore, let \( U_n \) with \( n \geq 2 \) be the set of utility functions such that the first derivative is positive and the other \((n - 1)\) derivatives alternate in sign:

\[ U_n = \{ u: (-1)^{i+1} u^{(i)} \geq 0 \text{ for } i = 1, 2, \ldots, n \} \quad (7) \]

where \( u^{(i)} \) denotes the \( i^{th} \) derivative of the utility function \( u \).

**Proposition 2.** Suppose that any increase in the insurable loss \( x \) induces the policyholder’s uninsurable random initial wealth to be riskier by the \( n^{th} \)-degree stochastic dominance, where \( n \geq 1 \). If his utility function belongs to the set \( U_{n+1} \), then the optimal insurance contract takes the following form. There exists \( D \geq 0 \) such that:

\[ I^*(x) \begin{cases} > 0 & \text{if } x > D \\ = 0 & \text{otherwise} \end{cases} \quad (8) \]

and the marginal coverage function satisfies for all \( x > D \):

\[ I^*'(x) = \frac{1}{\Sigma} \left[ -Eu''(w + \tilde{\epsilon}) + \int_a^b u'(w + \epsilon) \, dG_1(\epsilon/x) \right] > 0, \quad (9) \]

with

\[ w = w_0 - x + I^*(x) - P, \pi = \pi_0 + P - I^*(x) - c[I^*(x)], \]

\[ \Sigma = -Eu''(w + \tilde{\epsilon}) + \left[ R_v(\pi)(1 + c') + \frac{c''}{(1 + c')} \right] Eu'(w + \tilde{\epsilon}) > 0, \]

\[ \int_a^b u'(w + \epsilon) \, dG_1(\epsilon/x) > 0 \]

**Proof.** Problem (2) where the stochastic dependency between both sources of risk is based on a specific degree of stochastic dominance can be solved via optimal control theory. Since constraint (3) is binding at the optimum, the Hamiltonian of problem (2) is

\[ H = \left\{ \int_a^b u(w + \epsilon) \, dG(\epsilon/x) + \lambda u(\pi) \right\} f(x) \quad (10) \]
where $f(x)$ is the density function of $\bar{x}$ and the multiplier function $\lambda$ is constant with respect to $x$, as already shown by Raviv (1979). For $0 < I^*(x) < x$, the first-order necessary condition is

$$
\int_a^b u'(w + e) dG(e/x) - \lambda[1 + c'(I^*(x))]u'(\pi) = 0
$$

(11)

Differentiating Eq. (11) with respect to $x$ and rearranging the terms lead to Eq. (9). Carrying out several integrations by parts, the following relationship is obtained:

$$
\int_a^b u'(w + e) dG(e/x) = \sum_{i=1}^{n} (-1)^{i+1} u^{(i)}(w + b)T_i(b/x) + (-1)^{i} \int_a^b u^{(i+1)}(w + e)T_i(e/x) d\varepsilon
$$

(12)

If the stochastic relationship between $\bar{x}$ and $\bar{\varepsilon}$ is based on the $n$th-degree stochastic dominance and $u \in U_{n+1}$, then Eq. (12) is positive and thus Eq. (9) is positive. It is easy to demonstrate that the constraint $I^*(x) \leq x$ is always satisfied in this model. For $I^*(x) = 0$, the first-order necessary condition is

$$
K(x) = \int_a^b u'(w_0 + e - x - \pi)dG(e/x) - \lambda[1 + c'(0)]u'(\pi_0 + \pi) \leq 0
$$

(13)

From Eq. (12), $K(x)$ is easily seen to increase with $x$. Therefore, a deductible $D \geq 0$ exists such that the optimal insurance contract takes the form expressed in Eq. (8). Finally, notice that $I^*(x)$ also satisfies the sufficient condition because the Hamiltonian does not depend on the state variable (Kamien and Schwartz, 1971). Q.E.D.

A marginal increase in $x$ affects the optimal coverage in both ways. Firstly, the final wealth of the risk-averse policyholder decreases and thus he is induced to increase his coverage. This corresponds to the first right-hand side (RHS) term in brackets in Eq. (9). The second effect is due to the impact of $x$ on the conditional distribution of the $\varepsilon$ background risk. Let the CDF $H_1(e)$ and $H_2(e)$ be defined by $H_1(e) = G(e/x)$ and $H_2(e) = G(e/x + \delta)$ with $\delta > 0$ and consider, for example, an increase in risk according to the second-degree stochastic dominance. Hence, $H_2(e)$ is riskier than $H_1(e)$ according to this criterion. The second RHS term in brackets in Eq. (9) can be rewritten as

$$
\int_a^b u'(w + e) dG_1(e/x) = \frac{1}{\delta} \left\{ \int_a^b z(w + e) dH_1(e) - \int_a^b z(w + e) dH_2(e) \right\}
$$

(14)

where $z(w) = -u'(w)$ for all $w$. Hadar and Russell (1969) show that under second-degree stochastic dominance $H_1(x)$ is preferred to $H_2(x)$ if $z$ is increasing and concave. Therefore Eq. (14) is positive if the policyholder’s marginal utility function is decreasing ($u'' < 0$) and convex ($u'' > 0$).

The marginal coverage expressed in Eq. (9) may be larger than unity. This means that the optimal insurance contract may display a disappearing deductible. The insured individual thus hedges against the uninsurable risk by increasing his coverage against the insurable risk. When the premium is proportional to the expected indemnity, the optimal insurance contract always contains a disappearing deductible, as shown by Gollier (1996) in the particular case of a mean-preserving increase in risk.
The risk exchange between the policyholder and the insurer as expressed by Eqs. (8) and (9) is thus optimal under the following assumptions: firstly, an increase in $x$ makes $G(e/x)$ riskier according to the first-degree stochastic dominance and the policyholder exhibits risk-aversion ($u'' < 0$); secondly, an increase in $x$ makes $G(e/x)$ riskier according to the second-degree stochastic dominance and the insured individual is risk-averse and prudent ($u'' > 0$); thirdly, an increase in $x$ makes $G(e/x)$ riskier according to the third-degree stochastic dominance and the policyholder exhibits risk aversion, prudence and temperance ($u''' < 0$). The notions of prudence and temperance have been introduced by Kimball (1990, 1992). Prudence is meant to suggest the propensity to prepare and forewarn oneself in the face of uncertainty, and temperance expresses the desire to reduce exposure to risks following the introduction of an additional source of risk. It is a necessary condition for decreasing absolute risk aversion and decreasing absolute prudence, respectively. Finally, when the policyholder’s utility function is HARA, this risk exchange is optimal if an increase in $x$ makes $G(e/x)$ riskier according to any degree of stochastic dominance.

5. Conclusion

An optimal insurance policy has been designed when the policyholder’s wealth is affected by both an insurable risk and an uninsurable background risk and when the risk-averse insurer faces increasing and convex administrative costs. This paper has reexamined not only Raviv’s result (Theorem 1, 1979) in the presence of an uninsurable background risk, but also Gollier’s results (Propositions 1 and 3, 1996) under insurer’s risk aversion and weaker correlation between both sources of risk.

References