An all-pay auction with a pure-strategy equilibrium

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Abstract

I present an all-pay auction where the valuations of the contestants is a function of their bids (expenditures). I show that there exists a unique Nash equilibrium in pure strategies under certain conditions. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction and analysis

It is a standard result that there is no equilibrium in pure strategies in all-pay auctions (Baye et al., 1996; Ellingsen, 1991; Hillman and Riley, 1989; Hillman and Samet, 1987 and Konrad, 1999a).

In this paper, I present an all-pay auction which has an equilibrium in pure strategies. Let me state, from the onset, that what drives my result is the fact that the valuations of the contestants are functions of their bids as opposed to the standard case in the literature where the valuations are exogenous (i.e., independent of the contestants’ bids). I present an example of this game, which is similar to the games that Konrad and Schlesinger (1997) refer to as ‘rent-augmenting’ games.

As in Amegashie (1999), consider a contest for a job position between two risk-neutral applicants, A and B. Without loss of generality, but for the sake of exposition, assume that they are identical. Let y and x be the expenditures (bids or efforts) of A and B respectively. Suppose that there is a minimum salary which will be paid to whoever gets the job. Let this minimum salary give a rent, \( S > 0 \).

Suppose that the successful applicant gets the minimum salary plus some salary increase or bonus, \( R' \) (\( i = A, B \)), which is an increasing function of his effort (e.g., his performance at the interview). This is usually the case in practice since most jobs positions have a salary range (a minimum salary and a

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maximum salary), with any increase above the minimum depending on the successful applicant’s experience, how personable he is, his performance at the interview and on aptitude tests, reference letters, etc. Let $R^A = \beta y^a$ and $R^B = \varphi x^a$, where $\varphi \geq 0$, $\beta \geq 0$ and $0 < \alpha < 1$. Suppose that there is some bias in favor of $A$ due to, for example, race or gender. Specifically, suppose $\beta > \varphi \geq 0$. That is, for the same effort $A$ gets a bigger salary increase, if he is successful. In the extreme case of $\varphi = 0$, $B$ gets no salary increase, if he is successful.

Suppose that the contest is an all-pay auction, so that the contestant with the higher expenditure wins with certainty and ties are broken randomly. Let $p(x, y)$ be the probability that $A$ wins the contest. Then $p(x, y) = \begin{cases} 1 & \text{if } y > x \\ 1/2 & \text{if } y = x \\ 0 & \text{if } y < x \end{cases}$

In the literature, $p(x, y)$ is also referred to as the contest success function.

Clearly, $B$’s valuation is $V^B = S + \varphi x^a$ and $A$’s valuation is $V^A = S + \beta y^a$, both of which are functions of their expenditures. Indeed, the fact that the contestants’ expenditures may influence the size of the rent has also been noted and examined by Konrad and Schlesinger (1997) and Lee (1985). See also Amegashie (1999, section 1) for further examples.

In what follows, I assume complete information. Before I show that there could be an equilibrium in pure strategies, let me present the argument for the non-existence of a pure strategy equilibrium when the valuations are fixed and independent of the contestants’ expenditures. Suppose the contestants have fixed valuations, $\hat{V}^A$ and $\hat{V}^B$. Assume, for the sake of exposition, that $\hat{V}^B < \hat{V}^A$. Suppose $B$ bids $x^*$, such that $0 < x^* \leq \hat{V}^B$. Then $A$’s optimal response is $y^* = x^* + \theta < \hat{V}^A$ (i.e., marginally higher than $x^*$). But then $x^* > 0$ cannot be an optimal response to $y^* = x^* + \theta$. Also it is obvious that $x^* = y^* = 0$ cannot be an equilibrium. Hence, there is no equilibrium in pure strategies.

Consider first the case of $\beta > \varphi = 0$. Suppose that $y^{**}$ maximizes $\pi^A = S + \beta y^a - y$. This will be $A$’s bid, if he were the only contestant. Assume that $y^{**} = (\alpha \beta)^{1/(1-\alpha)} S$. Then when $A$ bids $y^{**}$, $B$’s optimal reply is zero and $A$ wins the contest with certainty. Since $y^{**}$ maximizes $\pi^A$, $A$ has no incentive to bid less than $y^{**}$ or deviate from $y^{**}$, given that $B$ bids zero. The contest could be won cheaply by bidding less than $y^{**}$, but unlike the case of fixed valuations, $A$’s payoff will be smaller (not bigger) if he bids less than $y^{**}$. Thus there is a Nash equilibrium in pure strategies, where $A$ bids $y^{**}$ and $B$ bids zero, if $y^{**} \geq S$.

Note that the pure strategy equilibrium $(y^{**}, 0)$ may not be the outcome of the game if $B$ can pre-empt $A$’s bid. I now show that even if $B$ has a first-mover advantage, he will still bid zero, given $y^{**} \geq S$. First, consider the case of $y^{**} > S$. If $B$ bids zero and $A$ bids $y^{**}$, $B$’s payoff is zero. Suppose that $B$ bids $x > 0$ and assume that this bid ensures that he wins the contest with certainty, then we require that his payoff, $S - x \geq 0$. That is, $x \leq S$. But if $y^{**} > S$, then $A$ can out-bid any $x \leq S$. Hence bidding $x > 0$ is not optimal. It follows that if $y^{**} > S$, then there exists a Nash equilibrium, in which $A$ bids $y^{**}$ and $B$ bids zero, even if $B$ has a first-mover advantage.

Now consider $y^{**} = S$. It is obvious that it is not optimal for $B$ to bid any $x < S$. Suppose that $B$ were to bid his maximum of $S$. Suppose that $A$ also bids $S$. Then $A$’s expected payoff is

\footnote{Needless to say that there is no pure strategy equilibrium if $y^{**} < S$.}
\[(1/2)[S + \beta S^\alpha] - S. \]

Since \(y^{**} = (\alpha \beta)^{1/(1-\alpha)} = S\), it follows that \(\beta = S^{1-\alpha}/\alpha\). Putting this into (1), it is easy to show that the expression in (1) is positive, given \(0 < \alpha < 1\). Then it follows that \(A\) will bid, at least, \(S\) if \(B\) bids \(S\). It then does not make sense for \(B\) to bid \(S\), because his (B’s) payoff will be \((1/2) S - S < 0\) if \(A\) bids \(S\), and is obviously negative if \(A\) bids marginally more than \(S\). Hence \(B\) will bid zero, even if he has a first-mover advantage and \(A\) will bid \(y^{**} = S\).

Since \(B\) will not bid more than \(S\), bidding \(y^{**}\) is a strictly dominant strategy for \(A\), if \(y^{**} > S\). Given that \(B\) has a unique reply (i.e., \(x=0\)) to \(y^{**} > S\), it follows that the pure-strategy equilibrium \((y^{**}, 0)\) is unique, if \(y^{**} > S\).

Note that in the case of exogenous valuations, if one contestant bids zero, it is not optimal for the other contestant to also bid zero. However, he should just bid a small but positive amount. In our case, there exists a bid which maximizes his payoff, if he is the only contestant. If this bid is such that the other contestant bids zero, then it is indeed the optimal bid.

Consider now \(\beta > \varphi > 0\). Suppose \(A\) bids \(y^{**}\). If \(B\) matches this bid, then his expected payoff is \((1/2)[S + \varphi (y^{**})^\alpha] - y^{**}\). If \(\beta\) is sufficiently large, then \(B\)’s expected payoff could be negative. Thus \(B\) will not match \(A\)’s bid, if \(\beta\) is sufficiently large. The other strategy then is for \(B\) to bid more than \(y^{**}\). Let \(\tilde{x}\) be the maximum that \(B\) will bid. Then \(S + \varphi \tilde{x}^{\alpha} - \tilde{x} = 0\). But \(\tilde{x}\) could also be less than \(y^{**}\), if \(\beta\) is sufficiently large. Hence under these conditions, \(B\) will bid zero, if \(A\) bids \(y^{**}\) and this is a Nash equilibrium.

Since \(B\) will not bid more than \(\tilde{x}\), a similar argument, as in the first case of \(y^{**} > S\) (with \(\varphi = 0\)), establishes that the pure-strategy equilibrium is unique if \(y^{**} > \tilde{x}\).

The analysis shows that the dependence of the rent (valuation) on the contestants’ bids and the differential treatment of the contestants account for the existence of the pure-strategy equilibrium in this all-pay auction.

To conclude, I summarize the paper’s result in the following proposition:

**Proposition.** In all-pay auctions, where the size of the rent is an increasing function of the contestants’ expenditures, if there is one contestant for whom the relationship between his expenditure and the size of the rent is sufficiently stronger than that of all the other contestants, then there exists a

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\(^1\)Indeed, \(A\) will bid marginally more than \(S\), if \(B\) bids \(S\). To see this note that if \(A\) bids \((S + \epsilon)\), where \(\epsilon\) is positive, then his payoff is \(S + \beta(S + \epsilon)^\alpha - (S + \epsilon)\). This gives a higher payoff than the payoff from bidding \(S\), if \(S + \beta(S + \epsilon)^\alpha - (S + \epsilon) > (1/2)[S + \beta S^\alpha] - S\). That is, if \(\beta(S + \epsilon)^\alpha - \beta S^\alpha > (1/2)S - \epsilon > 0\). It is easy to see, that any \(\epsilon \in (0, 0.5S]\), is a sufficient but not necessary condition for \(A\)’s payoff to be higher when he bids \((S + \epsilon)\) than when he bids \(S\). However, since \(R^\alpha\) is strictly concave in \(y\), \(A\) will try to get as close as possible to \(y^{**} = S\), by bidding \((S + \epsilon)\), such that \(\epsilon\) is very small but positive.

\(^2\)Note that this is not possible if \(\beta = \varphi\).

\(^3\)In a two-player contest, Konrad (1999b) examines a model with bias. He also gets the possibility of a pure-strategy equilibrium. But the conditions under which his contest success function generates a pure-strategy equilibrium makes his function different from the contest success function in this paper and in the literature on all-pay auctions. That is, for his contest success function to generate a pure strategy equilibrium, a necessary condition is that one contestant (i.e., the one who enjoys the favorable bias) can bid less than the other contestant and still win the contest with certainty. Also in his equilibrium, both contestants bid zero.
pure-strategy equilibrium in which this contestant bids a strictly positive amount and the other contestants bid zero.

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