Volatility and stock prices: implications from a production model of asset pricing

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Abstract

The general equilibrium foundation for the inverse association between return volatility and stock price is explored using a production model of asset pricing. The return volatility obeys the well-known constant elasticity of variance (CEV) diffusion process. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A fundamental relationship documented and often used in the finance literature is the negative association between stock prices and volatility of returns. A simple mean-variance intuition tells us that the rational investor would be willing to pay a lower price to a claim that is uncertain as against a sure claim. While this is an intuitively appealing explanation for the well known inverse relation between volatility and stock prices, one is not sure why an investor’s willingness to pay for an asset will necessarily depend on the variance of portfolio returns regardless of its expected payoff stream. To justify this inverse relationship between volatility and stock prices, financial and operating leverage arguments are often invoked (see Beckers, 1980; Black, 1976; Christie, 1982; Schmalensee and Trippi, 1978). Firms with a high debt-to-equity ratio issue riskier assets that sell at a lower price in financial markets. However, at a macro or aggregate level it is difficult to explain this inverse association between stock prices and volatility without knowing the exact composition of leveraged and unleveraged firms in the economy. Even at the micro level the negative relation between volatility

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and stock prices cannot be fully explained in terms of leverage (see Campbell and Hentschel, 1992; Dubofsky and French, 1988; Schwert, 1989).

The purpose of this note is to understand the general equilibrium foundations of this inverse relationship between volatility and stock prices from a macroeconomic perspective. This inquiry is important because this observed inverse relationship must be driven by some economic fundamentals. Although there are several papers that address this inverse association between stock prices and variance, very few explore its general equilibrium foundation. Since a financial asset is nothing but a claim to the output produced in the economy, this economic fundamental is closely related to the production technology. With the advent of the real business cycle theory, it is now well known that the production technology, particularly its returns to scale property, is important in determining the persistence and volatility of cycles. A number of papers following the pioneering work of Brock (1982) have explored how uncertainty originating from the technology influences the equilibrium asset pricing behavior.

One can invoke several stochastic specifications to describe the dynamics of stock price movements. A commonly used representation is the constant elasticity of variance (CEV) diffusion, proposed in Cox (1975) and Cox and Ross (1976), where the variance of returns (or volatility) varies inversely with the asset price. Algebraically, the following stochastic differential equation describes a CEV process:

$$\Delta S_t = \mu S_t \Delta_t + \sigma S_t^{\gamma/2} \Delta Z_t$$

where \(S_t\) is the stock price, \(\mu\) is the expected return on the stock over the interval \([t, t + \Delta t]\) and \(\Delta Z_t\) is a standard Wiener process. The stock return is thus \(\Delta S_t / S_t\). The elasticity parameter is \(0 < \gamma \leq 2\). From (1) the variance of the stock’s return is equal to \(\sigma^2 S_t^{\gamma-2}\), which is an inverse function of the stock price. Notice that the elasticity of the return variance with respect to the stock price is constant and equal to \(\gamma - 2\). The parameter \(\gamma\) thus captures the strength of the association between stock prices and return volatility.

In this paper, we will focus on the CEV process as a parametric representation of the inverse association between stock prices and return volatility. Our primary goal is to understand the general equilibrium foundation of the CEV parameter. In order to accomplish this we use a variant of Brock’s (1982) production model of asset pricing to derive a reduced form relationship between stock price and volatility which precisely turns out to be of the CEV form as in (1). We show that the inverse relationship between stock price and return volatility is driven by diminishing returns in the production technology. The intuition for this result stems from the fact that in a frictionless complete

\[^{1}\]To the best of our knowledge, the only paper that indirectly addresses this issue is Abel (1988). However, because of the exogenous specification of the dividend process in Abel’s model, it is not clear how the relationship between stock price and volatility is driven by the production fundamentals.

\[^{2}\]Empirically, a test of the inverse association between the stock price and the variance of the stock return can be carried out by estimating the elasticity coefficient \(\gamma\) in (1). If the estimate of \(\gamma\) is not significantly different from 2, then one can conclude that the dynamics of the stock price is not significantly different from log normality. Alternately, if the estimate of \(\gamma\) is closer to either unity or zero, then one may conclude that the square root process (\(\gamma = 1\)) or the absolute process (\(\gamma = 0\)) better fits the diffusion of stock prices. Beckers (1980), Black (1976), Christie (1982), MacBeth and Merville (1980) have documented evidence in support of a CEV process for individual stocks. Samanta (1995) found evidence in support of a CEV process for the dynamics of stock index prices.
market environment the basic arbitrage condition requires that the stock return closely reflects the return on physical investment. A diminishing returns technology means that a higher ex-post return on physical investment will be eventually followed by a low return because a high return will induce more capital investment which lowers the prospective return to investment. Since, in equilibrium, stock returns are related to investment returns, stock returns will be negatively autocorrelated because of diminishing returns. This negative serial correlation in stock returns makes stock prices mean reverting and thus reduces the volatility of stock returns. On the other hand, because of a larger capital stock due to increased physical investment the stock price responds positively to the level of physical investment. The result is an inverse association between stock price and volatility of stock returns. The extent of this inverse correlation between stock prices and volatility is thus governed by the size of the returns to scale in the technology.

The rest of the paper is organized as follows. In the next section, we lay out the basic production model of asset pricing and derive the reduced form relationship between stock price and volatility. In Section 3 we end with concluding comments.

2. The model

We consider a parametric form \(^3\) of the asset pricing model with production a la Brock (1982). Consider a representative household studied by Lucas (1978) whose source of income at date \(t\) is dividend \(D_t\) and the resale value \(P_t\) per share. Each share represents a perfectly divisible claim on the output produced by the firm. The first order condition for a typical household is given by the following valuation rule a la Lucas (1978):

\[
P_t U'(C_t) = \beta E[U'(C_{t+1})(P_{t+1} + D_{t+1})]
\]

where \(U'(C_t)\) is the marginal utility of consumption at date, and \(E_t\) is the mathematical expectation conditioned on the information at time. The first order condition simply indicates that the price of a share must be such that it equates the current utility cost of buying the share to the discounted future utility benefits which involve dividends and the resale value of the share.

The representative firm solves the following present value maximization problem:

\[
\text{Max } E_0 \sum_{t=0}^{\infty} [\mu_t \varepsilon_t K_t^\alpha - K_{t+1}] \prod_{i=0}^{t} d_i
\]

where \(\mu_t\) is an exogenously specified neutral technical progress component. The technology parameter \(\alpha \in (0, 1)\) represents capital’s share in national income and \(E_0\) is the expectation operator conditional on date 0 information. \(K_t\) is the capital stock prevailing at the beginning of date \(t\). The uncertainty in this economy arises entirely from productivity sources, which is captured by the serially uncorrelated productivity shock \(\varepsilon_t\). After observing the realization for \(\varepsilon_t\), the firm decides about the purchase of new capital goods. The difference between the gross output \(Y_t\) and the investment spending \(K_{t+1}\) is distributed to the shareholder household as dividend. While solving the capital accumulation problem in period \(t\), the firm faces a sequence of time-varying discount factors \(\{d_i\}, i = 1, 2 \ldots t\). In a state of

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\(^3\)Balvers et al. (1990) develop a similar parametric model to demonstrate the predictability of stock returns.
rational expectations equilibrium, these discount factors are endogenously determined by the household’s marginal rate of substitution in consumption. In other words:

\[
d_{t+1}(\omega) = \frac{BU'(C_{t+1}(\omega))}{U'(C_t)}
\]

where \( \omega \) is the state of nature prevailing at \( t+1 \).

Following the same principles as in Brock (1982) it is possible to show that the rational expectations equilibrium of this asset-pricing problem can be solved by constructing the following social planner’s problem:

\[
\max E_0 \sum_{i=0}^{\infty} \beta^i U(C_i)
\]

s.t.

\[
C_t + K_{t+1} = \mu_t e_t K_t^a
\]

Assuming \( U(C_t) = \log C_t \), it is straightforward to verify that the following equations represent the equilibrium law of motion of the asset price and the capital stock:

\[
P_t = \frac{\beta}{1 - \beta} D_t
\]

\[
D_t = (1 - \alpha \beta) \mu_t K_t^a e_t
\]

\[
K_t = \alpha \beta \mu_t^{t-1} K_{t-1}^a e_{t-1}
\]

The stock return between \( t \) and \( t+1 \) (denoted by \( R_{t+1} \)) includes the rate of dividend and capital gains. It is given by the following equation:

\[
R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}
\]

Using (7), (8) and (9) the stock returns can be rewritten as:

\[
R_{t+1} = \frac{1}{\beta} Y_{t+1}/Y_t
\]

where \( Y_t = \mu_t K_t^a e_t \). In other words the stock return is proportional to the growth rate of the economy-wide output. Next, using (9) observe that the process for output evolves as:

\[
Y_t = (\alpha \beta)^a \mu_t Y_{t-1}^a e_t
\]

Using (11) and (12), it is immediate that the conditional variance of returns \( (\nu_t) \) is:

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See Basu and Vinod (1994) appendix for derivation.
\[ v_t = \left(1/\beta^2\right) Y_t^{2(\alpha - 1)} \cdot (\alpha \beta)^{2\alpha} \mu^{2(t+1)} \text{Var}(\varepsilon_{t+1}) \]  

(13)

Notice that the variance is time-varying because of the time-varying nature of the output, \( Y \). Finally, using (7) through (13) and defining \( \text{Var}(\varepsilon_{t+1}) = \sigma^2 \), we obtain the following reduced form relationship between the stock price and the variance of returns:

\[ \ln v_t = -2(\alpha - 1) \ln \left(1 - \frac{\alpha \beta}{(1 - \beta) \cdot \sigma^{1/(\alpha - 1)} \cdot \alpha^{(\alpha - 1)}}\right) + 2 \ln \mu + 2 \ln(\mu) t + 2(\alpha - 1) \ln P_t \]  

(14)

The striking feature of this general equilibrium asset pricing model is that it replicates the CEV feature of the stock price process. To see this, notice that from (14) the elasticity of the variance with respect to the contemporaneous stock price is constant and equal to \( 2(\alpha - 1) \) which immediately implies that the CEV parameter \( \gamma \) in Eq. (1) equals \( 2\alpha \). Since \( \alpha \) is less than 1 due to diminishing returns, the elasticity is negative in sign. The extent of diminishing returns parameterized by \( \alpha \) determines the size of the elasticity coefficient in the CEV process. Also notice from (7) and (8) that the stock price is growing at the rate \( \mu \). Hence, the neutral technical progress accounts for the deterministic growth of the stock price as shown in (1) and the diminishing returns to capital explains the empirically observed inverse CEV-type relationship between variance of returns and stock prices.

To see more clearly why diminishing returns to production may be at the root of the inverse relation between stock prices and volatility, observe that the equilibrium returns, \( R_t \), follow the process:

\[ \ln R_t = - (1 - \alpha) \ln \beta + \alpha \ln R_{t-1} + \ln \varepsilon_t - \ln \varepsilon_{t-1} \]  

(15)

which means that the returns are negatively autocorrelated, implying that stock prices are mean reverting. The extent of mean reversion is governed by \( \alpha \). The smaller the \( \alpha \), the greater the degree of mean reversion which tends to stabilize the variability of returns. A positive productivity shock \( \varepsilon_t \) at date \( t \) by raising the economy-wide output promotes the stock market price in the same period. On the other hand, the volatility of returns goes down (see Eq. (13)) because of the negative serial correlation in stock returns induced by the positive shock to productivity. The result is an inverse association between stock prices and volatility whose strength depends crucially on the diminishing returns to technology parameterized by \( \alpha \).

This parametric example vividly illustrates the role of the technology in determining the relationship between volatility and stock prices. To put it another way, the relationship between stock prices and volatility is intimately connected to the nature of the growth process in the economy. A vast body of growth literature (see Barro and Sala-i-Martin (1995) for a good survey) documents conditional convergence of the growth rates among countries, which indirectly confirms the fact that there is indeed diminishing returns to capital even when capital is broadly defined to include human capital. Since a financial asset in the form of a share certificate represents an ownership title to the

\[ \text{Notice that the first order autocorrelation for } \ln R_t \text{ is } (\alpha - 1)/2. \text{ For details of the derivation of (15) and the autocorrelation property of stock returns, see Basu and Vinod (1994).} \]
capital stock, the equilibrium process for stock prices should also reflect the same diminishing returns property of the capital stock.

3. Conclusion

The principal objective of this paper is to provide a general equilibrium foundation for the relationship between volatility and stock prices. We accomplish this by using the production model of asset prices as in Brock (1982). A parametric specification of this production model reproduces a CEV-type relationship between stock price and volatility. Previous research on this used operating and financial leverage arguments to explain the inverse relationship between stock prices and volatility. Although these arguments may be more relevant for individual stock prices, they do not necessarily apply to aggregate stock prices. For aggregate stock price, general equilibrium consideration is imperative because a financial asset at an aggregate level is ultimately a claim to the aggregate output of the economy. It is, therefore, important to understand the role of production fundamentals in determining the relationship between stock prices and volatility. We argue that the inverse relationship between stock prices and volatility is driven by diminishing returns in the production technology. A diminishing returns technology makes stock prices mean reverting which lies at the foundation of the negative relationship between stock prices and volatility.

Given our parametric specifications for preferences and technology, the reduced form relationship between volatility and stock prices turns out to be of the CEV form. From our reduced form relationship it is clear that the extent of diminishing returns determines the size of the elasticity coefficient in the CEV process.

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