Local equilibria in economic games

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Received 14 December 1998; received in revised form 19 April 2000; accepted 17 September 2000

Abstract

We study solution concepts for economic games that are resistant to local deviations. Strategy spaces are subsets of $\mathbb{R}^n$ and local deviations are small in the Euclidean metric. We define local Nash equilibrium and local evolutionarily stable strategy, and present applications to Walrasian outcomes in Cournot games and separating outcomes in screening models. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Nash equilibrium; ESS; Cournot; Screening

JEL classification: C72

1. Introduction

The idea that solution concepts for games should be resistant to small deviations — but not necessarily to large ones — is already established in economics. The standard concept of Nash equilibrium requires resistance to both small and arbitrarily large deviations by only one player. Refinements like trembling-hand perfection require stability against arbitrarily large deviations on the side of any player which occur with small probability. Other concepts, like the one of Evolutionarily Stable Strategy (see e.g. Weibull, 1995), require a symmetric profile to be resistant to the appearance of a small proportion of mutants (who play otherwise) in a large population. Dynamical stochastic models as the one, by now well-known, due to Kandori et al. (1993) analyze long-run outcomes when players have small probabilities of choosing any new strategy at random.

The aim of this paper is to propose an alternative formalization of small deviations in $n$-player games with continuum strategy spaces played by a finite population of players. Hence, we depart from the previous literature in the following respects.

First, in economic models it is often convenient to consider infinite strategy spaces (a continuum)
and a finite population of agents. Prices, quantities, and the like are customarily modeled as real variables. Oligopolies are formed by a finite number of firms, and bargaining takes place between a few agents. This stands in contrast, for example, with the framework in which an Evolutionarily Stable Strategy is defined, namely a continuum of agents who are randomly matched to play an underlying two-player game with finite strategy space.

Second, if strategies are real variables and the population of players is finite, then there is another possible formalization of small deviations. A small deviation is a strategy close (in the usual metric) to the one being played. This departs from the previous literature where the notion of small deviations is not related to the strategy space, but rather to the probability of choosing any different strategy, or to the proportion of agents choosing it.

We define two solution concepts that arise with our interpretation of small deviations. Two applications are then presented, which give a rigorous meaning to facts already pointed out in the economic literature.

## 2. Local equilibria

### 2.1. Local Nash equilibria

We focus on games with a finite number of players and Euclidean (real) strategy spaces. There are numerous economic games that fall into this category. We have in mind firms competing in quantities, prices, and other economic variables.

**Definition 1.** We call a triple \( G = (N, \{S_i\}_{i \in N}, \{\pi_i\}_{i \in N}) \) real game if the set of players \( N = \{1, \ldots, n\} \) is finite, the strategy spaces \( S_i \) are connected subsets of finite-dimensional Euclidean spaces, and \( \pi_i: \Pi_{j=1}^n S_j \to \mathbb{R} \) are the payoff functions.

Given a real game, let \( l \) be the dimension of the minimal Euclidean space that contains each of the strategy spaces. Then, without loss of generality, it can be assumed that \( S_i \subset \mathbb{R}^l \) for all \( i \). As customary in game theory, we denote \( S = \Pi_{i=1}^n S_i \) and \( s_{-i} = (s_j)_{j \neq i} \). We call a real game *symmetric* if \( S_i = S_j \) for all \( i, j \) and \( \pi_i(s, s_{-i}) = \pi_j(s, s_{-j}) \) for all \( s \in S_i = S_j \), and for all \( i, j \), whenever \( s_{-i} \) and \( s_{-j} \) coincide up to a permutation.

A Nash equilibrium is a strategy profile where each player is choosing a best response to the strategies of the other players. If players are able to pick any strategy without any constraint or information problem, then each of them will consider any possible deviation when choosing a best response. Thus, a Nash equilibrium is robust against any possible unilateral deviation. We introduce now the concept of *local Nash equilibrium*, where the possible deviations are constrained.

Given \( s_i \in S_i \) and \( \varepsilon > 0 \), denote \( B(s_i, \varepsilon) = \{s_i' \in S_i \mid \|s_i' - s_i\| < \varepsilon\} \), being \( \|\cdot\| \) any real norm.

**Definition 2.** A local Nash equilibrium for a real game \( G \) is a strategy profile \( (s_i)_{i=1}^n \in S \) such that, for all \( i, s_i \) is a local maximum of \( \pi_i(\cdot, s_{-i}) \), that is, there exists \( \varepsilon > 0 \) such that, for all \( s_i' \in B(s_i, \varepsilon) \), \( \pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}) \).

In words, a local Nash equilibrium is a strategy profile such that no player has an incentive to
deviate to a similar (close) strategy. This implies that each player is using a local maximum of his payoff function given the strategies of other players.

Obviously, any Nash equilibrium is a local Nash equilibrium. Still there are some cases where the non-existence of pure-strategy Nash equilibria has made it hard to formally explain the properties of certain focal, pure-strategy outcomes. One such case is the screening model of Rothschild and Stiglitz (1976) which we consider below. These authors already suggested to consider a local notion of equilibrium (Rothschild and Stiglitz, 1976, p. 646).

Local deviations may be unjustified in games played by rational players with full information. Knowing the full strategy space, and the payoff function on the whole space, why would a rational player not deviate to the global best response? A local Nash equilibrium is sensible if the information about the payoff function is limited. Imagine that players, e.g. firms in an oligopoly, find themselves playing a local Nash equilibrium. Lacking data about the outcome of large deviations, they can only safely interpolate their payoff functions in a neighborhood of their equilibrium strategies. Hence, they will find no incentive to deviate.

2.2. Local evolutionarily stable strategies

Another solution concept where local properties play an important role is that of an Evolutionarily Stable Strategy (ESS). A strategy is said to be evolutionarily stable if, when adopted by all individuals in a population, it is uninvadable by rare mutants; that is, if it is resistant to the sudden appearance of a small proportion of individuals choosing a different strategy. Here, resistant means that, after their appearance, mutants earn lower payoffs than the ESS players. If payoffs are directly related to the probability that strategies endure, i.e. that they are either inherited or imitated, then mutants entering this population in small proportions always extinguish after some time.

The concept of an ESS was originally defined in the framework of an infinite (continuum) population of agents who are randomly matched to play a two-player game with a finite number of strategies. In this case, an ESS is always a perfect Nash equilibrium of the underlying game. Moreover, any strict Nash equilibrium of the game is an ESS.

Schaffer (1988) provides an analogous definition of an ESS for a finite population of randomly matched agents and a finite number of strategies, provided that mutants arise only one at a time. In contrast to the case of a continuum population, he shows that in a finite population an ESS may fail to be a Nash equilibrium (Schaffer, 1988, pp. 474–5).

Our interest is in finite populations and, hence, we start from the ESS concept in Schaffer (1988) and not from the infinite-population concept. Moreover, we focus on n-player symmetric real games (playing the field), instead of on a random matching setting.

We define a local ESS for a finite population and a continuum of strategies as a strategy which is resistant to single mutations to close strategies. As we will see, this concept is unrelated to the one of local Nash equilibrium.

**Definition 3.** A local Evolutionarily Stable Strategy (local ESS) for a symmetric real game G is a strategy \( s \in S \) such that there exists \( \varepsilon > 0 \) such that, for every \( s' \in B(s,\varepsilon) \setminus \{s\} \), \( \pi(s) < \pi(s') \), where \( \hat{s}_i = s' \) and \( \hat{s}_j = s \) for all \( j \neq i \).

\[ ^{1}\text{In this framework, an ESS may be a mixed strategy, which is then interpreted as a population profile.} \]
A global ESS is a strategy verifying the last inequality for all \( s' \in S \setminus \{s\} \).

Our definition of a global ESS coincides with what Schaffer (1988) calls a playing the field ESS. Our definition of a local ESS is here novel.

In a local ESS stability is required only against mutants entering with similar behavior to the one that they displayed before mutation. This definition turns out to be very useful to model the learning process of boundedly rational agents with limited information about the environment in which they interact. As we will see below, the Walrasian outcome in a market with a small number of firms is a local ESS.

3. Separating equilibria in screening models

Rothschild and Stiglitz (1976) pose a basic problem of non-existence of pure-strategy Nash equilibrium in a competitive insurance market. They model such a market as an asymmetric information game with a finite number of uninformed individuals (insurance firms) and two types of informed individuals (customers of high and low risk). Firms offer insurance contracts consisting of a premium and an indemnity to be paid in case of accident.

It is then shown that a pooling equilibrium does not exist, because pooling profiles can always be destabilized by a deviation to an arbitrarily close contract. Only a particular pair of zero-profit, separating contracts may be an equilibrium. However, if the proportion of high-risk individuals in the population is low, there may exist a contract (which may lie far away from the two candidate contracts) that can destabilize the separating profile.

In this section we show that the separating outcome proposed by Rothschild and Stiglitz (1976) is always a local Nash equilibrium, whereas the pooling one is not.\(^2\)

Consider a large population of individuals with initial wealth \( W \), who face the risk of an accident that would decrease their wealth in an amount \( L \). Individuals are of one of two types. A proportion \( \lambda \) of them have high accident probability \( p_h \), whereas the rest have low accident probability \( p_l < p_h \). All individuals are risk-averse. Their direct utility of money is given by a real function \( U: \mathbb{R} \to \mathbb{R} \), such that \( U'(\cdot) > 0 \) and \( U''(\cdot) < 0 \). Thus, they are willing to buy insurance against accident. Each of \( n > 2 \) risk-neutral firms, \( N = \{1, \ldots, n\} \), offers one insurance contract \((P, I)\), where \( P \) is the premium paid by the insured and \( I \) is the indemnity to be paid by the insurer in case of accident. Given the contracts posted, individuals have to decide whether to buy an insurance policy and which one. Both, individuals and firms, are expected utility maximizers. The expected utility of an individual of type \( i = l, h \) after signing a contract \((P, I)\) is given by

\[
V_i(P, I) = (1 - p_i) \cdot U(W - P) + p_i \cdot U(W - P - L + I).
\]

The expected profit of a firm that sells contract \((P, I)\) to an individual with probability of accident \( p \) is given by

\(^2\)Ania et al. (1998) propose a finite stochastic evolutionary model based on imitation, withdrawal, and local experimentation to study this problem. In their model, local experimentation means precisely that mutating firms enter the market with new contracts that are close to the ones currently offered. They conclude that the separating outcome would be the only one observed in the long run.
\[ \pi(p, (P, I)) = (1 - p) \cdot P + p \cdot (P - I) = P - p \cdot I. \]

The strategy space of each firm is \( S_i = \{ (P, I) \in \mathbb{R}^2_+ \mid V_h(P, I) > V_5(0, 0) \} \), i.e. the set of contracts that are preferred to no insurance by at least one of the types.

In this framework, Rothschild and Stiglitz (1976) show that the only candidates for an equilibrium are profiles such that the only contracts offered are \((P_h, I_h) = (P_h \cdot L, L)\) and \((P_i, I_i)\) such that \(V_h(P_i, I_i) = V_h(P_h, I_h)\) and \(\pi(p_i, (P_i, I_i)) = 0\). They also show that this pair of contracts constitutes a Nash equilibrium if and only if there exists no contract \((P, I)\) such that \(V_h(P, I) > V_i(P_i, I_i)\) and \(\pi(p_h, (P, I)) > 0\), where \(p_h = (1 - \lambda) \cdot p + \lambda \cdot p_h\).

These contracts can be equivalently characterized as follows. \((P_h, I_h)\) is the best contract for high-risk individuals, subject to the constraint that profits per contract are non-negative when sold only to high risks. \((P_i, I_i)\) is the best contract for low-risk individuals, subject to the analogous non-negative profits constraint, and the additional constraint that \((P_i, I_i)\) is not strictly preferred to \((P_h, I_h)\) by the high risk.

Any profile such that at least one firm offers contract \((P_h, I_h)\), and at least two firms offer contract \((P_i, I_i)\) is called a Rothschild–Stiglitz separating profile.

**Proposition 1.** No pooling profile can be a local Nash equilibrium. The Rothschild–Stiglitz separating profiles are always the only local Nash equilibria.

**Proof.** First, note that, at any \((P, I)\), the high-risk indifference curve is steeper than the low-risk one.

Then, for any pooling profile \(s = ((P, I), \ldots, (P, I))\) with \(P > 0\), and for any \(\varepsilon > 0\), there exists a policy \(P' \in B((P, I), \varepsilon)\), such that \(V_h(P', I') > V_h(P, I)\), but \(V_h(P, I) > V_h(P', I')\). By continuity, we can take \(\varepsilon\) small enough that \((P' - p_hI') \in (P - p_hI - \delta, P - p_hI + \delta)\), where \(\delta = (p_h - p_hI)\). If one firm offers \((P', I')\), it will attract all low-risk individuals and realize profits \((P' - p_iI')\) per contract. Since \((P' - p_hI') > (P - p_hI - \delta) = (P - p_hI), (P, I)\) is not a local Nash equilibrium. It is trivial to observe that this is also true if \(P = 0\).

Moreover, any profile in which firms earn non-zero profits is not a local Nash equilibrium. If a firm earns positive profits, it has incentives to either undercut premium or increase indemnity slightly, getting all demand of the corresponding type and increasing profits. Profiles that entail losses for any firm are destabilized if that firm increases premium or lowers indemnity slightly, losing all customers and earning zero profits.

Consider now zero-profit separating outcomes. If the contract bought by the high-risk types is not \((P_h, I_h)\), there exist profitable contracts arbitrarily close, which are preferred by the high risk, and hence this separating outcome is not a local Nash equilibrium.

Given any zero-profit, separating outcome with contract \((P_h, I_h)\) for the high-risk types and any contract different from \((P_i, I_i)\) for the low-risk types, there exist profitable contracts, arbitrarily close, such that they are preferred by the low risk only.

Now consider the Rothschild–Stiglitz separating outcome, where contracts \((P_i, I_i)\) and \((P_h, I_h)\) are the only ones offered, and profits per contract are zero.

First, recall that, at \((P_i, I_i)\), the high-risk indifference curve is steeper than the low-risk one. Since

\[1\text{If only one firm offers } (P_i, I_i), \text{this firm can find close profitable deviations such that the low-risk types are forced to choose the new contract just because the previous one has disappeared, while the high-risk types still prefer } (P_h, I_h). \text{This problem is also present in Rothschild and Stiglitz (1976).}\]
these two curves only intersect at \((P_i, I_i)\), it follows that \((P_h, I_h)\) is in the (open) lower contour set of \((P_i, I_i)\) for the low-risk types. Hence, there exists \(\varepsilon_h > 0\) such that \(V_h(P, I) < V_i(P, I)\) for all \((P, I) \in B((P_h, I_h), \varepsilon_h)\). Thus, a local deviation from \((P_h, I_h)\) to a new contract in \(B((P_h, I_h), \varepsilon_h)\) will either attract all high-risk types, or leave the firm with no customers at all (in the latter case, high risk will either remain with \((P_h, I_h)\), if still available, or buy \((P_i, I_i)\), if there was only one firm offering \((P_h, I_h)\)). It follows from the definition of \((P_h, I_h)\) that, if \(P - 1 \cdot p_h > 0\), then \(V_h(P_h, I_h) > V_h(P, I)\). This implies that deviations of that type yield either losses or zero-profits. Hence, there exists no profitable deviation from \((P_h, I_h)\) in \(B((P_h, I_h), \varepsilon_h)\).

Second, since \(\pi(p_i, (P_i, I_i)) = 0\) and \(p_i < p_h\), there exists \(\varepsilon_i > 0\) such that \(\lambda \cdot \pi(p_h, (P, I)) + (1 - \lambda) \cdot \pi(p_i, (P, I)) < 0\) for all \((P, I) \in B((P_i, I_i), \varepsilon_i)\). By the single-crossing property mentioned above, we can take this \(\varepsilon_i\) small enough that for all \((P, I) \in B((P_i, I_i), \varepsilon_i)\), \(V_i(P, I) > V_i(P_h, I_h)\) (i.e. \(B((P_i, I_i), \varepsilon_i)\) is in the (open) upper contour set of \((P_h, I_h)\) for the low-risk types). Consider now a deviation of a firm offering \((P_i, I_i)\) to a new contract \((P, I) \in B((P_i, I_i), \varepsilon_i)\). There are four possible cases.

Suppose \(V_h(P, I) < V_i(P_i, I_i)\). If \(V_i(P, I) > V_i(P_i, I_i)\), the mutant attracts only low-risk individuals, earning negative profits by definition of \((P_i, I_i)\). If \(V_i(P, I) < V_i(P_i, I_i)\), the mutant attracts no customers (and earns zero profits), because there is at least another firm still offering \((P_i, I_i)\). Suppose now that \(V_h(P, I) > V_i(P_i, I_i)\). If \(V_i(P, I) < V_i(P_i, I_i)\), then the mutant firm would attract all high-risk individuals, but \(P - p_h \cdot I < 0\) by definition of \((P_h, I_h)\). If \(V_i(P, I) > V_i(P_i, I_i)\), then both types prefer \((P, I)\) to \((P_h, I_h)\) and \((P_i, I_i)\), leading to losses. This shows that there are no profitable local deviations from the Rothschild–Stiglitz separating profile.

Now, it is enough to take \(\varepsilon = \min\{\varepsilon_h, \varepsilon_i\}\) and the proof is complete. \(\square\)

4. Walrasian outcomes in Cournot games

Vega-Redondo (1997) studies a stochastic dynamical model of a market in discrete time, in which identical firms, with convex costs, choose quantities from a finite grid. Each period, firms imitate the quantity that yielded highest profits in the previous period. Occasionally, they also experiment with any quantity in the grid, chosen at random. This defines a perturbed Markov chain in the spirit of Kandori et al. (1993). Vega-Redondo shows that the long-run outcome of this process is the Walrasian equilibrium. Alós-Ferrer et al. (1999) then show that this conclusion can be generalized to the case of increasing returns, obtaining price equal marginal cost as the prediction of the corresponding market, when mutation is local, i.e. experimenting firms only change quantities slightly.

Without recurring to the explicitly dynamic and stochastic formalization, these results can be shown to follow from two simple facts. First, with convex cost functions, the Walrasian outcome is a global ESS. This is implicitly shown in the proof of the main result in Vega-Redondo (1997). Second, the price-equal-marginal-cost outcome is a local ESS in general. We verify this below.

Consider the symmetric real game where players are \(n \geq 2\) firms, \(N = \{1, \ldots, n\}\), who decide on quantities \((l = 1), S_i = \mathbb{R}_+\), and payoff functions are

\[
\pi_i(s_i, s_{-i}) = P\left( \sum_{j=1}^{n} s_j \right) \cdot s_i - C(s_i),
\]

where \(C: \mathbb{R}_+ \to \mathbb{R}_+\) is the cost function, \(C'(\cdot) > 0\), \(P: \mathbb{R}_+ \to \mathbb{R}_+\) is the inverse demand function, \(P'(\cdot) < 0\), both functions are twice differentiable, and
\[ 2P'(n \cdot x) < C''(x) \quad \forall \ x \in \mathbb{R}_+, \tag{1} \]

Condition (1) requires that marginal costs do not decrease too rapidly relative to the demand function. It allows for increasing returns to scale (i.e. decreasing marginal costs), provided that they are not too acute and, moreover, it is trivially satisfied under decreasing returns to scale (i.e. if \(C''(\cdot) \geq 0\)). Standard Cournot oligopoly models typically require \(P'(\sum_{i=1}^{n}x_i) - C''(x_i) < 0\) for all \((x_1, \ldots, x_n)\) for equilibrium existence, which implies (1).

**Definition 4.** A (symmetric) Marginal Cost Pricing Equilibrium (MCPE) is a quantity \(y^* \in \mathbb{R}_+\) such that \(P(n \cdot y^*) = C'(y^*)\).

If \(C''(x) \geq 0\) for all \(x\), then \(y^*\) can be thought of as a Walrasian Equilibrium. Otherwise, Walrasian equilibria may not exist (e.g. if \(C''(x) < 0\) for every \(x\)), but an MCPE still exists under quite general conditions.

**Proposition 2.** In the Cournot Game, the MCPE is the unique local ESS.

**Proof.** Define \(\Phi: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}\) by

\[
\Phi(x, y) = [P((n-1)y + x) - C(x)] - [P((n-1)y + x) - C(y)]
= P(n-1)y + x)(x - y) + (C(y) - C(x))
\]

i.e. the differential profits of a mutant (relative to non-mutants) when deviating from a symmetric situation \((y, \ldots, y)\) to a new output level \(x\).

Note that, obviously, \(\Phi(y, y) = 0 \ \forall y\), and that \(\Phi\) is a continuously differentiable function with successive partial derivatives:

\[
\frac{\partial \Phi}{\partial x}(x, y) = P((n-1)y + x) + P'((n-1)y + x)(x - y) - C'(x)
\]
\[
\frac{\partial^2 \Phi}{\partial x^2}(x, y) = 2P'((n-1)y + x) + P''((n-1)y + x)(x - y) - C''(x)
\]

Consider the function \(\Phi'_y: \mathbb{R}_+ \rightarrow \mathbb{R}\), given by \(\Phi'_y(x) = \Phi(x, y)\). If an output \(y\) is a local maximum of its own \(\Phi\), then no close experimentation will destabilize \(y\) by achieving better profits. Such an output would be a local ESS. From the First Order Condition,

\[
\Phi'_y(y) = \frac{\partial \Phi}{\partial x}(y, y) = P(ny) - C'(y) = 0 \Leftrightarrow P(ny) = C'(y),
\]

that is, for an interior output, the equality of price and marginal cost is a necessary condition for local ESS. The Second Order Condition is automatically satisfied by (1):

\[
\Phi''_y(y) = \frac{\partial^2 \Phi}{\partial x^2}(y, y) = 2P'(ny) - C''(y) < 0
\]

This means that the only local ESS output is \(y^*\). Moreover, define
\[ f(y) = \Phi_y'(y) = \frac{\partial \Phi}{\partial x}(y, y) = P(ny) - C'(y) \]
as the slope at \( y \) of the differential profit \( \Phi_y' \). Then, \( f(y^*) = 0 \) and, since \( n \geq 2 \),
\[ f'(y) = nP'(ny) - C''(y) = (n - 2)P'(ny) + \Phi_y''(y) \]
which is negative by (1). It follows that
\[ \text{Sign}\left(\frac{\partial \Phi}{\partial x}(y, y)\right) = \text{Sign}(y^*-y) \]
This means that close experimentations in the direction of \( y^* \) are always profitable. \( \square \)

Note that, if the profit function is strictly concave, the Cournot equilibrium of this game is a strict Nash equilibrium. However, the only local ESS is still the MCPE. This proves that the concepts of local ESS and (local) strict Nash equilibrium are in general unrelated.

5. Conclusion

We have shown that local solution concepts can be fruitful in real games with economic content. Such concepts incorporate the idea of local deviations, which can be regarded as a form of boundedly rational behavior. The framework has been deliberately kept simple for the sake of clarity, but it should be apparent that these concepts have much wider implications than those shown here. For instance, it seems clear that local Nash equilibria should be rest points of any continuous-time dynamics based on local best reply. Analogously, in Section 4 we discussed some evolutionary dynamics whose long-run outcomes turn out to be local ESS.

Acknowledgements

We thank Manfred Nermuth and an anonymous referee for helpful comments and suggestions.

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