Viscoplasticity based on an additive split of the conjugated forces

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ABSTRACT. – A generalized form of viscoplasticity is proposed, within a thermodynamic concept. The theory developed is based on an expansion of the dissipation inequality, where additional quantities are introduced. These quantities are obtained by an additive split of the conjugated thermodynamic forces. Additional potential functions that depend on these quantities can be introduced, which enables one to achieve a generalized form of non-associative viscoplastic theory. Within this concept, the Duvaut–Lions formulation follows naturally as a special case and other important possibilities are also discussed. It is shown that the proposed concept can be generalized to the case where corners exist on the yield and potential functions. Finally, some specific models that all take the Tresca criterion as the yield surface are discussed and used to illustrate some of the findings. © Elsevier, Paris

Introduction

Today there exist two major concepts when formulating viscoplastic models: the Perzyna and the Duvaut-Lions format. A recent review of various viscoplastic formulations is given by Krausz and Krausz (1996). In the Perzyna model (1971), the direction of viscoplastic flow is, in general, determined by the gradient of the plastic potential function calculated at the current stress point. In the Duvaut-Lions model (1972), the direction of viscoplastic flow is determined by the gradient of the static yield surface evaluated at the state, defined by the closest-point-projection, of the current stress, in stress space on the static yield surface. In some simple cases, like linear von Mises hardening these formulations coincide, cf. Runesson et al. (1996). Usually, the evolutions laws for the Duvaut-Lions model are postulated and not derived from a potential function. The reason that the evolution laws are postulated is that the Duvaut-Lions model does not fit into the standard way of treating the dissipation inequality. Here we will present a generalization of viscoplasticity, which includes the Duvaut-Lions model as a special case and which allows evolution laws to be derived from potential functions such that the dissipation inequality is fulfilled. The essential concept in this theory is based on an additive split of the thermodynamic conjugated forces.

The Duvaut-Lions type of model has been used in numerical investigations for some time. Mroz and Sharma (1980) used the somewhat similar model of Phillips and Wu (1973) and Ortiz et al. (1983) adopted the model of Duvaut-Lions (1972). However,
use of this type of model was brought into focus by Simo et al. (1988), where they discussed several numerical aspects of the Duvaut-Lions model. They also generalized the original Duvaut-Lions model to allow for hardening viscoplastic materials. This model will be termed the generalized Duvaut-Lions model. However, according to the derivation given by Simo et al. (1988) this model seems to exclude softening viscoplasticity. We shall show that the generalized Duvaut-Lions formulation certainly can mimic softening effects while fulfilling the dissipation inequality. Turning to numerical issues, it has been shown, cf. Simo et al. (1988), that the generalized Duvaut-Lions model is very attractive, since advantage can be taken of existing drivers for the corresponding inviscid plasticity problem, thereby reducing the solution of the viscoplastic problem to a simple scaling of the thermodynamic forces. Although the model is attractive, a concise theoretical outline of the model is lacking.

This paper is concerned with the theoretical framework found by utilizing an additive split of the conjugated forces entering the dissipation inequality. This general framework allows 4 potential functions to be used in the formulation. Within this concept, the generalized Duvaut-Lions model follows naturally as a special case. Moreover, other interesting formulations which allow for more freedom of the directions of the viscoplastic strain rate can be derived; this is especially important for soil and concrete materials. The concept proposed even allows a consistent theory for corner viscoplasticity to be derived where the number of yield and potential surfaces may differ. Furthermore, it is shown that the Duvaut-Lions model for corner loading can be considered as special case of this general model.

Finally, we illustrate some of the findings by considering some specific models that all take the Tresca criterion as yield surface. Smooth and corner loading situations are considered and the specific models include the generalized Duvaut-Lions models as well as another formulation where the problem of corner loading never arises despite the use of Tresca yield surface.

**Thermodynamic basis**

Small strains are assumed. This allows a decomposition of the total strain tensor into an elastic part $\epsilon_{ij}^e$ and a viscoplastic part $\epsilon_{ij}^p$, i.e.

$$
\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^p
$$

With $\theta$ being the absolute temperature, let us consider the following form of Helmholtz's free energy function $\psi$ per unit volume

$$
\psi = \psi^e (\epsilon_{ij}^e - \epsilon_{ij}^p, \theta) + \psi^p (\kappa_{\alpha}, \theta)
$$

where $\kappa_{\alpha}$ denotes a set of internal variables which may be scalars or second-order tensors. Moreover, the number of internal variables may be one, two or more and this is denoted by the Greek subscript $\alpha$. The decomposition (2) corresponds to the assumption that the instantaneous elastic response does not depend on the internal variables $\kappa_{\alpha}$.
cf. Lubliner (1972). This decomposition is not necessary, but it has been chosen as it facilitates the exposition.

With $s$ being the entropy per unit volume, the Clausius-Duhem inequality takes the form

$$-\dot{\psi} - \dot{\theta} s + \sigma_{ij} \dot{\epsilon}_{ij} \geq 0$$

for any admissible process; here a dot denotes the time rate. Whereas (3) expresses the non-negative mechanical entropy production, the non-negative thermal entropy production is expressed by $-q_i \theta_i / \theta \geq 0$ where $q_i$ is the heat flux vector. As usual, this latter inequality shall be assumed to be fulfilled trivially. Taking the rate of (2) and substituting into (3), we obtain that an allowable solution is given by

$$(4a) \quad \sigma_{ij} = \frac{\partial \psi^e}{\partial \epsilon_{ij}^p};$$

$$(4b) \quad s = -\frac{\partial \psi}{\partial \theta} = -\left( \frac{\partial \psi^e}{\partial \theta} + \frac{\partial \psi^p}{\partial \theta} \right);$$

and the dissipation inequality

$$\gamma \equiv -\frac{\partial \psi^e}{\partial \epsilon_{ij}^p} \dot{\epsilon}_{ij}^p - \frac{\partial \psi^p}{\partial \kappa_\alpha} \dot{\kappa}_\alpha \geq 0$$

Define the thermodynamic forces $\sigma_{ij}^e$ conjugated to the flux $\dot{\epsilon}_{ij}^e$ and the thermodynamic forces $K_\alpha$ conjugated to the fluxes $\dot{\kappa}_\alpha$ by

$$(6a) \quad \sigma_{ij}^e = -\frac{\partial \psi^e}{\partial \epsilon_{ij}^e};$$

$$(6b) \quad K_\alpha = -\frac{\partial \psi^p}{\partial \kappa_\alpha}$$

Since $\partial \psi^e / \partial \epsilon_{ij}^e = -\partial \psi^e / \partial \epsilon_{ij}$, it appears that

$$\sigma_{ij}^e = \sigma_{ij}$$

i.e., the dissipation inequality takes the form

$$\gamma \equiv \sigma_{ij} \dot{\epsilon}_{ij}^e + K_\alpha \dot{\kappa}_\alpha \geq 0$$

Differentiation of (4a) then gives the stress rate, i.e.

$$\dot{\sigma}_{ij} = D_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^p) + M_{ij} \dot{\theta} \quad \text{where} \quad D_{ijkl} = \frac{\partial^2 \psi^e}{\partial \epsilon_{ij}^e \partial \epsilon_{kl}^e}; \quad M_{ij} = \frac{\partial^2 \psi^e}{\partial \epsilon_{ij} \partial \theta}$$
We note that the elastic stiffness tensor $D_{ijkl}$ is symmetric and in general not constant. However, we shall assume $D_{ijkl}$ to be positive definite and to depend only on the temperature $\theta$, i.e.

$$D_{ijkl} = D_{ijkl}(\theta)$$

The rate of the thermodynamic conjugated forces $K_{\alpha}$ is obtained by differentiation of (6b), i.e.

$$\dot{K}_{\alpha} = -d_{\alpha,\beta} \dot{\kappa}_{\beta} - m_{\alpha} \dot{\theta} \quad \text{where} \quad d_{\alpha,\beta} = \frac{\partial^2 \psi_p}{\partial \kappa_{\alpha} \partial \kappa_{\beta}} ; \quad m_{\alpha} = \frac{\partial^2 \psi_p}{\partial \kappa_{\alpha} \partial \theta}$$

It follows that $d_{\alpha,\beta}$ is a symmetric tensor. In general, we have $d_{\alpha,\beta} = d_{\alpha,\beta}(\kappa_{\gamma}, \theta)$, but here we shall assume that $d_{\alpha,\beta}$ only depends on the temperature $\theta$, i.e.

$$d_{\alpha,\beta} = d_{\alpha,\beta}(\theta)$$

**Evolution laws**

Let us first define when viscoplastic loading takes place. The elastic domain is defined by the set

$$B = \{ (\sigma_{ij}, K_{\alpha}) | F(\sigma_{ij}, K_{\alpha}, \theta) \leq 0 \}$$

where $F = 0$ describes the static yield surface. Thus we have viscoplastic loading when

$$F > 0$$

otherwise, we have elastic loading.

From the thermodynamic formulation, the laws for the stress tensor $\sigma_{ij}$ (4a) and the thermodynamic forces $K_{\alpha}$ (6b) were obtained, but no information is given about the evolution laws for the viscoplastic strains and the internal variables. The only restriction on these evolution laws is that the dissipation inequality (8) must be fulfilled.

The usual way to achieve this is to introduce a potential function $G(\sigma_{ij}, K_{\alpha}, \theta)$ generating the following evolution laws

$$\dot{\varepsilon}_{ij} = \lambda \frac{\partial G}{\partial \sigma_{ij}} ; \quad \dot{\kappa}_{\alpha} = \lambda \frac{\partial G}{\partial K_{\alpha}}$$

where $\lambda$ is a known non-negative quantity normally dependent on $F$. In order to fulfill the dissipation inequality, use can be made of the elegant procedure utilizing convex functions; this has been adopted in the context of constitutive mechanics by Edelen (1972) who treated the more general problem involving functionals. Eringen (1975) then established the corresponding procedure suitable for us, i.e. when only functions are involved. Within this procedure we need $G(\sigma_{ij}, K_{\alpha}, \theta)$ to be a convex function in the $\sigma_{ij}, K_{\alpha}$-space and that $G(\sigma_{ij}, K_{\alpha}, \theta) - G(0,0, \theta) \geq 0$. The format known as an associated
viscoplastic model is found by replacing $G$ with $F$ and transferring the requirement on $G$ to $F$. The above model is of course the well known Perzyna model (1971). Note that the only restriction put on $F$ is that it should be a closed set, i.e. that in the general case we allow $F$ to be non-convex.

It appears that the usual procedure consists of establishing the evolution laws from a single potential function that depends on the thermodynamic forces conjugated to $\epsilon_{ij}^{vp}$ and $\kappa_\alpha$, i.e. $\sigma_{ij}$ and $K_\alpha$, cf. (8). Let us deviate from this procedure and instead make an additive decomposition of these thermodynamic forces. In turn, this will imply that two potential functions will generate the evolution equations. Let us write $\sigma_{ij}$ and $K_\alpha$ as

\begin{equation}
\sigma_{ij} = \sigma_{ij}^* + \sigma_{ij}^* \quad \text{where} \quad \sigma_{ij}^* = \sigma_{ij} - \sigma_{ij}^*,
\end{equation}

\begin{equation}
K_\alpha = K_\alpha^* + K_\alpha^* \quad \text{where} \quad K_\alpha^* = K_\alpha - K_\alpha^*,
\end{equation}

where $\sigma_{ij}^*$ and $K_\alpha^*$ are quantities not yet specified. The dissipation inequality (8) now takes the form

\begin{equation}
\gamma \equiv \sigma_{ij}^* \epsilon_{ij}^{vp} + K_\alpha^* \kappa_\alpha + \sigma_{ij}^* \epsilon_{ij}^{vp} + K_\alpha^* \kappa_\alpha \geq 0
\end{equation}

This approach may be generalized to include more terms, but (17) suffices for our purpose. To specify the quantities $\sigma_{ij}^*$ and $K_\alpha^*$, we assume that there exists a hyper-surface $F^\gamma$ such that

\begin{equation}
(\sigma_{ij}^*, K_\alpha^*) = \{(\sigma_{ij}, K_\alpha) \mid F^\gamma(\sigma_{ij}, K_\alpha, \theta) = 0\}
\end{equation}

i.e. $(\sigma_{ij}^*, K_\alpha^*)$ is a state defined such that $F^\gamma(\sigma_{ij}, K_\alpha^*, \theta) = 0$ is satisfied. The dissipation inequality (17) will now be decomposed into two parts, both assumed to be greater than or equal to zero, i.e.

\begin{equation}
\gamma_1 \equiv \sigma_{ij}^* \epsilon_{ij}^{vp} + K_\alpha^* \kappa_\alpha \geq 0 \quad \text{and} \quad \gamma_2 \equiv \sigma_{ij}^* \epsilon_{ij}^{vp} + K_\alpha^* \kappa_\alpha \geq 0
\end{equation}

Again, we shall take advantage of the procedure utilizing convex functions, Eringen (1975). Therefore, we assume that there exists a convex function $G^\gamma(\sigma_{ij}, K_\alpha^*, \theta)$ in the $\sigma_{ij}^*, K_\alpha^*$-space with the property $G^\gamma(\sigma_{ij}, K_\alpha^*, \theta) - G^\gamma(0, 0, \theta) \geq 0$. Then (19a) is satisfied if

\begin{equation}
\dot{\epsilon}_{ij}^{vp} = \chi \frac{\partial G^\gamma}{\partial \sigma_{ij}^*} \quad \chi \geq 0
\end{equation}

\begin{equation}
\dot{\kappa}_\alpha = \chi \frac{\partial G^\gamma}{\partial K_\alpha^*}
\end{equation}

where $\chi$ is an arbitrary non-negative quantity. Clearly, we can choose $G^\gamma = F^\gamma$, but then we must require that $F^\gamma$ fulfills the convexity requirement and $F^\gamma(\sigma_{ij}^*, K_\alpha^*, \theta) - F^\gamma(0, 0, \theta) \geq 0$. Now turning to the second inequality (19b), we can proceed in the same way as above. Therefore, we seek a convex function $G(\sigma_{ij}^*, K_\alpha^*, \theta)$ in the admissible $\sigma_{ij}^*, K_\alpha^*$-space with the property $G(\sigma_{ij}^*, K_\alpha^*, \theta) - G(0, 0, \theta) \geq 0$. We are considering the
situation of viscoplastic loading, i.e. \( F(\sigma_{ij}, K_{\alpha}, \theta) > 0 \). Moreover, \( \dot{\epsilon}_{ij}^p, K_{\alpha}^c \) fulfil (18) and the admissible \( \sigma_{ij}^*, K_{\alpha}^* \)-space is formed by these restrictions. Then \( \gamma_2 \geq 0 \) is satisfied if

\[
\dot{\epsilon}_{ij}^p = \Lambda \frac{\partial G}{\partial \sigma_{ij}^*}, \quad \Lambda \geq 0 \\
\kappa_{\alpha} = \Lambda \frac{\partial G}{\partial K_{\alpha}^*}
\]

(21)

In principle, \( \Lambda \) is an arbitrary non-negative quantity and we shall here take \( \Lambda \) to be a known non-negative quantity. Since we have treated \( \gamma_1 \) and \( \gamma_2 \) separately, we must make sure that all relations are fulfilled at the same time. This can only be achieved if the evolutions laws for \( \dot{\epsilon}_{ij}^p \) and \( \kappa_{\alpha} \) defined by (20) and (21) display the same response for all values of \( \sigma_{ij}^*, \sigma_{ij}^c, K_{\alpha}^* \) and \( K_{\alpha}^c \), i.e. we require that

\[
\chi \frac{\partial G^c}{\partial \sigma_{ij}^c} = \Lambda \frac{\partial G}{\partial \sigma_{ij}^*} \\
\chi \frac{\partial G^c}{\partial K_{\alpha}^c} = \Lambda \frac{\partial G}{\partial K_{\alpha}^*}
\]

(22)

These equations together with (18) comprise \((7 + \alpha)\) equations; with \( \sigma_{ij}, K_{\alpha} \) and \( \Lambda \) being known, then these equations determine the \((7 + \alpha)\) unknown quantities \( \sigma_{ij}^*, K_{\alpha}^* \) and \( \chi \).

In general, however, the existence of a solution to (22) cannot always be ensured. As an example, if \( G^c \) is chosen as a von Mises function and \( G \) as a Drucker-Prager function, a solution to (22) does not exist. Therefore, care must be taken when choosing the functions \( G^c \) and \( G \) so that these choices allow a solution.

As previously discussed, fulfilment of the dissipation inequality puts restrictions on the two potential functions \( G^c \) and \( G \); among other things that they are convex functions. It is of considerable interest that no restrictions are placed on the static yield function \( F \) or on the function \( F^c \); in general, therefore, in order to fulfil the dissipation inequality the static yield surface need not be convex. It is also clear that the convexity requirement on \( G^c \) and \( G \) is a sufficient condition and could be relaxed, but this sufficient condition suits our purpose when deriving specific formulations.

We also mention that associated viscoplasticity is defined as \( \dot{\epsilon}_{ij}^p = \lambda \frac{\partial F}{\partial \sigma_{ij}} \) and \( \kappa_{\alpha} = \lambda \frac{\partial F}{\partial K_{\alpha}} \). Therefore, it is evident that the viscoplastic formulation presented above is inherently non-associative. An associated model can only be obtained in very special cases, but it is clear that such a model must correspond to (15), i.e. we obtain the Perzyna model.

With these remarks and within the framework presented above, let us now turn our attention to some specific formulations.

**Generalized Duvaut-Lions formulation**

To derive this specific model let us start with choosing proper forms for the function \( F^c \) and the potential functions \( G^c \) and \( G \) defined previously. For \( F^c \) entering (18) we choose

\[
F^c = F(\sigma_{ij}^c, K_{\alpha}^c, \theta)
\]

(23)

where \( F \) is the static yield surface. Moreover, we also choose the convex function appearing in (20) as

\[
G^c = F(\sigma^c_{ij}, K^c_\alpha, \theta)
\]

With these definitions we now require that \( F \) is a convex function in the \( \sigma_{ij}, K_\alpha \)-space with the property \( F(\sigma^c_{ij}, K^c_\alpha, \theta) - F(0,0, \theta) \geq 0 \); this ensures that \( \gamma_1 \geq 0 \). The choice (24) could be interpreted such that the part related to quantities with superscript \( c \) forms an associated viscoplastic theory, cf. (20). Before choosing a specific form for the potential function \( G \), we note the definitions (9) and (11) for \( D_{ijkl} \) and \( d_{ij, \alpha} \), respectively, and introduce the inverse tensors according to

\[
C_{ijkl} D_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{ij} \delta_{kl}) ; \quad c_{ij, \alpha} d_{ij, \alpha} = \delta_{ij, \alpha}
\]

where \( C_{ijkl} \) is the elastic flexibility tensor which for a given temperature is constant, cf. (10). Let us next choose the function \( G(\sigma^c_{ij}, K^c_\alpha, \theta) \) present in (21) as

\[
G(\sigma^c_{ij}, K^c_\alpha, \theta) = \frac{1}{2} \sigma^c_{ij} C_{ijkl}(\theta) \sigma^c_{kl} + \frac{1}{2} K^c_\alpha c_{ij, \alpha}(\theta) K^c_\alpha
\]

Since \( C_{ijkl} \) is positive definite, it is evident that the function \( G(\sigma^c_{ij}, K^c_\alpha, \theta) \) is convex in the \( \sigma^c_{ij}, K^c_\alpha \)-space and that \( G(\sigma^c_{ij}, K^c_\alpha, \theta) - G(0,0, \theta) \geq 0 \) if \( c_{ij, \alpha} \) is positive semi-definite. However, there are even cases where \( c_{ij, \alpha} \) is allowed be negative definite. This is a consequence of the admissible space for \( \sigma^c_{ij}, K^c_\alpha \) being restricted, as discussed in relation to (21); we shall return to this aspect.

Now that proper forms of all potential functions are established, although some of them are not specified in detail, the model can be outlined. The evolution laws (21) then become

\[
\dot{\epsilon}^p_{ij} = \Lambda C_{ijkl}(\sigma_{kl} - \sigma^c_{kl}) ; \quad \dot{\kappa}_\alpha = \Lambda c_{ij, \alpha}(K_\beta - K^c_\beta)
\]

where \( \Lambda \) is any non-negative quantity chosen by us. We may for instance choose

\[
\Lambda = \Lambda(\sigma_{ij} - \sigma^c_{ij}, K_\alpha - K^c_\alpha, \theta)
\]

The final task is to establish relations such that quantities with superscript \( c \) can be determined. From (22), (24) and (26) as well as (18) and (23) we obtain

\[
\lambda^c \frac{\partial F}{\partial \sigma^c_{ij}} = \Lambda C_{ijkl}(\sigma_{kl} - \sigma^c_{kl})
\]

\[
\lambda^c \frac{\partial F}{\partial K^c_\alpha} = \Lambda c_{ij, \alpha}(K_\beta - K^c_\beta)
\]

\[
F(\sigma^c_{ij}, K^c_\alpha, \theta) = 0
\]

With \( (\sigma_{ij}, K_\alpha, \theta) \) being known, equation system (29) comprises \( (7 + \alpha) \) equations with \( (7 + \alpha) \) unknowns \( (\sigma^c_{ij}, K^c_\alpha, \lambda^c) \) and it can therefore be solved. For completeness we note that combination of (9), (11) and (27) yields

\[
\dot{\sigma}_{ij} = D_{ijkl} \dot{\epsilon}_{kl} - \Lambda(\sigma_{ij} - \sigma^c_{ij}) + M_{ij} \dot{\theta}
\]

\[
\dot{K}_\alpha = -\Lambda(K_\alpha - K^c_\alpha) - m_\alpha \dot{\theta}
\]

\[
\]
It is evident that we have recovered the formulation of Duvaut and Lion (1972), here presented in a form identical with that of Simo et al. (1988) although they did not derive the model using the present concept. The difference in results is that we have allowed $C_{ijkl}$ and $c_{\alpha\beta}$ to depend on the temperature $\theta$ whereas Simo et al. (1988) assume these quantities to be constant. Moreover, Simo et al. (1988) required $c_{\alpha\beta}$ to be positive definite, thereby excluding the important case of softening viscoplasticity. As previously touched upon, this requirement can be relaxed.

The restriction that $G$ is convex and the $G(\sigma^c_{ij}, K^c_{\alpha}, \theta) - G(0, 0, 0) \geq 0$ both boil down to the requirement that $G(\sigma^c_{ij}, K^c_{\alpha}, \theta) \geq 0$. Now, use of (29) in (26) yields

$$
G = \frac{1}{2} \left( \frac{\lambda^c}{\lambda} \right)^2 A
$$

where

$$
A = \frac{\partial F}{\partial \sigma^c_{ij}} D_{ijkl} \frac{\partial F}{\partial \sigma^c_{kl}} + \frac{\partial F}{\partial K^c_{\alpha}} d_{\alpha\beta} \frac{\partial F}{\partial K^c_{\gamma}}
$$

This means, we must require that the quantity $A$ is non-negative. However, this quantity is the same as encountered in inviscid plasticity where the plastic modulus is defined by $H = \partial F / \partial K^c_{\alpha} d_{\alpha\beta} \partial F / \partial K^c_{\gamma}$. Considering, for instance, inviscid von Mises plasticity, we obtain the requirement $A = 3G + H > 0$. It then appears that requiring $D_{ijkl}$ and $d_{\alpha\beta}$ to be positive definite matrices is a sufficient condition, but not a necessary one. A sufficient and necessary condition is that $A > 0$ and this allows softening viscoplasticity to be considered.

It is recalled that since the formulation of Duvaut and Lions fits into the present framework, this formulation is intrinsically non-associated. Let us now turn to the derivations found in the literature, where the evolution laws for internal variables, corresponding to (27), are postulated without introducing the potential function $G$ given by (26). Therefore, in the literature the necessary condition (22) is not introduced and instead the motivation of the model is based on the concept of closest-point-projection of $\sigma_{ij}, K_{\alpha}$ onto the static yield surface $F$ which determines the quantities $\sigma^c_{ij}$ and $K^c_{\alpha}$. The main arguments for this approach seem to be based on numerical issues. Naturally, it does not seem to be attractive to motivate a physical model by numerical viewpoints. However, the derivation above illustrates clearly that the model follows from sound thermodynamic considerations.

Let us now show that the quantities $\sigma^c_{ij}$ and $K^c_{\alpha}$ in the specific case considered, are the closest-point projection of $\sigma_{ij}$ and $K_{\alpha}$ on the static yield surface $F$. For this purpose, we introduce a metric related to the $\sigma_{ij}, K_{\alpha}$-space defined by

$$
\begin{bmatrix}
C_{ijkl} & 0 \\
0 & c_{\alpha\beta}
\end{bmatrix}
$$

In this $\sigma_{ij}, K_{\alpha}$-space, the distance $ds$ between the state $\sigma_{ij}, K_{\alpha}$ and the state $\sigma^c_{ij}, K^c_{\alpha}$ is, per definition, given by

$$
ds^2 = \frac{1}{2} (\sigma_{ij} - \sigma^c_{ij}) C_{ijkl}(\theta)(\sigma_{kl} - \sigma^c_{kl}) + \frac{1}{2} (K_{\alpha} - K^c_{\alpha}) c_{\alpha\beta}(\theta)(K_{\gamma} - K^c_{\gamma})
$$
where the factor 1/2 has been introduced for convenience. The state \( \sigma'_{ij}, K'_n \) is now defined to be the closest-point-projection of \( \sigma_{ij}, K_n \) on the static yield surface \( F \).

With these preliminaries, we may now determine the state \( \sigma'_{ij}, K'_n \) according to the following constrained minimization problem: For given \( \sigma_{ij}, K_n \) and temperature \( \theta \), find \( \sigma'_{ij}, K'_n \) such that \( ds^2 \), defined by (34), is minimized under the constraint given by \( F(\sigma'_{ij}, K'_n, \theta) = 0 \). The Lagrangian multiplier method (e.g. Luenberger (1984) p. 314) then provides the following solution

\[
-C_{ijkl}(\sigma_{kl} - \sigma'_{kl}) + \mu \frac{\partial F}{\partial \sigma'_{ij}} = 0
\]

(35)

\[
-c_{n,j}(K_{,j} - K'_j) + \mu \frac{\partial F}{\partial K'_n} = 0
\]

\[
F(\sigma'_{ij}, K'_n, \theta) = 0
\]

where \( \mu \) is a Lagrangian multiplier. A comparison with (29) shows that we can identify \( \mu \) as \( \lambda' / \Lambda \). It appears that in the present case, the quantities \( \sigma'_{ij} \) and \( K'_n \) are the closest-point-projection of \( \sigma_{ij} \) and \( K_n \) on the static yield surface \( F \).

With the metric defined by (33) and introduced by Simo et al. (1988), an interesting analogy emerges. To see this, multiply the first two equations of (35) by \( D_{ijkl} \) and \( d_{n,j} \), respectively to obtain

\[
\sigma'_{ij} = \sigma_{ij} - \mu D_{ijkl} \frac{\partial F}{\partial \sigma'_{ij}}
\]

(36)

\[
K'_n = K_n - \mu d_{n,j} \frac{\partial F}{\partial K'_j}
\]

\[
F(\sigma'_{ij}, K'_n, \theta) = 0
\]

Consider now a fully implicit Euler backward scheme when integrating the constitutive equations in inviscid associated plasticity. Let \( \sigma_{ij} \) and \( K_n \) be interpreted as the trial stress and hardening parameters, respectively, in the load step considered, whereas \( \sigma'_{ij} \) and \( K'_n \) are interpreted as the final stress state and the final hardening parameters, respectively; moreover, let \( \mu \) be viewed as the increment \( \Delta \lambda \) of the plastic multiplier. It then appears that (36) precisely expresses the fully implicit radial return scheme. This interesting aspect may be of advantage when implementing a viscoplastic formulation in a computer program, as discussed by Simo et al. (1988).

Some clarifying remarks related to the Duvaut-Lions type of viscoplasticity may be in order. Duvaut and Lions (1972) considered perfect viscoplasticity, i.e. no internal variables \( \kappa_n \), and made the following proposal

\[
\dot{\varepsilon}^{\text{vp}}_{ij} = \frac{1}{\tau^*} (\sigma_{ij} - \sigma'_{ij})
\]

(37)

where \( \sigma'_{ij} \) is the closest-point-projection in the \( \sigma_{ij} \)-space with the usual Euclidean metric. Moreover, \( \tau^* \) is some positive parameter. This formulation is illustrated in Fig. 1 and it
appears that $\dot{\varepsilon}_{ij}^{vp}$ is normal to the static yield surface. To obtain a formulation where $\dot{\varepsilon}_{ij}^{vp}$ depends nonlinearly on the stresses, we shall instead of (37) adopt

$$
\dot{\varepsilon}_{ij}^{vp} = \frac{h(\sigma_{ij} - \sigma_{ij}^c)}{\tau^*}(\sigma_{ij} - \sigma_{ij}^c)
$$

Phillips and Wu (1973) assumed incompressible viscoplasticity and proposed the following model

$$
\dot{\varepsilon}_{ij}^{vp} = \gamma \chi(q)(s_{ij} - s_{ij}^c)
$$

where $s_{ij}$ is the deviatoric stress tensor and $s_{ij}^c$ is the closest-point-projection on the static yield surface using the Euclidean metric. Moreover, $\gamma$ is some parameter and the function $\chi(q)$ depends on the distance $q = [(s_{ij} - s_{ij}^c)(s_{ij} - s_{ij}^c)]^{1/2}$. The model is illustrated in Fig. 2 and it appears to have much in common with the original Duvaut-Lions formulation.

Simo et al. (1988) generalized the Duvaut-Lions formulation by considering the effects of internal variables and by using the metric defined by (33). It is of interest that by comparing the first equation of (29) with the evolution law for $\dot{\varepsilon}_{ij}^{vp}$ given by (27), it appears that $\dot{\varepsilon}_{ij}^{vp}$ is normal to the static yield surface at the closest-point-projection.
just like the Duvaut-Lions and Phillips-Wu formulations. The essential difference is the definition of the closest-point-projection, cf. Fig. 3 with Figs. 1 and 2.

\[ F = \sigma_{eff} - \sigma_{yo} = 0 : \quad \sigma_{eff} = \left( \frac{3}{2} \frac{s_{ij}}{s_{ij}} \right)^{1/2} \]

where \( \sigma_{yo} \) is the constant static yield stress. For all three formulations, we then obtain

\[ \sigma_{k_k} = \sigma_{kk} : \quad s_{ij}^l = \frac{\sigma_{yo}}{\sigma_{eff}} s_{ij} \]

and (27) then reduces to

\[ \dot{\varepsilon}_{ij}^{ep} = \frac{\Lambda}{2G} (s_{ij} - s_{ij}^l) = \frac{\Lambda}{2G} \frac{\sigma_{eff} - \sigma_{yo}}{\sigma_{eff}} s_{ij} \]

which evidently is of the same form as (38) and (39). Choosing \( \Lambda = \Lambda(q) \) where

\[ q = \left[ (s_{ij} - s_{ij}^l)(s_{ij} - s_{ij}^l) \right]^{1/2} \]

we obtain

\[ \Lambda = \Lambda(\sigma_{eff} - \sigma_{yo}) \]

Moreover, the format given by (42) and (43) turns out to be identical to the associated Perzyna model (1971) which states that

\[ \dot{\varepsilon}_{ij}^{ep} = \frac{\phi(F) > 0 \partial F}{\eta \partial \sigma_{ij}} \]

where \( \phi(F) > 0 \) if \( F > 0 \); otherwise \( \phi(F) = 0 \). With \( F \) given by (40), the Perzyna model then becomes

\[ \dot{\varepsilon}_{ij}^{ep} = \frac{3}{2} \frac{\phi(\sigma_{eff} - \sigma_{yo})}{\eta \sigma_{eff}} s_{ij} \]
where the notation \( \langle \phi \rangle \) has been dropped, since only viscoplastic development is considered. A comparison of (42) with (43) shows a complete correspondance, implying that the generalized Duvaut-Lions model coincides with the Perzyna model.

Other formulations

Apart from the Duvaut-Lions model discussed above, an arsenal of other models fits into the framework proposed, which hinges intimately on an additive split of the conjugated forces.

The Duvaut-Lions model emerges for the choice \( G^r = F^r = F \) and with \( G \) given by (26). An immediate generalization suggests itself by choosing

\[
F^r = F(\tau_{ij}^r, K_i, \theta)
\]

and

\[
G^r = F(\sigma_{ij}^r, K_i, \theta) + H(K_i)
\]

whereas \( G \) still is chosen in accordance with (26), i.e. the evolution laws becomes identical to (27), i.e.

\[
\varepsilon_{ij}^\text{th} = \Lambda G_{ijkl}(\sigma_{kl} - \tau_{kl}^r) \quad ; \quad \dot{\epsilon}_i = \Lambda a_i (K_i - K_i^r)
\]

Let us choose the function \( H(K_i) \) as

\[
H = \frac{1}{2} K_i h_{ij} K_j
\]

where \( h_{ij} \) is a constant positive semi-definite matrix. If the yield function \( F \) then is a convex function in the \( \sigma_{ij}^r, K_i \)-space, so is the function \( G^r \). Moreover, if \( F \) fulfills the requirement \( F(\sigma_{ij}^r, K_i, \theta) - F(0, 0, \theta) \geq 0 \), as it usually does, it follows that \( G^r(\sigma_{ij}^r, K_i, \theta) - G^r(0, 0, \theta) \geq 0 \), i.e. all requirements related to \( G^r \) and discussed in relation to (19) are fulfilled. As \( G \) given by (26) also fulfills all requirements, the dissipation inequality is fulfilled.

By this choice, we have, in fact, arrived at the formulation adopted by Johansson (1996) who demonstrated that close fits can be obtained to experimental data for INCONEL 718 loaded by various strain rates, even though they did not make use of the present framework.

To further explore the characteristics of the formulation discussed, we make use of (26), (47) and (49) in (22) to obtain

\[
\lambda' \frac{\partial F}{\partial \sigma_{ij}} = \Lambda G_{ijkl}(\sigma_{kl} - \tau_{kl}^r)
\]

\[
\lambda' \left( \frac{\partial F}{\partial K_i} + \frac{\partial H}{\partial K_i} \right) = \Lambda a_i (K_i - K_i^r)
\]
a comparison with (29) shows immediately that the state \((\sigma_{ij}, K_{\alpha})\) can no longer be viewed as the closest-point-projection on \(F^c = F = 0\).

A further generalization is possible if the functions \(F^c\) and \(G^c\) are chosen independently of each other. Let us choose \(F^c\) and \(G^c\) as

\[
\begin{align*}
F^c &= F(\sigma_{ij}, K_{\alpha}, \theta) \\
G^c &= G^c(\sigma_{ij}, K_{\alpha}, \theta)
\end{align*}
\]

(51)

The function \(G\) is again chosen as (26), i.e. the evolution laws (48) hold. However, relations (22) become

\[
\begin{align*}
\lambda^c \frac{\partial G^c}{\partial \sigma_{ij}} &= \Lambda C_{ijkl}(\sigma_{kl} - \sigma_{kl}') \\
\lambda^c \frac{\partial G^c}{\partial K^c} &= \Lambda c_{\alpha}\delta(K_{\alpha} - K_{\alpha}')
\end{align*}
\]

(52)

A comparison with (29) shows again that the state \((\sigma_{ij}, K_{\alpha})\) can no longer be viewed as the closest-point-projection on \(F^c = F = 0\). Moreover, insertion of the evolution laws (48) into (52) gives

\[
\dot{\epsilon}_{ij}^{\text{up}} = \lambda^c \frac{\partial G^c}{\partial \sigma_{ij}} ; \quad \dot{\kappa}_{\alpha} = \lambda^c \frac{\partial G^c}{\partial K^c}
\]

(53)

In the Duvaut-Lions formulation \(G^c = F\) and \(\dot{\epsilon}_{ij}^{\text{up}}\) and \(\dot{\kappa}_{\alpha}\) are then normal to the \(F\)-surface evaluated at the state \((\sigma_{ij}, K_{\alpha})\). With the format given by (53), however, we are free to choose the function \(G^c\) and this implies that there is a freedom in the direction of \(\dot{\epsilon}_{ij}^{\text{up}}\) and \(\dot{\kappa}_{\alpha}\) and thereby a significant flexibility to better fit experimental data, especially for soil and concrete materials. For these materials, this freedom can be illustrated by choosing the yield function \(F\) as a Drucker-Prager function and the potential function \(G^c\) as another Drucker-Prager function with another slope; thereby modelling of realistic dilatancy properties is enhanced.

Let us finally consider a model where

\[
G^c = F^c \neq F
\]

(54)

whereas the function \(G\) still is given by (26), i.e. the evolution laws (48) hold again. With (26) and (54), relations (22) become

\[
\begin{align*}
\lambda^c \frac{\partial F^c}{\partial \sigma_{ij}} &= \Lambda C_{ijkl}(\sigma_{kl} - \sigma_{kl}') \\
\lambda^c \frac{\partial F^c}{\partial K_{\alpha}} &= \Lambda c_{\alpha}\delta(K_{\alpha} - K_{\alpha}')
\end{align*}
\]

(55)

Now the state \((\sigma_{ij}, K_{\alpha})\) can again be viewed as the closest point-projection on \(F^c = 0\). However, the important issue is that if \(F^c = 0\) is a smooth surface whereas the yield function \(F\) possesses corners, these corners will not imply any complications of the
formulation. We shall later return to this issue and illustrate the situation where the yield function is a Tresca-surface whereas $F^c$ is chosen as a von Mises function.

It has been illustrated that the Duvaut-Lions models, as well as the model discussed above, fit into the general framework proposed. Evidently, since the concept involves the four functions $F^c, G^c, G$ and $F$ there are more possibilities for specific models, but the models discussed above suffice for illustration of the generality of the proposed concept.

**Multiple yield and potential functions**

Previously, we assumed the potential and static yield surfaces to be smooth. Let us next generalize the concept to include the situation where multiple potential and static yield surfaces meet at a corner. Let us first define when viscoplastic loading can take place in this generalized situation. The elastic domain is now defined by the set

$$
B = \{ (\sigma_{ij}, K_\alpha) \mid F_I(\sigma_{ij}, K_\alpha, \theta) \leq 0 \} \quad I = 1, 2, \ldots, F_{total}
$$

where $F_I$ is the static yield surface with number $I$ and $I$ ranges from 1 to the total number of the static yield surfaces $F_{total}$ involved. It is assumed that each of the individual static yield surfaces is smooth. With the definition (56), we then have viscoplastic loading if

$$
F_I(\sigma_{ij}, K_\alpha, \theta) > 0 \quad \text{for at least one } I\text{-value}
$$

otherwise, we have elastic loading. As previously discussed in connection with smooth surfaces, $B$ is a closed set and it is allowed to be non-convex. Later, restrictions will be imposed on $B$ when specific models are established.

Let us now recall the concept of additive split of the conjugated forces. From (19) we have the expanded form of the dissipation inequality given as

$$
\gamma_1 \equiv \sigma_{ij}^c \varepsilon_{ij}^{vp} + K_\alpha^c \kappa_\alpha \geq 0 \quad \text{and} \quad \gamma_2 \equiv \sigma_{ij}^* \varepsilon_{ij}^{vp} + K_\alpha^* \kappa_\alpha \geq 0
$$

where

$$
\sigma_{ij}^* = \sigma_{ij} - \sigma_{ij}^c \quad ; \quad K_\alpha^* = K_\alpha - K_\alpha^c
$$

and where the additional quantities $\sigma_{ij}^c$ and $K_\alpha^c$ have been introduced. To define the admissible space for these quantities, we assume that there exists a domain defined by the set

$$
B^c = \{ (\sigma_{ij}, K_\alpha) \mid F^c_I(\sigma_{ij}, K_\alpha, \theta) \leq 0 \} \quad J = 1, 2, \ldots, F_{total}^c
$$

where $J$ ranges from 1 to the total number of functions $F_{total}^c$ involved. From (60) a hyper-surface is defined by the boundary of $B^c$, i.e. $\partial B^c$. With these definitions we can define the admissible space for the quantities with superscript $c$ as

$$
(\sigma_{ij}^c, K_\alpha^c, \theta) \in \partial B^c
$$
Let us now return to the dissipation inequality (58) and proceed in the same way as for the case of smooth yield and potential functions. To fulfil the inequality \( \gamma_1 \geq 0 \), we first introduce the following potential functions

\[
G^c_\Phi = G^c_{\Phi} (\sigma^c_{ij}, K^c_\alpha, \theta) \quad \Phi = 1, 2, \ldots, G^c_{\text{max}, r}
\]

where \( G^c_{\text{max}, r} \) is the total number of potential functions meeting at the corner in question. We then assume that each of the potential functions is convex and obeys

\[
G^c_{\Phi} (\sigma^c_{ij}, K^c_\alpha, \theta) - G^c_\Phi (0, 0, \theta) \geq 0.
\]

The evolution laws then become

\[
\begin{align*}
\dot{\varepsilon}^{c t}_{i j} &= \sum_{\Phi=1}^{G^c_{\text{max}, r}} \lambda^c_\Phi \frac{\partial G^c_\Phi}{\partial \sigma^c_{i j}} \\
\dot{\kappa}^c_\alpha &= \sum_{\Phi=1}^{G^c_{\text{max}, r}} \lambda^c_\Phi \frac{\partial G^c_\Phi}{\partial K^c_\alpha} \\
\lambda^c_\Phi &\geq 0
\end{align*}
\]

where \( \lambda^c_\Phi \) are arbitrary non-negative quantities. To appreciate these evolution laws, insert (63) into the expression for \( \gamma_1 \). Since every potential function fulfils the convexity requirement as well as

\[
G^c_{\Phi} (\sigma^c_{ij}, K^c_\alpha, \theta) - G^c_\Phi (0, 0, \theta) \geq 0,
\]

it follows that every term \( \sigma^c_{ij} \frac{\partial G^c_\Phi}{\partial \sigma^c_{ij}} + K^c_\alpha \frac{\partial G^c_\Phi}{\partial K^c_\alpha} \) is non-negative and thereby that the inequality \( \gamma_1 \geq 0 \) is fulfilled.

Let us next turn to the inequality \( \gamma_2 \geq 0 \). We first introduce the following potential functions

\[
G_{\Theta} = G_{\Theta} (\sigma^*_{ij}, K^*_\alpha, \theta) \quad \Theta = 1, 2, \ldots, G_{\text{max}, r}
\]

where \( G_{\text{max}, r} \) is the total number of potential functions meeting at the corner in question. Proceeding as above, we assume that each of these potential functions is convex and obeys

\[
G_{\Theta} (\sigma^*_{ij}, K^*_\alpha, \theta) - G_{\Theta} (0, 0, \theta) \geq 0.
\]

The evolution laws then become

\[
\begin{align*}
\dot{\varepsilon}^{* t}_{i j} &= \sum_{\Theta=1}^{G_{\text{max}, r}} \Lambda^*_{\Theta} \frac{\partial G_{\Theta}}{\partial \sigma^*_{i j}} \\
\dot{\kappa}^*_\alpha &= \sum_{\Theta=1}^{G_{\text{max}, r}} \Lambda^*_{\Theta} \frac{\partial G_{\Theta}}{\partial K^*_\alpha} \\
\Lambda^*_{\Theta} &\geq 0
\end{align*}
\]

In principle, \( \Lambda^*_{\Theta} \) are arbitrary non-negative quantities and it is therefore allowable to take \( \Lambda^*_{\Theta} \) as known non-negative quantities.

Since we have treated the evolution laws (63) and (65) separately, we must make sure that all relations are fulfilled at the same time. This can only be achieved if the following relations are fulfilled

\[
\begin{align*}
\sum_{\Phi=1}^{G^c_{\text{max}, r}} \lambda^c_\Phi \frac{\partial G^c_\Phi}{\partial \sigma^c_{i j}} &= \sum_{\Theta=1}^{G_{\text{max}, r}} \Lambda^*_{\Theta} \frac{\partial G_{\Theta}}{\partial \sigma^*_{i j}} \\
\sum_{\Phi=1}^{G^c_{\text{max}, r}} \lambda^c_\Phi \frac{\partial G^c_\Phi}{\partial K^c_\alpha} &= \sum_{\Theta=1}^{G_{\text{max}, r}} \Lambda^*_{\Theta} \frac{\partial G_{\Theta}}{\partial K^*_\alpha} \\
F^c_J (\sigma^c_{ij}, K^c_\alpha, \theta) &= 0 \quad J = 1, 2, \ldots, F^c_{\text{max}, r}
\end{align*}
\]
The last equation above arises from (61). However, whereas (61) just defines the boundary \( \partial B^c \) of the region \( B^c \), the last equation in (66) involves the specific position of \((\sigma_{ij}^c, K^c_\alpha)\) on \( \partial B^c \) and this specific position may be located on a smooth part of \( \partial B^c \) whereby \( F_{max}^c = 1 \) or on a corner of \( \partial B^c \) whereby \( F_{max}^c > 1 \). The number \( F_{max}^c \) is therefore the number of active \( F^c \)-surfaces at the corner in question. With \((\sigma_{ij}, K_\alpha, \theta)\) and \( \lambda_\phi \) being known, the above equations comprise \((6 + \alpha + F_{max}^c)\) equations involving the \((6 + \alpha + G_{max}^c)\) unknown quantities \( \sigma_{ij}^c, K^c_\alpha \) and \( \lambda_\phi^c \). From this it appears immediately that we must require

(67) \[ F_{max}^c \geq G_{max}^c \]

where the condition \( F_{max}^c > G_{max}^c \) calls for special considerations. Relation (67) shows that one or more functions \( F_j^c \) may be related to each potential function \( G_\phi^c \). It turns out that an analogous relation is found for inviscid plasticity, cf. Ottosen and Ristinmaa (1996). In principle, the numbers \( G_{max}^c \) and \( F_{max}^c \) are not known and the solution of (66) simultaneously involves the determination of these numbers; we shall return to this delicate problem when specific models are discussed.

**Generalized Duvaut-Lions formulation at a corner**

To generalize the Duvaut-Lions model to corner loading, let us start with choosing proper forms for the potential functions \( F_j^c, G_\phi^c \) and \( G_{\Theta}^c \) appearing in (66). For \( F_j^c \) defined by (60) and (61) we choose

(68) \[ F_j^c = F_j(\sigma_{ij}^c, K^c_\alpha, \theta) \]

where \( F_j \) are the static yield surfaces. Moreover, we choose the convex function \( G_\phi^c \) defined in (62) as

(69) \[ G_\phi^c = F_\phi(\sigma_{ij}^c, K^c_\alpha, \theta) \]

where we now have that \( G_{max}^c = F_{max}^c \), i.e. we fulfil the (equality) requirement imposed by (67). With these choices we now require the static yield functions \( F_j \) to be convex functions with the property \( F_j(\sigma_{ij}, K_\alpha, \theta) = F_j(0,0,\theta) \geq 0 \).

Let us next choose the function \( G(\sigma_{ij}^c, K^c_\alpha, \theta) \) present in (64) identical to (26), i.e.

(70) \[ G(\sigma_{ij}^c, K^c_\alpha, \theta) = \frac{1}{2} \sigma_{ij}^c C_{ijkl}(\theta) \sigma_{kl}^c + \frac{1}{2} K^c_\alpha c_{\alpha,\beta}(\theta) K^c_\beta \]

Note that this formulation is non-associated and based on one potential function and several static yield functions active at the same time, i.e. each of the yield functions is connected to the same potential function. The evolution laws (65) then become

(71) \[ \dot{\sigma}_{ij}^c = \Lambda C_{ijkl}(\sigma_{kl} - \sigma_{kl}^c) + \dot{\lambda}_\zeta_\alpha (K_j - K^c_j) \]
Finally, (68)-(70) imply that (66) takes the form

\[ \sum_{j=1}^{F_{\text{max}}} \lambda_j \frac{\partial F_I}{\partial \sigma_{ij}^c} = \Lambda C_{ijkl} (\sigma_{kl} - \sigma_{kl}^c) \]

(72)

\[ \sum_{j=1}^{F_{\text{max}}} \lambda_j \frac{\partial F_I}{\partial K_{\alpha}^c} = \Lambda c_{\alpha, s} (K_I - K_{\alpha}^c) \]

\[ F_I (\sigma_{ij}^c, K_{\alpha}^c, \theta) = 0 \quad I = 1, 2, \ldots, F_{\text{max}} \]

With \((\sigma_{ij}, K_{\alpha})\) being known, the equation system (72) comprises \((6 + \alpha + F_{\text{max}})\) equations with the \((6 + \alpha + F_{\text{max}})\) unknowns \((\sigma_{ij}^c, K_{\alpha}^c, \lambda_j)\) and it can therefore be solved.

Expressions (72) are identical to those proposed by Simo et al. (1988), who directly postulated the evolution equations (71) and interpreted \((\sigma_{ij}^c, K_{\alpha}^c)\) as the closest-point-projection of \((\sigma_{ij}, K_{\alpha})\) on the static yield surface. To appreciate their approach, we again use the metric defined by (33). Then the distance measured in \((\sigma_{ij}, K_{\alpha})\)-space is given by (34). With these preliminaries, we may now determine the state \((\sigma_{ij}^c, K_{\alpha}^c)\) according to the following constrained minimization problem: For given \((\sigma_{ij}, K_{\alpha})\) and temperature \(\theta\), find \((\sigma_{ij}^c, K_{\alpha}^c)\) such that distance defined by (34) is minimized under the constraint given by \(F_I (\sigma_{ij}^c, K_{\alpha}^c, \theta) = 0\). The Lagrangian multiplier method then provides the following solution

\[ - C_{ijkl} (\sigma_{kl} - \sigma_{kl}^c) + \sum_{j=1}^{F_{\text{max}}} \mu_I \frac{\partial F_I}{\partial \sigma_{ij}^c} = 0 \]

(73)

\[ - c_{\alpha, s} (K_I - K_{\alpha}^c) + \sum_{j=1}^{F_{\text{max}}} \mu_I \frac{\partial F_I}{\partial K_{\alpha}^c} = 0 \]

\[ F_I (\sigma_{ij}^c, K_{\alpha}^c, \theta) = 0 \quad I = 1, 2, \ldots, F_{\text{max}} \]

where \(\mu_I\) are Lagrangian multipliers. A comparison with (72) shows that we can identify \(\mu_I\) as \(\lambda_j / \Lambda\).

While Simo et al. (1988) also assumed for corner loading \(c_{\alpha, s}\) to be positive definite and thereby restricting the formulation to hardening viscoplasticity, we will prove that (72) allows softening to be considered. Similar to the expressions (31) and (32), we now obtain from (72) and (70) that

\[ G = \frac{1}{2 \lambda^2} \sum_{I,J} A_{IJ} \lambda_I \lambda_J \]

(74)

where

\[ A_{IJ} = \frac{\partial F_I}{\partial \sigma_{ij}^c} D_{ijkl} \frac{\partial F_J}{\partial \sigma_{kl}^c} + \frac{\partial F_I}{\partial K_{\alpha}^c} d_{\alpha, s} \frac{\partial F_J}{\partial K_{\alpha}^c} \]

(75)

As for smooth loading, all requirements are fulfilled if \(G \geq 0\), which is fulfilled if \(A_{IJ}\) is positive definite. This is exactly the same requirement as emerged from inviscid corner
plasticity, cf. Ottosen and Ristinmaa (1996). Therefore requiring $A_{ij}$ to be positive
definite, the viscoplastic formulation allows softening to be included.

As in the discussion about smooth surfaces, it turns out that (73) may be interpreted
as an implicit backward Euler scheme when integrating the evolution laws in inviscid
associated corner plasticity. This appears immediately when multiplying (73) by $D_{ijkl}$
and $d_{ij}$. Without going into a detail discussion, it is evident that the above generalization
can be transferred to the other formulations discussed previously in combination with
smooth functions.

**Application to some specific models**

With the background presented, we shall now discuss some specific models. In order
to be able to discuss both the case of no corners (smooth loading) and the case of corner
loading, we shall in all cases adopt the Tresca criterion as the yield surface $F$. Then
the Duvaut-Lions model shall be evaluated both for smooth loading and corner loading.
Finally, we shall present a model where the situation of corner loading never arises even
though the yield criterion is taken as the Tresca criterion.

An essential part of the formulation is concerned with establishment of the relation
between $(\sigma_{ij}, K, \eta)$ and $(\sigma_{ij}^*, K^*, \eta^*)$, since this will allow us to determine $(\sigma_{ij}^*, K^*)$ and
thereby $(\sigma_{ij}^{*o}, K^0)$. In the specific models discussed we shall, in all cases, take the potential
function $G$ as (26), i.e. the evolution equations are given by (27).

**Preliminaries - Tresca yield function**

With $\sigma_1, \sigma_2$ and $\sigma_3$ being the principal stresses, two neighbouring yield functions, out
of six, yield functions that comprise the Tresca yield function are given by

\[
F_1 = \sigma_1 - \sigma_3 + k_{1,1}K_j - \sigma_{yo} \\
F_2 = \sigma_2 - \sigma_3 + k_{2,1}K_j - \sigma_{yo}
\]

where $k_{1,1}$ and $k_{2,1}$ are constant quantities and $\sigma_{yo}$ denotes the initial yield stress.
Moreover, we have $\sigma_1 > \sigma_3$ and $\sigma_2 > \sigma_3$ where tension is considered positive. Before
any viscoplasticity has been initiated, the conjugated forces $K_j$ are zero and we have
$\sigma_1 = \sigma_2 > \sigma_3$ at the intersection between $F_1 = 0$ and $F_2 = 0$. Later on, when
viscoplasticity has developed a stress point located at the intersection of $F_1 = 0$ and
$F_2 = 0$ is given by

\[
\sigma_1 = \sigma_2 + (k_{2,1} - k_{1,1})K_j
\]

It turns out that it is sufficient to consider the two yield functions given above, since the
rest of the yield functions can be found by simple permutation of indices. The format
(76) implies that each yield surface hardens in an isotropic manner. Since six yield
functions exist in the general case, it is natural to assume the existence of six conjugated
forces $K_1, K_2, ..., K_6$. Expression (8) then shows the existence of six internal variables $\kappa_1, \kappa_2, ..., \kappa_6$, i.e. we have

$$K_\alpha \in \{K_1, K_2, ..., K_6\} ; \quad \kappa_\alpha \in \{\kappa_1, \kappa_2, ..., \kappa_6\}$$

According to (11), we then have for isothermal conditions that

$$\dot{K}_\alpha = -d_{\alpha, \beta} \kappa_\beta$$

Let us return to (76) and observe that, depending on the specific choice of the parameters $k_{\alpha, \beta}$, different types of isotropic hardening emerge. A natural choice is to assume that all six yield functions harden isotropically by the same amount irrespective of the loading history. This type of hardening is called dependent hardening and it is illustrated in Fig. 4. It can be modelled by assuming that

$$k_{\alpha, \beta} = k I_{\alpha, \beta} \quad \text{dependent hardening}$$

where $I_{\alpha, \beta}$ is a matrix with all components equal to unity. It appears that $k_{\alpha, \beta} K_\beta = k(K_1 + K_2 + ... + K_6)$ for all $\alpha$, i.e. irrespective of how the hardening parameters $K_\alpha$ develop, we have the same amount of isotropic hardening of all six yield functions. Note that we do not necessarily require that $K_1, K_2, ..., K_6$ develop in the same way.

![Fig. 4 - Deviatoric plane. Dependent hardening, i.e. same amount.](image)

Another type of isotropic hardening is illustrated in Fig. 5, it is called independent hardening. It is characterized by the yield function being able to develop its own amount
of isotropic hardening. Dependent on the loading history, we may, for instance, have the development shown in Fig. 5a and b. It can be modelled by assuming that

\[ k_{\alpha,\beta} = k_{\alpha,\beta} \]

which implies that \( k_{1,3} K_3 = k K_1 \) and \( k_{2,3} K_3 = k K_2 \) etc. Therefore, if the hardening parameters \( K_\alpha \) are allowed to develop in an individual manner, we obtain independent
hardening. Referring to (79), this can be achieved if we choose \( d_{\alpha,3} \) as the following simple expression

\[
(82) \quad d_{\alpha,3} = d \delta_{\alpha,3}
\]

For simplicity, we shall adopt this expression for dependent hardening.

We may note that dependent hardening can also be achieved if we choose \( k_{\alpha,3} = k \delta_{\alpha,3} \) and \( d_{\alpha,3} = d \delta_{\alpha,3} \) since it implies \( k_{\alpha,3} K_{\alpha} = k K_{\alpha} \) and \( K_{1} = K_{2} = \ldots = K_{6} = d(\dot{\kappa}_{1} + \dot{\kappa}_{2} + \ldots + \dot{\kappa}_{6}) \), i.e. all hardening parameters develop in the same manner. In that case there is, in fact, no need to deal with 6 internal variables \( \kappa_{\alpha} \) and 6 hardening parameters \( K_{\alpha} \) since one internal variable and one hardening parameter are sufficient. However, apart from a different interpretation of the parameters \( k \) and \( d \) we obtain the same result as with (80) and (82) and we shall therefore stick to these expressions.

Returning to (76) it follows that

\[
(83) \quad \frac{\partial F_{1}}{\partial \sigma_{ij}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} ; \quad \frac{\partial F_{2}}{\partial \sigma_{ij}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} ; \quad \frac{\partial F_{\alpha}}{\partial K_{\alpha}} = k_{\alpha,3}
\]

Moreover, considering isotropic elasticity, we have

\[
(84) \quad C_{ijkl} = \frac{1}{2G} \left\{ \frac{1}{2} \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) - \frac{\nu}{1+\nu} \delta_{ij} \delta_{kl} \right\}
\]

With these preliminaries let us next turn our attention to the Duvaut-Lions model with the yield criterion given by the Tresca criterion and both consider smooth loading and corner loading.

**Duvaut-Lions model**

**Smooth loading**

Smooth loading means no corner loading and we adopt for convenience the yield function \( F_{1} \) where \( \sigma_{1} > \sigma_{2} > \sigma_{3} \). Viscoelastic loading then requires that \( F_{1} > 0 \). With (83) and (84), (29a) becomes

\[
\sigma_{1} - \sigma_{1}^{\prime} - \frac{\nu}{1+\nu} (\sigma_{kk} - \sigma_{kk}^{\prime}) = 2G \frac{\chi}{\Lambda}
\]

\[
(85)
\]

\[
\sigma_{2} - \sigma_{2}^{\prime} - \frac{\nu}{1+\nu} (\sigma_{kk} - \sigma_{kk}^{\prime}) = 0
\]

\[
\sigma_{3} - \sigma_{3}^{\prime} - \frac{\nu}{1+\nu} (\sigma_{kk} - \sigma_{kk}^{\prime}) = -2G \frac{\chi}{\Lambda}
\]

Recalling that \( d_{\alpha,3} \) is the inverse of \( c_{\alpha,3} \), (29b) gives

\[
(86) \quad K_{3} - K_{3}^{\prime} = d_{3\alpha} k_{3\alpha} \frac{\chi}{\Lambda}
\]
Taking the sum of the three equations in (85) provides the following simple relation for the hydrostatic stress

\[(87) \quad \sigma'_{kk} = \sigma_{kk}\]

Then, determination of \(\sigma'_1\) and \(\sigma'_2\) from (85) and \(K'_{ij}\) from (86) and insertion into (29c) gives

\[(88) \quad \lambda' = \Lambda \frac{F_1}{A} \geq 0\]

where

\[(89) \quad A = 4G + k_{1\alpha} d_{\alpha,\beta} k_{1,\beta} > 0\]

and \(F_1\) is given by (76a). Since it is required that \(\lambda' \geq 0\), it follows from (88) that we must require that \(A > 0\). It is interesting that this also ensures that we have fulfilled all requirements imposed on the potential function \(G\) in order that the dissipation function is fulfilled, cf. the discussion of (31) and (32). Finally, insertion of (87) and (88) into (85) and (88) into (86) provide the following solution

\[(90) \quad \sigma'_1 = \sigma_1 - 2G \frac{F_1}{A} ; \quad \sigma'_2 = \sigma_2 ; \quad \sigma'_3 = \sigma_3 + 2G \frac{F_1}{A}\]

and

\[(91) \quad K'_{ij} = K_{ij} - d_{\alpha,\beta} k_{1\alpha} \frac{F_1}{A}\]

Let us next examine some different types of hardening. For the ideal viscoplastic model, i.e. \(k_{n,\beta} = 0\), we find that \(A > 0\) is trivially fulfilled since \(A = 4G\). For dependent hardening shown in Fig. 4, we obtain with (80) and (82) that (89) implies

\[(92) \quad d > -\frac{2G}{3k^2}\]

Likewise, for independent hardening shown in Fig. 5, we obtain with (81) and (82) that

\[(93) \quad d > -\frac{4G}{k^2}\]

To evaluate the significance of the sign of the parameter \(d\), we observe that (76a) implies

\[(94) \quad k_{1,\beta} \tilde{K}_{j,\beta} < 0 \Rightarrow \text{hardening} ; \quad k_{1,\beta} \tilde{K}_{j,\beta} > 0 \Rightarrow \text{softening}\]

and it is easily shown that this leads to

\[(95) \quad d > 0 \Rightarrow \text{hardening} ; \quad d < 0 \Rightarrow \text{softening}\]

Therefore, (92) and (93) put restrictions on the amount of allowable softening.
CORNER LOADING

Let us next consider the situation of corner loading for the Duvaut-Lions model, i.e. both \( F_1 \) and \( F_2 \) are active. From (72a) we find by means of (83) and (84) that

\[
\sigma_1 - \sigma_1' = \frac{\nu}{1 + \nu} (\sigma_{kk} - \sigma_{kk}') = \frac{2G}{A} \lambda_1'
\]

(96)

\[
\sigma_2 - \sigma_2' = \frac{\nu}{1 + \nu} (\sigma_{kk} - \sigma_{kk}') = \frac{2G}{A} \lambda_2'
\]

\[
\sigma_3 - \sigma_3' = \frac{\nu}{1 + \nu} (\sigma_{kk} - \sigma_{kk}') = \frac{2G}{A} (-\lambda_1' - \lambda_2')
\]

whereas (72b) gives

(97)

\[K_3' = K_3 - \frac{d}{A} (k_{12} + k_{22} \lambda_2')\]

Taking the sum of the three equations in (96) gives the following simple relation for the hydrostatic stress

(98)

\[\sigma_{kk}' = \sigma_{kk}\]

It then follows from (96) that

(99) \[\sigma_1' = \sigma_1 - \frac{2G}{A} \lambda_1' ; \quad \sigma_2' = \sigma_2 - \frac{2G}{A} \lambda_2' ; \quad \sigma_3' = \sigma_3 + \frac{2G}{A} (\lambda_1' + \lambda_2')\]

To determine the quantities \( \lambda_1' \) and \( \lambda_2' \), we insert (97) and (99) into (72c) to obtain

(100) \[\frac{1}{A} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\]

where

(101) \[A = 4G + k_{12} d_{13} d_{13} ; \quad B = 2G + k_{12} d_{13} d_{23} ; \quad C = 4G + k_{22} d_{13} d_{23}\]

and where the definitions (76) were used; we may note that the parameter \( A \) is identical to that defined by (89). It then follows that a solution to (100) can be found if the determinant \( D = AC - B^2 \neq 0 \). Then

(102) \[\begin{bmatrix} \lambda_1' \\ \lambda_2' \end{bmatrix} = \frac{A}{D} \begin{bmatrix} CF_1 - BF_2 \\ AF_2 - BF_1 \end{bmatrix} \text{ where } \lambda_1' \geq 0 \]

\[\lambda_2' \geq 0\]

Now, it is observed that \( D = \det A_{IJ} \) where \( A_{IJ} \) is defined by (75) and as \( A_{IJ} \) must be positive definite it follows that

(103) \[D = AC - B^2 > 0\]
Let us now consider different hardening situations. For ideal viscoplasticity where \( k_{\alpha, \beta} = 0 \) we find that \( \lambda_1^\alpha \geq 0 \) and \( \lambda_2^\alpha \geq 0 \) imply that

\[
2F_1 - F_2 \geq 0 \quad \text{and} \quad 2F_2 - F_1 \geq 0
\]

must be fulfilled at the same time; otherwise we have the previously described situation with only one static yield function active. In the principal stress space, the planes defined by \( 2F_1 - F_2 = 0 \) and \( 2F_2 - F_1 = 0 \) evidently intersect the yield surface at the corner \( C \). The normal to the plane \( 2F_2 - F_1 = 0 \) is given by \((-1,2,-1)\) whereas the normal to the plane \( F_1 = 0 \) is given by \((-1,0,1)\). These two normals are orthogonal to each other, which implies that plane \( 2F_2 - F_1 = 0 \) is orthogonal to plane \( F_1 = 0 \); likewise, it follows that the planes \( 2F_1 - F_2 = 0 \) and \( F_2 = 0 \) are orthogonal. Consequently, conditions (104) take the form illustrated in Fig. 6 and it appears that \( F_1 > 0 \) and \( F_2 > 0 \) does not necessarily mean that we have corner loading.

![Fig. 6. Deviatoric plane; conditions for corner loading for ideal viscoplasticity.](image)

For dependent hardening given by (80) and (82) we obtain

\[
A = C = 4G + 6k^2d \quad ; \quad B = 2G + 6k^2d
\]

i.e. restriction (103) becomes

\[
d > -\frac{G}{2k^2}
\]
For corner loading to exist, we have $\lambda'_1 \geq 0$ and $\lambda'_2 \geq 0$ which with (102) and (105) provide that

\[(107)\quad (4G + 6k^2 d)F_1 - (2G + 6k^2 d)F_2 \geq 0 \quad \text{and} \quad (4G + 6k^2 d)F_2 - (2G + 6k^2 d)F_1 \geq 0\]

These conditions are illustrated in Fig. 7.

Finally, the case of independent hardening described by (81) and (82) implies

\[(108)\quad A = C = 4G + k^2 d \quad ; \quad B = 2G\]

i.e. restriction (103) leads to

\[(109)\quad d > -\frac{2G}{k^2}\]

(In fact, we also obtain the solution $d < -6G/k^2$, but this restriction is rejected since the solution above provides the first restriction when the behaviour changes from hardening to more and more softening). For corner loading to exist, it is required that $\lambda'_1 \geq 0$ and $\lambda'_2 \geq 0$. With (102) and (108) this results in

\[(110)\quad (4G + k^2 d)F_1 - 2GF_2 > 0 \quad \text{and} \quad (4G + k^2 d)F_2 - 2GF_1 > 0\]

These conditions are illustrated in Fig. 8. It is interesting to observe that hardening and softening have different implications for the region of corner loading when dependent hardening or independent hardening is considered, cf. Figs. 7 and 8.

**Two-potential model**

We shall now illustrate a specific case of the formulation proposed by (54). Again we choose $G$ as in the Duvaut-Lions model, i.e. given by (26). However, whereas the Duvaut-Lions model follows if we choose $G^\nu = F^\nu = \varepsilon$ (i.e. Tresca in the present situation), we now choose $G^\nu = F^\nu \neq \varepsilon$. In this spirit, the Duvaut-Lions model may be termed a one potential model (in addition to $G$) whereas the present model may be termed a two-potential model (in addition to $G$).

Whereas the yield surface is still chosen as the Tresca surface, we shall choose $G^\nu = F^\nu$ as the von Mises function, i.e.

\[(111)\quad F^\nu = G^\nu = \sigma'_{ff} + K^\nu - \bar{\sigma}_y \quad ; \quad \sigma'_{ff} = \left(\frac{3}{2}s'_{ij}s'_{ij}\right)^{1/2}\]

Here $s'_{ij}$ is the deviatoric part of the stress tensor $\sigma'_{ij}$ and $\bar{\sigma}_y$ is a constant. The Tresca yield surface is assumed to exhibit dependent hardening and one conjugated force $K$ is then sufficient for the description, cf. the discussion following (82). As an example the Tresca yield function $F_1$ is given by $F_1 = \sigma_1 - \sigma_3 + kK - \sigma_y$. As will be shown, the advantage of this model is that the question of corner loading never arises even though the yield function is given by the Tresca function.
Fig. 7. – Deviatoric plane; conditions for corner loading for dependent hardening. a) hardening and b) softening.
Fig. 8. - Deviatoric plane; conditions for corner loading for independent hardening. 
a) hardening and b) softening.
As before we have viscoplastic loading for any $F_I > 0$, $I = 1, 2, \ldots, 6$ where $F_I$ denote any of the Tresca functions. Using (111) in (55) provides

$$\chi' \frac{3}{2} s_{ij}' \sigma_{eff}' = \frac{\Lambda}{2G} \left\{ \sigma_{ij} - \sigma_{ij}' - \frac{\nu}{1+\nu} (\sigma_{kk} - \sigma_{kk}') \delta_{ij} \right\}$$

$$\chi' = \frac{\Lambda}{d} \{K - K'\}$$

(112)

Contraction of (112a) gives

$$\sigma_{kk}' = \sigma_{kk}$$

(113)

Use of (113) in (112a) results in

$$s_{ij}' = \frac{\Lambda}{\chi' 3G} s_{ij}$$

(114)

Then multiplying each side with itself implies that

$$\sigma_{eff}' = \sigma_{eff} - \frac{\chi'}{\Lambda} 3G : \sigma_{eff} = \left( \frac{3}{2} s_{ij} s_{ij} \right)^{1/2}$$

(115)

With $K'$ given by (112b) insertion of (115a) into $F' = 0$ where $F'$ is given by (111) provides

$$\chi' = \Lambda \frac{f}{3G + d} : f = \sigma_{eff} + K' - \bar{\sigma}_{go}$$

(116)

Finally, use of (116a) in (114) and (112b) and taking advantage of (115a) we obtain

$$s_{ij}' = (1 - \frac{3G}{3G + d} \frac{f}{\sigma_{eff}}) s_{ij}$$

(117)

$$K' = K' - \frac{d}{3G + d} f$$

From (116) and since $\chi' \geq 0$ and $\Lambda \geq 0$ it follows that when yielding takes place we must, in addition to $F_I \geq 0$, require that $f \geq 0$ as well as $d > -3G$. The requirement that $f \geq 0$ is fulfilled, if the von Mises function $F'$ = 0 always is located inside and touches the Tresca yield functions $F_I = 0$, we therefore require that

$$k = \frac{2}{\sqrt{3}} : \bar{\sigma}_{go} = \frac{\sqrt{3}}{2} \sigma_{go}$$

(118)

These relations are conveniently found by considering pure shear loading.

It appears that a significant advantage of the present model is that the question of smooth loading or corner loading never arises even though the yield function is given as Tresca's yield function.
Conclusions

A theory of viscoplasticity based on an additive split of the conjugated forces entering the dissipation inequality was developed. In addition to unifying the viscoplastic concept it is shown that the approach opens for new ways to define viscoplastic models. Within this framework, the generalized Duvaut-Lions model follows naturally and it was shown that softening viscoplasticity is included. Moreover, other important models allowing a larger freedom of the direction of the viscoplastic strain rate direction are shown to be included in the proposed framework.

The proposed theory was shown to include the important case of viscoplastic corner loading. Even the highly complex case, where the number of active potential functions may differ from the active yield functions, can be treated. For the case of corner loading, it was also shown that the Duvaut-Lions model can be derived in a consistent manner.

Finally, to illustrate some of the findings, the Duvaut-Lions model (one potential formulation) as well as a two-potential formulation were evaluated for the case where the Tresca yield condition is used. Both corner loading and smooth loading were considered. The two-potential formulation turns out to be of interest since the question of corner loading never arises despite the use of Tresca’s yield criterion.

REFERENCES


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