Frame analysis and continuum damage mechanics

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ABSTRACT. – In this paper a unified formulation for damage analysis of steel and reinforced concrete frame members is presented. This formulation is based on the methods of continuum damage mechanics and on the notion of inelastic hinges. Particular models for frames under monotonic, low cycle and cyclic loading are described. © Elsevier, Paris

1. Introduction

The analysis of inelastic frames is a very important subject in earthquake and civil engineering. Basically, there are two different approaches: the use of multilayer methods or frame theories. The term “frame theory” is used in this paper to indicate models that are based on the concept of plastic hinges or other related notions. Frame theories need specific constitutive laws different from the continuum models, although based on the same general concepts. Multilayered methods are too cumbersome and time-consuming when analyzing large or complex structures, on the other hand frame theories provide a simple, efficient and nevertheless accurate modeling of the phenomena taking place in the structure. This paper considers frame theory only.

The conventional frame theory (see for instance Takeda et al., 1970; Maier et al., 1973; Cohn and Franchi 1979; Roufaiel and Meyer 1987) is based on plasticity theory. However for many applications, perhaps for most, the main objective of the analysis is the damage assessment of the structure under exceptional overloads. Most of the best-known frame analysis programs do not make a damage analysis. Others perform a damage evaluation in two steps: first, a conventional plastic analysis and then, a damage analysis with a post-processor.

Continuum damage mechanics (see for instance Lemaitre 1992) is a theory within the framework of continuum mechanics that has been designed to make coupled damage-plasticity (or viscoplasticity) analysis. This paper describes a formulation for the analysis of frames based on concepts of damage mechanics. A general framework, independent of the material of the frame, is proposed in section 2. Sub-sections 2.1 and 2.2 do not contain original material and are dedicated to the introduction of the notation used in the paper. Some particular models for two different materials, reinforced concrete and steel, under monotonic (sections 3.1 and 3.2), fatigue (section 3.3) and cyclic loading (section 4), are also presented.

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2. Characterization of damage in a frame member

2.1. Constitutive models for a frame member

Frame analysis can be done using concepts that are equivalent to those of stress and strain in continuum mechanics:

Let us consider a frame member between nodes $i$ and $j$. Generalized stresses and deformations are defined as: $\{M\}' = (m_i, m_j, n)$ and $\{\Phi\}' = (\phi_i, \phi_j, b)$ (see Fig. 1). A constitutive model for a frame member can be defined as the set of equations that relates generalized stresses with the history of generalized deformations. For instance, an elastic frame member can be defined by the following constitutive equations (external forces are assumed to be applied on the nodes):

\[
\{M\} = \left\{ \frac{\partial W}{\partial \Phi} \right\} = [S']\{\Phi\} \quad \text{or} \quad \{\Phi\} = \left\{ \frac{\partial W^*}{\partial M} \right\} = [F']\{M\}
\]

where $W$ and $W^*$ are the strain and complementary strain energy and $[S']$ and $[F']$ are denoted elastic stiffness and flexibility matrices, respectively. These matrices may depend on stress $\{M\}$ if geometrically nonlinear effects are taken into account.

![Diagram of a frame member with generalized stress and deformation](image)

In the general case, a constitutive model and the conventional equilibrium and compatibility equations define the behavior of any frame.

2.2. Lumped dissipation model of a frame member

Inelastic effects can be included by considering the lumped energy dissipation model of a frame member as indicated in Figure 2. In this model, a frame member is represented...
as the assemblage of an elastic beam-column and two inelastic hinges. All the energy dissipation by the member is assumed to be concentrated in the hinges.

The conventional theory of elasto-plastic frames is obtained through the introduction of an additional set of variables: the generalized plastic deformations \( \Phi^p = (\phi^p_i, \phi^p_j, \delta^p) \). The first and second parameter of \( \{ \Phi^p \} \) represent the plastic rotations of hinges \( i \) and \( j \) and the third, the permanent elongation of the chord \( i-j \). The total deformations \( \{ \Phi \} \) of the member are now the sum of hinges deformations \( \{ \Phi^p \} \) and beam deformations \( \{ \Phi^b \} \). The state law of an elastoplastic frame member can be written taking into account that the beam is assumed to remain elastic and therefore to follow the elastic law (1):

\[
\{ \Phi^b \} = \{ \Phi - \Phi^p \} = [E^b] \{ M \}
\]

Expressions (1, 2) can not characterize the loss of stiffness in frame members due to the cracking of the concrete (in reinforced concrete frames) or local buckling and damage (in steel frames), since they assume constant flexibility and stiffness matrices. In the next sections, a model for damaged frame members based on the concepts of continuum damage mechanics, is proposed.

2.3. STRAIN AND ENERGY EQUIVALENCE PRINCIPLES IN FRAME THEORY

It is postulated the existence of a set of damage parameters \( \{ D \} = (d_i, d_j, d) \) that characterizes the state of damage of the member. The first and second variables in \( \{ D \} \) are associated to the damage due to flexural effects and the last term to the damage due to axial forces. These variables can take values between zero (no damage) and one (total damage) as the conventional continuum damage variables.

It is now possible to introduce the concept of "generalized effective stresses" \( \{ \widetilde{M} \} \) by analogy with the definition of the effective stress (Rabotnov, 1963) in continuum mechanics:

\[
\{ \widetilde{M} \} = \left( \frac{m_i}{1 - d_i}, \frac{m_j}{1 - d_j}, \frac{n}{1 - d_n} \right)
\]

The state law of damaged member could be obtained by the "deformation equivalence principle" (see Lemaitre 1992) as in conventional damage mechanics, i.e., the effective generalized stresses \( \{ \widetilde{M} \} \) would substitute the generalized stresses \( \{ M \} \) in Eq. (2).
However, such a procedure would lead to a nonsymmetric flexibility (or stiffness) matrix:

\[
\{\Phi - \Phi^p\} = [F^1(D)][M]; \quad [F^1(D)] = \begin{bmatrix}
\frac{f_{11}^0}{1 - d_i} & \frac{f_{12}^0}{1 - d_j} & 0 \\
\frac{f_{21}^0}{1 - d_i} & \frac{f_{22}^0}{1 - d_j} & 0 \\
0 & 0 & \frac{f_{33}^0}{1 - d_n}
\end{bmatrix}
\]

This is also the case of anisotropic damage theories, when a second order tensor is used for damage characterization (see Cordebois and Sidoroff, 1979). These authors propose the substitution of the strain (in the present case deformation) equivalence principle by the energy equivalence principle:

The effective stress would substitute the conventional stress in the potential \(W^e\) instead of in the elasticity law. In this case, the same state law (4) is obtained, but now with the following flexibility matrix:

\[
\{\Phi - \Phi^p\} = [F^2(D)][M]; \quad [F^2(D)] = \begin{bmatrix}
\frac{f_{11}^0}{1 - d_i} & \frac{f_{12}^0}{1 - d_j} & 0 \\
\frac{f_{21}^0}{(1 - d_i)(1 - d_j)} & \frac{f_{22}^0}{(1 - d_j)^2} & 0 \\
0 & 0 & \frac{f_{33}^0}{1 - d_n}
\end{bmatrix}
\]

The flexibility matrix (5) is symmetric but yields doubtful results in, at least, some limit cases:

Let us consider the particular case of a frame member of an homogeneous material of elastic modulus \(E\), length \(L\), and constant cross section of area \(A\), and inertia \(I\). If the member has a state of damage characterized by no axial damage \((d_n = 0)\), no flexural damage at hinge \(i\) \((d_i = 0)\) and total flexural damage at hinge \(j\) \((d_j = 1)\), it is reasonable to assume that the stiffness of this member should be the same as that of an elastic frame member with an internal hinge at the right hand end. It is well known that the stiffness of such a member is given by:

\[
[S^0] = \frac{EI}{L} \begin{bmatrix}
3 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{A}{I}
\end{bmatrix}
\]

It must be recalled that the term \(S^0_{11}\) of the stiffness matrix in an elastic member without internal hinges is \(4EI/L\). In other words, the existence of an internal hinge in a member reduces all the flexural-related terms in a stiffness matrix.

However, the inverse of the flexibility matrix (5) is:

\[
[S^2(D)] = \frac{EI}{L} \begin{bmatrix}
4(1 - d_i)^2 & 2(1 - d_i)(1 - d_j) & 0 \\
2(1 - d_i)(1 - d_j) & 4(1 - d_j)^2 & 0 \\
0 & 0 & \frac{A}{I}(1 - d_n)^2
\end{bmatrix}
\]
By substitution of the values of \{D\}, in the particular case under consideration, it is possible to verify that the term \(S_{11}\) in (7) is not reduced. In other words, there is no general stiffness redistribution in a damaged frame member if the energy equivalence principle is used.

In the following section, another procedure is proposed for obtaining flexibility matrices for damaged members that leads to symmetric matrices and damage-induced stiffness redistribution.

2.4. State laws by direct generalization of the uniaxial case

Let us consider again a member of constant cross section but subjected now to axial effects only. Conventional damage mechanics allows the determination of a very simple stress-deformation relationship, if we assume a constant state of damage \(\omega\) throughout the member:

\[
\delta - \delta^L = \frac{f_{33}^0}{1 - \omega}n; \quad f_{33}^0 = \frac{L}{AE}
\]

where \(\omega\) represents the conventional isotropic continuum damage variable such as defined in (Lemaitre 1992).

On the other hand, if we consider again the lumped dissipation model described in section 2.2, the total axial deformation of the member \(\delta\) can be expressed as:

\[
\delta = \delta^h + \delta^L + \delta^d = f_{33}^0 n + \delta^L + \delta^d
\]

where the term \(\delta^d\) represents an additional hinge deformation due to damage. Comparison of Eqs. (8) and (9) allows the definition of the damage deformation \(\delta^d\):

\[
\delta^d = \frac{\omega f_{33}^0}{(1 - \omega)} n
\]

Expression (10) gives a definition of the damage-induced deformation for a lumped dissipation model that is equivalent to that in conventional damage mechanics under axial loading. If the damage variable \(\omega\) takes the value zero, a rigid-plastic hinge without additional flexibility due to damage is obtained. The other limit case, \(\omega\) equal to one, results in a hinge with an infinite flexibility, i.e. hinges and beam-column can be considered as unconnected.

If flexural and axial effects are present simultaneously, the problem becomes too complex to obtain similar analytical results. Therefore, the existence of a set of damage-related hinge deformations \(\{\Phi^d\}^t = (\phi_j^d, \phi_j^d, \delta^d)\) is postulated that obeys the following law:

\[
\{\Phi^d\} = [C(D)]\{M\}; \quad [C(D)] = \begin{bmatrix}
\frac{d_i f_{11}^0}{(1 - d_i)} & 0 & 0 \\
0 & \frac{d_j f_{22}^0}{(1 - d_j)} & 0 \\
0 & 0 & \frac{d_n f_{33}^0}{(1 - d_n)}
\end{bmatrix}
\]
The zeros outside the diagonal of the flexibility matrix \([C(D)]\) mean that only the moment acting on a hinge induces additional deformations in the hinge under consideration. In other words, it is assumed that \(m_j\) does not produce damage-related rotations on hinge \(i\) and vice versa. Now, the total deformation \(\{\Phi\}\) of a damaged member can be expressed as:

\[
\{\Phi\} = \{\Phi^b\} + \{\Phi^p\} + \{\Phi^d\} \quad \text{or} \quad \{\Phi - \Phi^p\} = [F(D)]\{M\}
\]

where \([F(D)] = [F^0] + [C(D)]\) is the flexibility matrix of a damaged member. If there is no flexural effects, the state law (12) becomes (8) with \(\omega\) equal to \(d_n\). However, in the general case \(\omega\) does not correspond to any damage parameter of \(\{D\}\).

The inverse of \([F(D)]\) for a member of constant cross section is:

\[
[S(D)] = k \begin{bmatrix}
12(1 - d_i) & 6(1 - d_i)(1 - d_j) & 0 \\
6(1 - d_i)(1 - d_j) & 12(1 - d_j) & 0 \\
0 & 0 & EA(1 - d_n)
\end{bmatrix}
\]

where \(k = \frac{1}{4 - (1 - d_i)(1 - d_j)} \frac{EI}{L}\)

It can be noticed, by inspection of (13), that this matrix allows stiffness redistribution due to damage to occur.

The complementary strain energy of a damaged member can be expressed now as:

\[
U^* = \frac{1}{2} \{M\}^t [C(D)] \{M\} + W^*
\]

where \(W^*\) is the complementary strain energy of the elastic beam-column such as introduced in (1).

Using \(W^*\) as a thermodynamic potential, it is possible to define the thermodynamic forces conjugated to damage by:

\[
\{Y\} = - \left\{ \frac{\partial U^*}{\partial D} \right\}
\]

These forces are therefore the equivalent of the energy release rate introduced in fracture and conventional damage mechanics. For instance the energy release rate for hinge \(i\) has the following explicit expression:

\[
y_i = \frac{\partial U^*}{\partial d_i} = \frac{m_i^2 f_i^0}{2(1 - d_i)^2}
\]

2.5. ANOTHER EXPRESSION FOR THE STRAIN ENERGY OF A DAMAGED MEMBER

There is another possible expression for the energy of a damaged frame member derived by direct generalization of (10). The behavior of the hinges could be defined through a
stiffness matrix instead of a flexibility matrix as in Eq. (11), by the following expression:

\[
(M) = [R(D^*)][\Phi^d]; \quad [R(D^*)] = \begin{bmatrix}
  \frac{1 - d_i^e}{d_i} & 0 & 0 \\
  0 & \frac{1 - d_j^e}{d_j} & 0 \\
  0 & 0 & \frac{1 - d_n^e}{d_n}
\end{bmatrix}
\]

However, (17) and (11) become the same equation (i.e. $[R]^{-1} = [S]$) if we defined the damage variable \{D^e\} as:

\[
d_i^e = \frac{f_{11}^0 s_{11}^0 d_i}{1 - d_i + f_{11}^0 s_{11}^0 d_i}; \quad d_j^e = \frac{f_{22}^0 s_{22}^0 d_j}{1 - d_j + f_{22}^0 s_{22}^0 d_j}; \quad d_n^e = d_n
\]

In general the damage variable \{D\} used in (11-16) leads to algebraically simpler expressions. Therefore, for the models that will be described in this paper, the state laws (12) and (15) are adopted.

In order to complete the model, two internal-variable evolution laws must be proposed. Evolution laws are assumed to be material (steel or reinforced concrete) dependent. As usual, they must verify the second principle of thermodynamics. These laws are described in the following sections.

3. Internal-variable evolution laws

3.1. Time-independent laws for steel frame members

Very often, inelastic axial effects can be neglected in practical applications. In such a case, the conventional perfectly-plastic model without damage is defined by the state law (2) and the following yield functions for hinges i and j:

\[
f_i = |m_i| - m_y; \quad f_j = |m_j| - m_y
\]

Where $m_y$ is the yield moment of the member cross section.

The deformation equivalence principle gives the following yield function for a damaged frame member:

\[
f_i = \left| \frac{m_i}{1 - d_i} \right| - m_y; \quad f_j = \left| \frac{m_j}{1 - d_j} \right| - m_y
\]

Physically the damage variables $d_i$ and $d_j$ represent complex degrading phenomena that includes, first, local buckling and then, initiation and evolution of flexural cracks.

A constitutive model for damaged steel frame members is then composed by Eqs. (12), (20) and a damage evolution law yet to be defined.
In order to obtain this damage law, a procedure for the experimental determination of the flexural damage in a hinge should be specified. In continuum damage mechanics, one of the most widely used techniques is the method of the variation of the elasticity modulus. This method can be adapted easily to frame theory.

Let us consider the test indicated in Figure 3a. The state law (12) and the boundary conditions lead to the following force-displacement relationship:

\[ p = Z(d)(t - t_p) \quad Z(d) = (1 - d)Z^0; \quad Z^0 = \frac{3EI}{L} \]

where \( p \) is the force, \( t \) the deflection, \( t_p \) the permanent or plastic deflection and \( d \) is the damage in the hinge at the support. The function \( Z(d) \) can be interpreted as the slope of the elastic unloading branches in Figure 3b. From (21b) it follows:

\[ d = 1 - \frac{Z(d)}{Z^0} \]

With the aid of Eq. (22), it is now possible to express the hinge damage as a function of the plastic deflection, as shown in Figure 4. It can be noticed that these experimental results can be represented by a straight line, which suggests the following damage-evolution law (Inglessis et al., 1998):

\[ d_i = c(p_i - p_{cr}) \]
where \( p_i \) represents the accumulated plastic deformation of hinge \( i \) \((i.e., \, dp_i = |d\phi^p_i|)\), \( c \) and \( p_{ct} \) are member dependent constants such as the yield moment. The symbols \( \langle \cdot \rangle \) are the MacAuley brackets \( \langle x \rangle = x \) if \( x > 0 \); \( \langle x \rangle = 0 \) otherwise.

Eqs. (12), (20) and (23) define a constitutive model for a steel frame member under monotonic loading. A comparison between test and model results can be seen in Figure 3b.

It can be noticed that the evolution law (23) is the exact equivalent of Lemaitre's ductile damage law for metals in continuum damage mechanics (see Lemaitre 1992). It is interesting to notice that both cases, damage in a volume element and damage in a hinge, can be represented by the same kind of law.

It can be easily shown that this model is, as Lemaitre's model of ductile damage, thermodynamically admissible.

3.2. TIME-INDEPENDENT LAWS FOR REINFORCED CONCRETE FRAME MEMBERS

In reinforced concrete (RC) members, damage (mainly cracking of the concrete) and plasticity (yield of the reinforcement) seem to be in some degree uncoupled. That is, it is possible to experimentally observe significant cracking at very small plastic deformations (for instance, at the beginning of the loading) and very large plastic yielding with small amounts of additional cracking (as in cyclic tests with constant limit displacements).

For the modeling of this kind of behavior, two inelastic functions for each hinge are introduced (Cipollina and Flórez-López, 1995): A yield or plastic function \( f \) and a damage function \( g \).

\[
(24) \quad f_i = \frac{m_i}{1 - d_i} - c \phi^p_i - m_y; \quad g_i = g_i - (g_{ct} + B(d_i))
\]

The function \( f \) corresponds to a yield function with linear kinematic hardening and is obtained from the deformation equivalence principle. The function \( g \) corresponds to a Griffith criterion with a crack resistance function, namely, the term \( B \).

In Eq. (24) \( c \) and \( g_{ct} \) and \( m_y \) can be considered as member dependent constants. The crack resistance function can be obtained from a Damage vs. Energy release rate plot.
(see Fig. 6), where the damage is measured by the same experimental procedure described in the previous section. After experimental identification, the following expression for function $B$ is proposed (Cipollina and Flórez-López, 1995):

$$B(d_i) = q \frac{\log(1 - d_i)}{1 - d_i}$$

where $q$ is again a member dependent constant. A comparison between test and model results is shown in Figure 5.

![Figure 5](image)

Fig. 5. – Force vs. deflection in a $RC$ member after (Cipollina and Flórez-López, 1995).

![Figure 6](image)

Fig. 6. – Damage as a function of the energy release rate.

It can be shown (see Cipollina and Flórez-López, 1995) that this model is thermodynamically admissible.

3.3. LOW CYCLE FATIGUE LAW FOR $RC$ FRAME MEMBERS

Low cycle fatigue in $RC$ frames is a very important issue when considering earthquake applications. In (Marigo, 1985), a procedure for the development of a low cycle fatigue
damage law from a time-independent one, was proposed. This procedure was described within the framework of continuum damage mechanics but can be adapted to the present context with little modifications.

A model describing low cycle fatigue is then obtained from the damage law presented in section 3.2, by the substitution of the Griffith criterion (24b) by the following expression (Puglisi and Flórez-López, 1994):

$$
\dot{d}_i = \begin{cases} 
0 & \text{if } y_i < y_{R} \\
-T(m_i, d_i) y_i & \text{otherwise}
\end{cases} \quad T(m_i, d_i) = \left( A(m_i, d_i) \right)' \frac{\partial y_i}{\partial d_i}
$$

where $A(m_i, d_i) = \frac{y_i}{y_{R} + B(d_i)}$, $y_{R}$ is the member dependent constant, $y_i$ the damage function and $B(d_i)$ the hardening term, introduced in the time-independent model of the previous section. In this model a new member dependent constant $\rho$ is needed. This constant is related to the additional amount of damage in each cycle.

This model gives the same response of the time-independent model for monotonic loading. When the parameter $\rho$ tends to infinite, this model tends asymptotically to the time independent equations of section 3.2. Figure 7 shows the comparison between experiment and model in a fatigue test.

4. Unilateral models

Unilateral effects are very important in the analysis of damaged frames. For instance, in the case of $RC$ members, cracks that appear under a positive moment tend to close when the moment reverses. In conventional continuum damage mechanics there is a well-established procedure to model these effects. In this procedure two different damage variables that increase the compliance of the material under positive and negative stress are introduced. In other words the material has different effective elastic modulus depending on the sign of the stress (see Ladeveze, 1983 and Mazars 1986). This kind of procedure can be easily adapted to frame analysis:

The existence of two sets of damage parameters $\{D^+\}$ and $\{D^-\}$ is postulated so that the complementary strain energy of a damaged member has the following expression:

$$
U^* = \frac{1}{2} \langle M \rangle^T [C(D^+)] \{M\} + \frac{1}{2} \{M\}^T [C(D^-)] \{-M\} + W^*
$$

where the symbols $\langle \rangle$ are again the MacAuley brackets and matrices $[C]$ were defined in Eq. (11). The state laws derived from $U^*$ are then:

$$
\{\Phi - \Phi^p\} = [F(D^+)]\{M\} - [F(D^-)]\{-M\}; \{Y^+\} = -\left\{ \frac{\partial U^*}{\partial D^+} \right\}; \{Y^-\} = -\left\{ \frac{\partial U^*}{\partial D^-} \right\}
$$

The yield function is obtained from the deformation equivalence principle as in the previous case, but now two effective stresses are introduced, one for positive and other for negative moments:
Therefore, the yield function has the following expression:

\[
f_i = \text{Sup} \left( f_i^+, f_i^- \right) \quad \text{where} \quad f_i^+ = \frac{m_i}{1 - d_i^+} - c\phi_i^p - m_y; \quad f_i^- = -\frac{m_i}{1 - d_i^-} + c\phi_i^p + m_y
\]

where \( c \) and \( m_y \) are the member dependent constants introduced in the R/C yield function of section 3.2. Different yields moments or plastic hardening constants could be chosen for positive and negative actions if the member has a nonsymmetric cross section.

In some cases, the use of kinematic hardening exclusively may not be enough for a good representation of the hysteresis loops. Therefore, it may be necessary to split the
plastic hardening into two parts: a kinematic hardening term and an isotropic hardening term as proposed in (Flórez-López, 1995).

Damage evolution laws are assumed to be the same as in the time independent model of section 3.2 or the fatigue law of section 3.3. In the former case, for instance, damage evolution laws derive from the damage functions:

\[
g_i^+ = y_i^+ - (y_{cr} + B(d_i^+)); \quad g_i^- = y_i^- - (y_{cr} + B(d_i^-))
\]

Figure 8 shows the experimental results of a RC specimen subjected to a cyclic loading. The numerical simulation of the test using this model is indicated in Figure 9.

![Figure 8](image)

5. Final remarks and conclusions

It has been shown that damage mechanics can be adapted to frame analysis. The resulting constitutive models can be implemented in conventional structural analysis programs simply as a new finite element (see Cipollina et al., 1995).

This general framework can be generalized to include biaxial flexure and torsion for both RC and steel frames, and to include other effects such as shear damage and pinching.
The main obstacle to using these models is the identification of the member dependent parameters. This is due the fact that they depend on the cross section of the member, which can have any shape, size or reinforcement. A general procedure for the determination of these constants as a function of the cross section and the length of the member is needed. For some cases this procedure has already been established (see Flórez-López, 1995; Bendito et al., 1997).

In the structural analyses performed so far (see Flórez-López, 1993; Cipollina et al., 1995 and López-Inojosa et al., 1996), the numerical instabilities that are typical of conventional damage models (due to strain localization) have not been observed. Not even in cases where significant softening behavior was present. This statement deserves a detailed mathematical study and further numerical verification.

Another point that could be brought up is the criteria for the discretization of a structure since this discretization will determine the points of damage localization in the structure. The same question arises in the analysis of conventional elastoplastic frames and is usually solved without difficulty. Typically, an element for each member of the frame is chosen. This will probably be the case in the present context.

REFERENCES


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