Boundary control of free and forced oscillation
of shearable thin-walled beam cantilevers

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ABSTRACT. – This paper deals with the problem of controlling the bending oscillations of cantilevers subjected to harmonic time-dependent excitations and with that of enhancing their eigenvibration response characteristics. The structural model consisting of a thin-walled beam of arbitrary cross-section includes a number of nonclassical effects, such as transverse shear, secondary warping and heterogeneity. The control is achieved via the action of a bending moment applied at the tip of the structure. A dynamic feedback control law relating the boundary bending moment to the angular velocity at the tip of the structure is implemented and its implications upon the closed-loop eigenvalues and dynamic response to harmonic excitations are revealed. A computational methodology based on the extended Galerkin’s technique aimed at determining the closed-loop response characteristics is used and numerical results revealing the efficiency of the adopted control methodology are displayed. © Elsevier, Paris

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1. Introduction

Although very relevant to the design process, the dynamic response of cantilevered thin-walled beams to external excitations has received little attention in the technical literature. A survey of pertinent investigations can be found in the paper by Song and Librescu, 1995. The problem is of particular importance to structures such as airplane wings, helicopter and turbine blades, robot manipulator arms, etc. As the flexibility of such structures increases, a trend likely to become more pronounced in the design of advanced structural systems, the vibration must be contained and its detrimental effects avoided. Moreover, the occurrence of resonance must be prevented. Still, it should be kept in mind that the enhancement of the dynamic response must be achieved without weight penalties. One of the possible ways towards achieving such goals consists in the implementation of a feedback control methodology. The one considered in this paper is related to the generation and use of a dynamic moment acting at the tip of the beam. This method which was mathematically substantiated by Lagnese (1988) and Lions and Lagnese (1989), is referred to as the boundary moment control methodology. The moment control at the beam tip, can be generated via incorporation into the structure of adaptive materials technology and can be related, via a prescribed functional relationship with one of the various kinematical quantities characterizing the response of the structure (see e.g. Bailey and Hubbard (1985), Tzou and Zhong (1992), Tzou (1993), Librescu et al. (1993, 1996)).

However, in contrast to previous studies (see Tzou and Zhong (1992), Librescu et al. (1993, 1996)), in the present investigation a dynamic feedback control law is implemented.

To enhance understanding of the vibration control via the adoption of the boundary moment control methodology, in general, and of the dynamic feedback control law, in particular, a beam structure capable of uncoupled vibrations is considered. To this end, it is assumed that the longitudinal z-axis of the beam

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constitutes an axis of symmetry of the cross sections. The illustration of the efficiency of this control technology should encourage the application to more complex cases, such as coupled vibrations, and to wider classes of structural systems.

2. Geometry and basic assumptions

The case of a cantilevered thin-walled beam of closed cross-section is considered.

Two coordinate systems are used in the forthcoming developments: (i) a global rectangular system $x, y, z$, where $x, y$ denote the cross-section beam coordinates and $z$ denotes the spanwise coordinate, and (ii) a local system $n, s, z$, where $n$ and $s$ denote the thicknesswise coordinate normal to the beam mid-surface and the tangential coordinate along the contour line of the beam cross-section, respectively (see Fig. 1). The analytical beam model considered in this paper is based on the following assumptions: (i) the structure is considered symmetrically laminated of transversely-isotropic material layers, the surface of isotropy being parallel to the reference surface of the beam. In such a way, a more comprehensive assessment of the implications played by transverse shear flexibility can be achieved, (ii) the cross-sections of the beam do not deform in their own planes, (iii) the hoop stress resultant $N_{ss}$ is negligibly small compared to the remaining stress resultants, (iv) in view of the importance of secondary warping, whose physical significance is discussed by Gjelsvik (1981), and because secondary warping induces transverse bending, this effect is also being considered. In the light of the considered anisotropy of the constituent material layers, of the structural symmetry and of the fact that the external loads are distributed along the beam $z$-axis only, an exact decoupling of transverse bending (flapping) described in terms of variables $v_0$ and $\theta_x$, lateral bending (lagging) described in terms of variables $u_0$ and $\theta_y$ and twist in terms of $\phi$, is obtained. In the present paper the analysis is confined to transverse bending only.

As a result of the incorporation of the transverse shear effect, the analysis presented here can also accommodate thick-walled beams for which $h_{\text{max}}/b \geq 0.1$, where $h_{\text{max}}$ denotes the maximum thickness of the beam and $b$ a typical cross-sectional dimension.

![Geometry of the wing structure](image)

Fig. 1. – Geometry of the wing structure.

3. The boundary-value problem

The equations of motions and the boundary conditions can be obtained via Hamilton’s principle. The transverse bending equations of motion constitute a part of the general coupled equations as derived in Song and Librescu
(1995) and Librescu et al. (1993). They are given by

\[(1a,b) \quad Q_y' + p_y = I_1, \quad M_x' - Q_y = I_2\]

while for cantilevered beams, the associated boundary conditions at the root \((z = 0)\) and beam tip \((z = L)\) are

\[(2a,b) \quad v_0 = 0, \quad \theta_x = 0\]

and

\[(3a,b) \quad Q_y = 0, \quad M_x = \dot{M}_x\]

respectively.

In Eq. (3b) \(\dot{M}_x\) denotes the boundary moment control, while in Eqs. (1) and (3) \(Q_y\) and \(M_x\) denote the transverse shear force in the \(y\)-direction and the bending moment about the \(x\)-axis, respectively. They are defined as

\[(4a) \quad Q_y(z; t) = \oint \left( N_{zz} \frac{dy}{ds} - N_{z} \frac{dx}{ds} \right) ds\]

\[(4b) \quad M_x(z; t) = \oint \left( y N_{zz} - L_{zz} \frac{dx}{ds} \right) ds\]

Herein \(N_{zz}, N_{z} \) and \(N_{z} \) denote the 2-D longitudinal, tangential shear and transverse shear stress resultants, respectively, and \(L_{zz}\) denotes the 2-D stress couple. Their expressions in terms of the strain components are:

\[(5a) \quad N_{zz}(s, z, t) = K_{11} e_{zz}^o\]

\[(5b) \quad N_{z}(s, z, t) = A_{66} \gamma_{sz}^o\]

\[(5c) \quad N_{z}(s, z, t) = A_{44} \gamma_{zn}\]

\[(5d) \quad L_{zz}(s, z, t) = \hat{K}_{11} e_{zz}^n\]

Herein, \(e_{zz}^o\) and \(e_{zz}^n\) denote the axial strain components associated with the primary and secondary warping, respectively, \(\gamma_{sz}^o\) denotes the membrane shear strain while \(\gamma_{zn}\) denotes the transverse shear strain. In addition, \(K_{11}\) and \(\hat{K}_{11}\) are stiffness quantities to be defined later. Associated with the flapping motion, the expression of the 2-D strain measures are (see Song and Librescu, 1995):

\[(6a,b) \quad e_{zz}^o(s, z, t) = \theta_x' y(s); \quad e_{zz}^n(s, z, t) = -\theta_x' \frac{dx}{ds}\]

\[(6c,d) \quad \gamma_{sz}^o(s, z, t) = (v_0' + \theta_x) \frac{dy}{ds}; \quad \gamma_{zn}(s, z, t) = -(v_0' + \theta_x) \frac{dx}{ds}\]

In addition, \(p_y = p_y(z, t)\) denotes the transverse distributed loading and \(I_1\) and \(I_2\) denote the bending and rotatory inertias, respectively, defined as

\[(7a,b) \quad I_1 = \int m_0 ds, \quad I_2 = \int \left[ m_0 y^2 + m_2 \left( \frac{dx}{ds} \right)^2 \right] ds\]

where

\[(8) \quad (m_0, m_2) = \sum_{k=1}^{N} \int_{h(k-1)}^{h(k)} \rho_n(1, n^2) dn.\]

denote the mass terms, \(v_0 = v_0(z, t)\) and \(\theta_x = \theta_x(z, t)\) denote the transverse displacement and rotation about the \(x\)-axis, respectively, \(N\) is the total number of constituents layers of the structure, \(\oint (\cdot) ds\) is the integral...
along the closed mid-line contour and \( \rho_{k} \) the mass density of the \( k \)th layer; primes and superposed dots denote derivatives with respect to \( z \) and \( t \), respectively.

Expressed in terms of displacement quantities, the equations for the bending motion become

\[
\begin{align*}
(9a) & \quad a_{55}(v''_0 + \theta'') + py = b_1 \ddot{v}_0 \\
(9b) & \quad a_{33} \theta''_x - a_{55}(v'_0 + \theta_x) = b_2 \ddot{\theta}_x
\end{align*}
\]

Herein \( a_{33} \) and \( a_{55} \) define the bending and transverse shear stiffness quantities, respectively, which can be expressed as (see Song and Librescu (1995) and Librescu et al. (1996)):

\[
\begin{align*}
(10a) & \quad a_{33} = \oint \left[ \overline{K}_{11} y^2 + \overline{K}_{11} \left( \frac{dx}{ds} \right)^2 \right] ds \\
(10b) & \quad a_{55} = \oint \left[ A_{66} \left( \frac{dy}{ds} \right)^2 + A_{44} \left( \frac{dx}{ds} \right)^2 \right] ds
\end{align*}
\]

In these expressions

\[
(11a, b) \quad \overline{K}_{11} = A_{11} - \frac{A_{12}^2}{A_{11}}; \quad \hat{K}_{11} = D_{11}
\]

denote the local reduced stiffness quantities where, \( A_{ij}(i, j = 1, 2, 6) \) and \( D_{11} \) denote local membrane and bending stiffness quantities, respectively, and \( A_{44} \) is the local transverse shear stiffness. They are defined in a generic form as

\[
\begin{align*}
(12a) & \quad A_{ij} = \sum_{k=1}^{N} C^{(k)}_{ij} \left( n_{i(k)} - n_{i(k-1)} \right) \\
(12b) & \quad D_{ij} = \frac{1}{3} \sum_{k=1}^{N} C^{(k)}_{ij} \left( n_{i(k)}^2 - n_{i(k-1)}^2 \right)
\end{align*}
\]

where the index \( (k) \) indicates that the quantity in question pertains to the \( k \)th layer. The boundary conditions for the cantilever beams expressed in terms of displacement quantities are

\[
(13a, b) \quad v_0 = 0; \quad \theta_x = 0 \text{ at } z = 0
\]

and

\[
(13c, d) \quad v'_0 + \theta_x = 0; \quad a_{33} \theta'_x = \dot{M}_x \text{ at } z = L.
\]

Corresponding to the adopted type of anisotropy of the structure, the elastic coefficients \( C_{ij} \) expressed in terms of the engineering constants (see Librescu (1975) and Song and Librescu (1995)) are

\[
\begin{align*}
(14a, b) & \quad C_{11} = (E \nu'^2 - E') E / \Delta; \quad C_{12} = - (E \nu'^2 + E' \nu) E / \Delta \\
(14c, d) & \quad C_{13} = - \nu'(1 + \nu) E E' / \Delta; \quad C_{33} = - (1 - \nu^2) E'^2 / \Delta \\
(14e, f) & \quad C_{44} = G'; \quad C_{66} = \frac{C_{11} - C_{12}}{2} = G = \frac{E}{2(1 + \nu)}
\end{align*}
\]

in which \( \Delta = (1 + \nu)(2E \nu'^2 + E' \nu - E) \), where \( E \) and \( \nu \) and \( E' \) and \( \nu' \) denote Young’s modulus and Poisson’s ratio in the plane of isotropy, and transverse to the plane of isotropy, respectively, while \( G' \) denotes the transverse shear modulus and \( G \) the in-plane shear modulus.
4. The control law

One of the possibilities of generating bending control moment at the beam tip.

The moment $\tilde{M}_x$ appearing in the boundary conditions at the beam tip, plays the role of boundary moment control. One of the possibilities of generating the control bending moment at the wing tip is via the implementation into the structure of piezoactuators and the use of the converse effect featured by these devices. The structures incorporating the capabilities provided by the piezoactuator devices are referred to as intelligent structures. For a comprehensive review of the achievements reached through the implementation of such capabilities, see Crawley, 1994.

As shown (see e.g. Tzou (1993), Librescu et al. (1993, 1996, 1997)), piezoactuators featuring in-plane isotropic properties, spread over the entire span of the beam, bonded symmetrically on the outer and inner faces of the beam but activated out-of-phase, generate a bending moment at the beam tip in response to the applied electric field $E_3$.

For feedback control, the applied electric field $E_3$ on which the piezoelectrically induced moment depends, should be expressed, through a prescribed functional relationship with one of the mechanical quantities characterizing the wing's response. In this regard, a number of control laws can be implemented. For the problem at hand, the goal of the control is to enhance the free vibration behavior, inhibit the forced vibration response to time-dependent external excitations and prevent resonance.

In contrast to previously used feedback control laws (see e.g. Tzou and Zhong (1992), Librescu et al. (1993, 1996)), which have merely a static character and in agreement with the control methodologies used by Bailey and Hubbard (1985), Tzou (1993) and Baz (1997), velocity feedback control appears to be highly indicated for the problem at hand. According to this control law, $\tilde{M}_x$ is proportional to the velocity $\theta_x(L,t)$ at the wing tip, so that boundary condition (3b) can be recast as

$$\theta_x'(L,t) = k_p \hat{\theta}_x(L,t)/a_{33}$$

where $k_p$ is the feedback gain. The advantage of this dynamic control law consists, among others, in the possibility to generate damping. A short description of the methodology used to solve the present eigenvalue and dynamic response problems is displayed in the Appendix.

5. Problem studied

For the sake of illustration, the case of a cylindrical thin-walled beam of a biconvex cross-section profile, is adopted. Such cross-section profile is typical for a supersonic wing aircraft. The geometry of the cantilevered beam is displayed in Figure 1.

Corresponding to the dimensions in Figure 1, the wing's aspect ratio $AR$ corresponds to $AR = 16$. For the cases considered in the numerical illustrations where $AR \neq 16$, the change involves the length of the beam, only.

The studied problems are related with the control of free vibration, and the enhancement of the dynamic response to external excitation as well.

In the case of free vibration, the field variables are represented in a generic form as

$$F(z,t) = \overline{F}(z)e^{\lambda t}$$

where $\lambda$ is the complex closed-loop eigenvalue.

Notice that the eigenvalue appears in both the governing equations and boundary conditions.
In the case of forced vibration, the beam is assumed to be excited by a concentrated time-dependent harmonic load located on the beam z-axis. The load $p_y$ is represented accordingly as

$$p_y(z, t) = F_o \delta(z - z_o)e^{i\omega t}$$

where $F_o$ is the amplitude, $\delta(z - z_o)$ a spatial Dirac delta function acting at $z = z_0$ and $\omega$ the excitation frequency.

Both the free and forced vibration problems are solved via the Extended Galerkin Method. In the forced vibration problem, following the determination of the frequency response functions $v_o(z_o, \omega)$ and $\theta_x(z_o, \omega)$, the time-domain response can be obtained as their inverse Fourier transform

$$\begin{bmatrix} v_o(z_o, t) \\ \theta_x(z_o, t) \end{bmatrix} = \mathcal{F}^{-1} \begin{bmatrix} v_o(z_o, \omega) \\ \theta_x(z_o, \omega) \end{bmatrix} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \begin{bmatrix} v_o(z_o, \omega) \\ \theta_x(z_o, \omega) \end{bmatrix} e^{i\omega t} d\omega$$

6. Numerical illustrations

For the controlled beam $k_p \neq 0$, the closed-loop eigenvalues can be expressed as

$$\lambda = \sigma + i\omega_d = -\zeta \omega_n + i(1 - \zeta^2)^{1/2}\omega_n$$

where $\zeta$ is the damping factor, $\omega_d$ the damped frequency and $i$ the imaginary unit. From Eq. (19), we conclude that

$$\zeta = -\sigma/(\sigma^2 + \omega_d^2)^{1/2}$$

Figures 2-4 depict the first two damped frequencies versus the normalized feedback gain $K_p \equiv k_p L \bar{\omega}/a_{33}$ for various transverse shear flexibility ratios $E/G'$ including the case of $E/G' = 0$ which corresponds to the Euler-Bernoulli beam model. In the expression of $K_p$, $\bar{\omega} = 128.16 \text{ rad/s}$ is the fundamental natural frequency of the uncontrolled beam characterized by $\mathcal{M} = 16$ and $E/G' = 50$.

![Fig. 2. - The first damped frequency versus the dimensionless feedback gain for different values of the transverse shear flexibility parameter $\mathcal{M} = 6$.](image-url)
In addition to the ability of the adopted control methodology to increase the closed-loop eigenfrequencies, the results reveal also that use of the classical Euler-Bernoulli model results in overestimations of the natural frequencies. Whereas Figures 2 and 3 reveal that an increase in the aspect ratio of the beam $\bar{R} (\equiv 2L/c)$, reduces the sensitivity of the frequencies to transverse-shear effects, Figure 4 shows that transverse shear flexibility plays a more significant role in the case of larger mode frequencies than of the lower ones. Figure 5, depicting the piezoelectrically induced damping factor $\zeta$ versus the transverse shear flexibility parameter, reveals that compared to the shearable beam, the classical Euler-Bernoulli beam counterpart overestimates the induced damping. It is also apparent that for larger feedback gains, this trend is further exacerbated. In connection with the selection of the velocity feedback gain, a word of caution is in order. This is related to the fact that within such a feedback control law, the damping factor increases with the increase of the feedback gain until a maximum value, beyond which a sharp drop of the damping is experienced. That specific value of $k_p$ at which $\zeta$ reaches a maximum depends upon the geometrical and physical characteristics of the structure. In this study the values of the gain $k_p$ are selected as not to cross that critical value yielding the precipitous decay of $\zeta$. The above mentioned trend was highlighted also by Tzou (1993).
Fig. 5. – Induced damping versus the (transverse shear flexibility) for different values of $K_p$. $R = 6$.

Fig. 6. – Steady-state dimensionless deflection $\bar{v}(\equiv v_0/L)$ versus the excitation frequency for shearable ($E/G' = 50; 100$) and non-shearable ($E/G'^2 = 0$) beams, and for various feedback gains. $R = 6$.

Fig. 7. – The same plot as in Fig. 6, but for a beam of $R = 16$. 
The dimensionless steady-state deflection $\tilde{V}(\equiv \overline{V}/L)$ and rotational amplitudes $\theta_r$ at the beam tip versus the excitation frequency for different beam aspect ratios, various transverse shear flexibilities and for the uncontrolled and controlled beams are displayed in Figures 6, 7 and 8, respectively. In all these cases, the force excitation is located at the beam tip.

The results reveal that: i) for relatively low beam aspect ratios, the resonance frequencies are grossly overestimated by the Euler-Bernoulli beam model, ii) for the same $K_p$, the classical beam model underestimates the amplitude of the response as compared to that predicted by the shear deformable beam model. However, with the increase of the beam aspect ratio, the latter effect tends to decay, iii) with the increase in the transverse shear flexibility there is a decrease in the resonance frequency and an increase in the deflection amplitude and, finally, iv) the efficiency of the control methodology of inhibiting resonance is clearly demonstrated for both shearable and non-shearable beams.

The distribution of the steady-state deflection amplitude along the beam span is presented in Figure 9 for the case of an oscillating load located at the beam tip, $\eta(\equiv z/L) = 1$, and at mid-span, $\eta = 0.5$. The results reveal that the classical Euler-Bernoulli beam model underestimates the deflection amplitude predicted by the
shear deformable counterpart model. This trend is more prominent for oscillatory loads located at the tip rather than at mid-span.

Figure 10 depicts the steady-state dimensionless bending moment \( \tilde{M}(\equiv ML/a_{33}) \) produced by a harmonically oscillating load at the beam tip versus the beam aspect ratio. Herein, the cases of shear-deformable/non-shear deformable and controlled/uncontrolled beam structures are considered. The results reveal again the ability of the adopted control methodology to attenuate the intensity of the bending moment at the wing root as well as the fact that the classical structural model inadvertently underestimates the bending moment, a trend becoming more pronounced at high to moderate aspect ratio beams.

In Figures 11 and 12 semi-logarithmic plots of the steady-state dimensionless deflection and acceleration amplitudes in the first three modes versus the excitation frequency for the controlled and uncontrolled beam structures are displayed.

![Graph](image1.png)

Fig. 10. – Dimensionless steady-state moment at the root versus beam aspect ratio for different transverse shear flexibility and for \( - - - K_p = 0 \) and \( - - - - - - K_p = 0.1 \)

![Graph](image2.png)

Fig. 11. – Steady-state dimensionless deflection in the first three modes versus the excitation frequency for the controlled \( - - - - - - K_p = 0.1 \), and uncontrolled beam \( - - - K_p = 0 \) \( \mathcal{R} = 16 \) and \( E/G' = 10 \).
Fig. 12. – Steady-state dimensionless accelerations \( \ddot{a} = a/g \) at the wing tip versus the excitation frequency, in the first three modes, for the uncontrolled \((-\cdots K_p = 0\) \) and controlled \((-\longrightarrow K_p = 0.1\) \) beam. \( \mathcal{A} = 16, \frac{E}{G'} = 10. \)

Fig. 13. – Normalized steady-state deflection versus twice for shear deformable \((-\cdots \frac{E}{G'} = 50)\) and non shearable \((-\longrightarrow \frac{E}{G'} = 0)\) beams and for various feedback gains \( \mathcal{A}_f = 6. \)

Finally, Figure 13 depicts the closed and open-loop deflection amplitudes for the classical and shear-deformable beam models versus time. The ability of the adopted control methodology to inhibit the oscillatory response is once again demonstrated. This plot also shows that in this case, the transverse shear flexibility has merely the role of shifting the curves towards larger times.

7. Conclusions

This paper is concerned with the dynamic bending response control of beam structures. The structure was modeled as a thin-walled beam whose theory incorporates a number of non-classical effects, such as material anisotropy, heterogeneity and transverse shear.

As the obtained results reveal, the classical Euler-Bernoulli beam model underestimates the bending moment at the beam root and overestimates the resonance frequencies. This clearly show that ignoring transverse shear

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flexibilities featured by the advanced composite materials, an unsafe design, with detrimental results on the operational capabilities of the mechanical system, can emerge.

It was shown that the use of a bending moment at the beam tip, in conjunction with a dynamic feedback control law, permits control and optimization of the structure from the dynamic response enhancement and resonance prevention points of view. Moreover, based on the obtained results it can be anticipated that the control methodology presented herein can play a significant role in the flutter control of aircraft wings as well.

APPENDIX

Several steps aiming at solving the open/closed loop eigenvalue and dynamic response problems.

The method used is based on the extended Galerkin’s method (see in this respect Librescu et al. (1997)). As a first step, Hamilton’s variational principle stating that

\[(A1) \quad \int_{t_1}^{t_2} (\delta T - \delta V + \delta W) dt = 0, \quad \delta v_0 = \delta \theta_x = 0 \text{ at } t_1, t_2\]

is used. Herin \(T\) and \(V\) denote the kinetic and strain energies, respectively while \(W\) is the work done by the external loads; \(t_1\) and \(t_2\) denote two arbitrary instants of time \(t\) while \(\delta\) denotes the variation operator. For the problem at hand it can be shown that \(V\) and \(T\) assume the form:

\[(A2) \quad V = \frac{1}{2} \int_{0}^{L} \left[ a_{33}(\theta_x')^2 + a_{55}\theta_x^2 + a_{55}(v_x')^2 \right] dz,\]

\[T = \frac{1}{2} \int_{0}^{L} \left[ b_1 v_0^2 + b_2 \theta_x^2 \right] dz,\]

where the coefficients in \((A2)\) are given by Eqs. (7) considered in conjunction with (9), and (10).

The virtual work due to the boundary moment \(\bar{M}_r\) and the load \(p_y\) is given by

\[(A3) \quad \delta W = \bar{M}_r \delta \theta_x(L, t) + p_y(z, t) \delta v_0(z, t)\]

where \(p_y\) is defined by Eq. (17).

Consideration of \((A2)\) and \((A3)\) in \((A1)\), performing the indicated operations one can obtain the Eqs. (9) and (13). However, for practical reasons, we discretize the boundary-value problem (see e.g. Meirovitch, 1997). This amounts to representing \(v_0\) and \(\theta_x\) by means of series of space-dependent trial functions multiplied by time-dependent generalized coordinates as

\[(A4) \quad v_0(z, t) = \phi_1^T(z)q_1(t), \quad \theta_x(z, t) = \phi_2^T(z)q_2(t)\]

In \((A4)\)

\[(A5) \quad \phi_1 = [\phi_1 \phi_2 \ldots \phi_N]^T, \quad \phi_2 = [\phi_{N+1} \phi_{N+2} \ldots \phi_{2N}]^T.\]

are the vectors of trial functions while \(q_1 = [q_1 \ q_2 \ldots \ q_N]\), \(q_2 = [q_{N+1} \ q_{N+2} \ldots \ q_{2N}]\) are vectors of generalized coordinates.
Inserting \((A4)\) into \((A1)\), performing the required integration with respect to the spanwise, and the time \(t\), coordinates and enforcing Hamilton’s condition \(\delta q = 0\) at \(t = t_1, t_2\) one obtains the discrete governing equations of motion

\[
M\ddot{q}(t) + H\dot{q}(t) + Kq(t) = Q(t)
\]

In Eq. \((A6)\) the mass \(M\), the damping \(H\) and stiffness \(K\) matrices as well as the generalized force vector \(Q\) are given respectively, by

\[
M = \int_0^L \begin{bmatrix} b_1 \phi_1 \phi_1^T & 0 \\ 0 & b_2 \phi_2 \phi_2^T \end{bmatrix} dz
\]

\[
H = \begin{bmatrix} 0 & 0 \\ 0 & k_p \phi_2(L) \phi_2^T(L) \end{bmatrix}
\]

\[
K = \int_0^L \begin{bmatrix} a_{55} \phi_1 \phi_1^T + a_{55} \phi_1 \phi_1^T & a_{55} \phi_1 \phi_1^T + a_{55} \phi_1 \phi_1^T \\ a_{55} \phi_1 \phi_1^T + a_{55} \phi_1 \phi_1^T & a_{55} \phi_1 \phi_1^T + a_{55} \phi_1 \phi_1^T \end{bmatrix} dz
\]

\[
Q = \int_0^L p_q \phi_q^T dz.
\]

A solution of \((A6)\) can be obtained by casting it in the state space form. To this end, we define the state vector \(X = [q^T \quad \dot{q}^T]^T\) and adjoin the identity \(I\) with this \((A6)\) becomes

\[
\dot{X}(t) = AX(t) + BQ(t)
\]

where

\[
A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}H \end{bmatrix}; \quad B = [0 \quad M^{-1}]^T
\]

For the free vibration problem, the homogeneous solution of \((A8)\) has the form \(X(t) = xe^{\lambda t}\) where \(x\) is a constant vector and \(\lambda\) is a constant scalar, both generally complex. With this, a standard eigenvalue problem is obtained

\[
AX = \lambda X
\]

which can be solved for the eigenvalues \(\lambda_r\) and eigenvectors \(x_r\) \((r = 1, 2, \ldots)\). The solution of the algebraic eigenvalue problem yields the closed-loop eigenvalues

\[
(\lambda_r, \bar{\lambda}_r) = \sigma_r \pm i\omega_{dr}
\]

where \(\sigma_r\) is a measure of the induced damping in the \(r\)th mode and \(\omega_{dr}\) is the \(r\)th frequency of damped oscillation.

For the frequency response problem, considering a particular solution of \((A8)\) in the form

\[
X(t) = x_p e^{i\omega t}
\]
we obtain that

(A13) \[ x_p = (i\omega I - A)^{-1} BQ \]

where \( I \) is the identity matrix.

Equation (A13) can be used in conjunction with (A5) to address the frequency response problem.

REFERENCES


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