Thermoelastic stress fluctuations in random-structure coated particulate composites

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Abstract – In this paper linearly thermoelastic composite media are considered, consisting of a homogeneous matrix containing a statistically homogeneous random set of ellipsoidal uncoated or coated inclusions. This study constitutes an extension of the theory regarding the purely isothermal elastic case and uncoated ellipsoidal inclusions. Effective properties (such as compliance, thermal expansion, stored energy) and both first and second statistical moments of stresses in the components are estimated for the general case of nonhomogeneity of the thermoelastic inclusion properties. The micromechanical approach is based on Green’s function techniques and on the generalization of the ‘multiparticle effective field’ method (MEFM), previously proposed for the estimation of stress field averages in the components. The application of this theory is demonstrated by calculating this overall yield surfaces of composite materials. The influence of the coating is analyzed both by the assumption of homogeneity of the stress field in the inclusion core and the thin-layer hypothesis. © Elsevier, Paris

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1. Introduction

In this paper composite media are considered which consist of a homogeneous matrix containing a random set of uncoated or coated inclusions of ellipsoidal form. Recently, there has been a growing demand for coated inclusions as a reinforcement in application areas such as electrical composites, metal matrix composites and ceramic matrix composites intended for high-temperature application. Improvement in the bonding between the inclusion and the matrix, preventing oxidation of the fiber and introducing transition properties are the basic functions of coating (Chang and Cheng, 1992).

More detailed consideration of the mechanical behavior of composite materials leads to the analysis of interfacial effects between the reinforcement and the matrix. Although in the general results obtained in the present paper no restrictions are imposed on geometrical and mechanical properties of the coating, the detailed calculations are based on a generalization of the method proposed by Cherkaoi et al. (1995) for inclusions with homogeneous coating based on Green’s function techniques as well as on interfacial Hill operators.

A considerable number of methods are available on the linear theory of uncoated and some coated composites which yield the effective elastic constants and stress field averages of the components. Pertinent but by no means exhaustive references are provided by Shermegor (1977), Willis (1982, 1983), Kunin (1983), Mura (1987), Kreher and Pompe (1989), Buryachenko and Parton (1992b) and Nemat-Nasser and Hori (1993). It now

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appears that variants of the effective medium (Kröner, 1961; Hill, 1965) and the mean field method (Mori and Tanaka, 1973; Benveniste, 1987) are the most widely used. A new approach has been proposed recently, i.e. the multiparticle effective field method (MEFM), developed by the first author of this paper (Buryachenko and Parton 1992b; Buryachenko and Kreher, 1995; Buryachenko and Rammerstorfer, 1997).

When an attempt is made to estimate the equivalent stress in strength and nonlinear creep theory, or when the plasticity theory yield function is considered, squares of the first or second invariants of the deviator of local stresses are frequently used (Buryachenko, 1996). In the extensive recent reviews of Ponte Castañeda (1997) and Suquet (1997), rigorous variational methods for analyses of different nonlinear problems are presented. The exact relations for all components of the second moment tensor of elastic stresses and internal residual stresses averaged over the volume of the components for anisotropic constituents are obtained by the perturbation method (Buryachenko and Kreher, 1995; Buryachenko, 1996; Buryachenko et al., 1996), based on the assumption that functional dependents of the effective compliance and effective stored energy on the compliance of the components are known. The above-mentioned assumptions significantly limit the applicability of the proposed methods for estimating second moments of stress in the matrix because they require an evaluation of the properties of the composite for the general case of matrix anisotropy (even for the case of the isotropic matrix). For the elastic case this disadvantage was eliminated via the method of integral equations as reported by Buryachenko and Rammerstorfer (1997).

In the present paper a generalization of the above-mentioned method of integral equations is proposed for the estimation of second moments of thermoelastic stress in the components of composites containing a statistically homogeneous field of ellipsoidal uncoated or coated inclusions. Considering both binary and triple interaction of the inclusions, explicit relations for second moments of thermoelastic stresses are obtained. The nonhomogeneity of the second moment of thermoelastic stress in coated inclusions is shown. A comparison between the second invariants of deviator stress in the matrix estimated via the perturbation method (Buryachenko and Kreher, 1995) and that estimated by the integral equation method has been made. Finally, the application of the proposed method to the prediction of certain limiting properties (e.g. strength and onset of yielding) of the composites is presented.

2. Preliminaries

2.1. Basic equations

In this paper a certain representative mesodomain $w$ with a characteristic function $W$ is considered, containing a set $X = (v_i)$ of inclusions $v_i$ with characteristic functions $V_i$ ($i = 1, 2, \ldots$). At the onset, no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions. It is assumed that the inclusions can be grouped into components $v^{(k)}$ ($k = 1, 2, \ldots, N$) with identical mechanical and geometrical properties. The local strain tensor $\varepsilon$ is related to the displacements $u$ via the linearized strain-displacement equation

$$\varepsilon = \frac{1}{2}[\nabla \otimes u + (\nabla \otimes u)^T]. \quad (1)$$

Here $\otimes$ denotes a tensor product, and $(\cdot)^T$ denotes a matrix transposition. The stress tensor $\sigma$, satisfies the equilibrium equation (no body forces acting)

$$\nabla \sigma = 0. \quad (2)$$

Stresses and strains are related to each other via the constitutive equation

$$\sigma(x) = L(x)\varepsilon(x) + \alpha(x) \quad \text{or} \quad \varepsilon(x) = M(x)\sigma(x) + \beta(x). \quad (3)$$
L(x) and M(x) ≡ L(x)^{-1} are known phase stiffness and compliance fourth-order tensors, and the common notation for tensor products has been employed: Lε = L_{ijkl}ε_{kl}, \ σε = σ_{ij}ε_{ij}, \ α \otimes β = α_{ij}β_{kl}. β(x) and α(x) ≡ -L(x)β(x) are second-order tensors of local eigenstrains and eigenstresses (frequently called transformation fields) respectively which may arise as a result of thermal expansion, phase transformation, twinning and other changes in shape or volume of the material. All tensors f (f = L, M, α, β) of material properties are decomposed as f ≡ f^{(0)} + f^{(k)}(x). f; is assumed to be constant in the matrix v^{(0)} = w \setminus v and is an inhomogeneous function within the inclusions, respectively:

\[ f(x) = \begin{cases} f^{(0)} & \text{for } x \in v^{(0)}, \\ f^{(0)} + f^{(k)}(x) & \text{for } x \in v^{(k)}. \end{cases} \tag{4} \]

Here and in the following the upper index \( (k) \) numbers the components and the lower index \( i \) numbers, the individual inclusions \( v \equiv \bigcup v^{(k)} \equiv \bigcup v_i, \quad (k = 1, 2, \ldots, N; \quad i = 1, 2, \ldots). \)

We assume that the phases are perfectly bonded so that the displacements and traction components of the stresses are continuous across the interphase boundaries. We take uniform traction boundary conditions for the mesodomain \( w \)

\[ \sigma^0(x)n(x) = T(x), \quad x \in \partial w, \tag{5} \]

where \( T(x) \) is the traction vector at the external boundary \( \partial w \), \( n \) is its unit outward normal, and \( \sigma^0 \) is the mesoscopic stress tensor, i.e. a given constant symmetric tensor.

2.2. Statistical description of the composite structure

It is assumed that the representative mesodomain \( w \) contains a statistically large number of inclusions \( v_i \subset v^{(k)} \quad (i = 1, 2, \ldots; \quad k = 1, 2, \ldots, N); \) all the random quantities under discussion are described by statistically homogeneous ergodic random fields and hence the ensemble averaging could be replaced by volume averaging

\[ \langle (...) \rangle = \bar{w}^{-1} \int \langle (...) \rangle W(x) \, dx, \quad \langle (...) \rangle^{(k)} = [\bar{v}^{(k)}]^{-1} \int \langle (...) \rangle^{(k)} V^{(k)}(x) \, dx, \tag{6} \]

where \( \sum V^{(k)} = \sum V_i \equiv V, \quad k = 1, 2, \ldots, N; \quad i = 1, 2, \ldots, V^{(k)} \) is the characteristic function of \( v^{(k)} \). The bar appearing above the region represents its measure, e.g. \( \bar{v} \equiv \text{mes } v \). We include in component \( v^{(k)} \quad (k = 1, 2, \ldots, N) \) those inclusions which have the same physical and geometric parameters. Therefore, the average for component \( v^{(k)} \) agrees with the general average for an individual inclusion \( v_i \subset v^{(k)} \quad (i = 1, 2, \ldots) : \quad \langle (...) \rangle_i = \langle (...) \rangle^{(k)} \); the notation \( \langle (...) \rangle_i(x) \) at \( x \in v_i \subset v^k \) indicates the average over an ensemble realization of surrounding inclusions (but not for the volume \( v_i \) of a particular inclusion, in contrast to \( \langle (...) \rangle \)).

For the description of the random structure of a composite material let us introduce a conditional probability density \( \varphi(v_i, x_i; v_1, x_1, \ldots, v_n, x_n) \), which is a probability density to find the \( i \)th inclusion with the center \( x_i \) in the domain \( v_i \) with fixed inclusions \( v_1, \ldots, v_n \) with centers \( x_1, \ldots, x_n \). The notation \( \varphi(v_i, x_i; v_1, x_1, \ldots, v_n, x_n) \) denotes the case \( x_i \neq x_1, \ldots, x_n \). Of course, \( \varphi(v_i, x_i; v_1, x_1, \ldots, v_n, x_n) = 0 \) for values of \( x_i \) lying inside the ‘included volumes’ \( \bigcup v_{im} \quad (m = 1, \ldots, n) \), where \( v_{im} \supseteq v_m \) with characteristic functions \( V_{im}^0 \) (since inclusions cannot overlap), and \( \varphi(v_i, x_i; v_1, x_1, \ldots, v_n, x_n) \rightarrow \varphi(v_i) \) at \( |x_i - x_m| \rightarrow \infty, \quad m = 1, \ldots, n \) (since no long-range order is assumed). \( \varphi(v_i) \) is a number density
2.3. General integral equations and definitions of the effective fields

From equations (1)–(4) a general integral equation for $\epsilon$ can be derived. Substituting (1) and (3) in the equilibrium equation (2), we obtain a differential equation with respect to the displacement $u$. By rearranging the latter into an integral equation and transforming it by a method developed earlier (Levin, 1976; Kröner, 1977; Willis, 1982; Buryachenko and Lipanov, 1986), we obtain

$$\sigma(x) = \sigma^0 + \int \Gamma(x - y) \{\eta(y) - \langle \eta \rangle\} dy$$  \hspace{1cm} (7)

where in a similarly to Willis (1982) we define

$$\eta(y) = M_1(y) \sigma(y) + \beta_1(y).$$  \hspace{1cm} (8)

The tensor $\eta$ is called the strain polarization tensor and is simply a notational convenience. In (8) $M_1(y)$ and $\beta_1(y)$ are the jumps in compliance $M^{(k)}$ and the eigenstrain $\beta^{(k)}$ respectively, within the component $\nu^{(0)}$, $M^{(k)} \equiv [L^{(k)}]^{-1}$ ($k = 0, \ldots, N$). The integral operator kernel

$$\Gamma(x - y) \equiv - [I \delta(x - y) + \nabla \nabla G(x - y) L^{(0)}],$$  \hspace{1cm} (9)

is defined by the Green tensor $G$ of Lame's equation of a homogeneous medium with an elasticity tensor $L^{(0)}$

$$\nabla \left\{ L^{(0)} \frac{1}{2} \left[ \nabla \otimes G(x) + (\nabla \otimes G(x))^T \right] \right\} = - \delta(x),$$  \hspace{1cm} (10)

$\delta(x)$ is the Dirac delta function, $\delta$ and $I$ are the unit second- and fourth-order tensors, respectively.

Let us consider some conditional statistical averages of the general integral equation (7), leading to an infinite system of integral equations ($n = 1, 2, \ldots$)

$$\langle \sigma(x) | \nu_1, \nu_1; \ldots; \nu_n, \nu_n \rangle_i = \sum_{i=1}^{n} \int \Gamma(x - y) \{V_i(y) \eta(y)|\nu_1, \nu_1; \ldots; \nu_n, \nu_n\} dy$$

$$= \sigma^0 + \int \Gamma(x - y) \{\eta(y)|\nu_1, \nu_1; \ldots; \nu_n, \nu_n\} - \langle \eta \rangle \} dy,$$  \hspace{1cm} (11)

where $x \in \nu_1, \ldots, \nu_n$ in the $n$th line of the system (Buryachenko and Kreher, 1995). Now we define the effective field $\tilde{\sigma}(x)_1, \ldots, n$ \hspace{1cm} (x \in \nu_1, \ldots, \nu_n) as a stress field in which the chosen fixed inclusions $\nu_1, \ldots, \nu_n$ are embedded. This effective field is a random function of all the other positions of the surrounding inhomogeneities,
and the average of $\tilde{\sigma}(x)_{1,...,n}$ over a random realization of these inclusions is equal to the right-hand side of the $n$th line of the system (11)

$$
\langle \tilde{\sigma}(x)_{1,...,n} \rangle = \sigma^0 + \int \Gamma(x-y) \{ \langle \eta(y) \rangle ; v_1, x_1; \ldots; v_n, x_n \} - \langle \eta \rangle \} dy.
$$

(12)

for $(x \in v_i, \quad i = 1, 2, \ldots, n)$. Consequently, each inclusion $v_i$ $(i = 1, \ldots, n)$ of the chosen fixed set is in a random (generally speaking nonhomogeneous) effective field

$$
\bar{\sigma}_i(x) = \tilde{\sigma}(x)_{1,...,n} + \sum_{i \neq j} \int \Gamma(x-y)V_j(y)\eta(y)dy,
$$

(13)

$(x \in v_i, \quad i \neq j, \quad i, j = 1, 2, \ldots, n)$ which is the superposition of the effective field $\tilde{\sigma}(x)_{1,...,n}$ and the disturbance caused by the other inclusions of the considered set (see figure 1).

![Figure 1. Schematic representation of the effective fields.](image)

### 2.4. Perturbation method for the estimation of second stress moments in the components

The fourth–rank tensor of the second moment of stress $\langle \sigma \otimes \sigma \rangle$, averaged over the volume of the component $v^{(k)}$, $(k = 0, \ldots, N)$ can be exactly determined by the perturbation method from the functional dependence of the compliance $M^*$, stored energy $U^*$ and effective eigenstrains $\beta^*$ on the compliance of the component $v^{(k)}$

$$
\langle \sigma \otimes \sigma \rangle^{(k)} = \frac{1}{c^{(k)}} \frac{\partial M^{*}}{\partial M^{(k)}} \sigma^0 \otimes \sigma^0 - \frac{2}{c^{(k)}} \frac{\partial U^{*}}{\partial M^{(k)}} + \frac{2}{c^{(k)}} \frac{\partial \beta^{*}}{\partial M^{(k)}} \sigma^0,
$$

(14)

or in index form

$$
\langle \sigma_{ij} \sigma_{mn} \rangle^{(k)} = \frac{1}{c^{(k)}} \frac{\partial M_{pq}^{*}}{\partial M_{ijmn}^{(k)}} \sigma_{pq}^{0} \sigma_{rs}^{0} - \frac{2}{c^{(k)}} \frac{\partial U^{*}}{\partial M_{ijmn}^{(k)}} + \frac{2}{c^{(k)}} \frac{\partial \beta_{pq}^{*}}{\partial M_{ijmn}^{(k)}} \sigma_{pq}^{0},
$$

(15)

where the partial derivatives are calculated under the assumption of fixed transformation fields $\beta(x)$, $x \in w$. Some results regarding relationships (14), (15) were obtained by Parton and Buryachenko (1990) (for $\beta \equiv 0$) and Buryachenko and Kreher (1995) (for $\sigma^0 \equiv 0$). Buryachenko and Shermergor (1995) considered the generalization of Eqs. (14) and (15) for thermo-elastic electric fields. Relations (14) and (15) have
been obtained for any degree of anisotropy of \( \mathbf{M}^*, \mathbf{M}^{(i)} \) \( (i = 0, 1, \ldots, N) \). For isotropic tensors 
\( \mathbf{M}^{(i)} = (3p^{(i)}, 2q^{(i)}) \equiv 3p^{(i)}\mathbf{N}_1 + 2q^{(i)}\mathbf{N}_2, \quad (\mathbf{N}_1 = \delta \otimes \delta /3, \quad \mathbf{N}_2 = \mathbf{I} - \mathbf{N}_1) \), we have

\[
(\sigma_0^{(k)}) = \frac{1}{9c^{(k)}} \left( \frac{\partial M^*_{pqrs}}{\partial p^{(k)}} \sigma_0^{pqrs} - \frac{2}{9c^{(k)}} \frac{\partial U^*}{\partial p^{(k)}} + \frac{2}{9c^{(k)}} \frac{\partial \beta^*_{pq}}{\partial p^{(k)}} \sigma_0^{pq} \right),
\]

\[
(\mathbf{s}s)^{(k)} = \frac{1}{2c^{(k)}} \left( \frac{\partial M^*_{pqrs}}{\partial q^{(k)}} \sigma_0^{pqrs} - \frac{1}{c^{(k)}} \frac{\partial U^*}{\partial q^{(k)}} + \frac{1}{c^{(k)}} \frac{\partial \beta^*_{pq}}{\partial q^{(k)}} \sigma_0^{pq} \right),
\]  

(16)  

where \( \sigma_0 = \delta \sigma /3, \quad s = \mathbf{N}_2\sigma \). The relations (16), (17) reduce to the results found by Bobeth and Diener (1987) and Kreher and Pompe (1989) for macroisotropic composites. The problem of estimating effective properties \( \mathbf{M}^*, \mathbf{U}^*, \beta^* \) is equivalent to that of evaluating average stresses in the components (e.g. Kreher and Pompe, 1989 Dvorak and Benveniste, 1992). Both the effective compliance \( \mathbf{M}^* \) and the effective eigenstrains \( \beta^* \) are defined by general relations

\[
\mathbf{M}^* = \langle \mathbf{MB}^* \rangle, \quad \beta^* = \langle \mathbf{B}^*\mathbf{T} \beta \rangle,
\]

(18)

where \( \mathbf{B}^* = \mathbf{B}^*(\mathbf{x}) \) is a local stress concentration tensor obtained under pure mechanical loading

\[
\sigma(\mathbf{x}) = \mathbf{B}^*(\mathbf{x})\sigma^0 \quad \text{for} \quad \beta(\mathbf{x}) \equiv 0.
\]

(19)

Conversely, the estimation of the residual stresses (for \( \sigma^0 \equiv 0 \)) can be used for the calculation of the stored energy \( \mathbf{U}^* \) in the transformed stress field as well as of the effective eigenstrains \( \beta^* \)

\[
\mathbf{U}^* = -\frac{1}{2} \langle \beta \sigma \rangle,
\]

(20)

\[
\beta^* = \langle \beta + \eta \rangle.
\]

(21)

3. Effective field hypotheses and average stresses in the components

3.1. Approximative effective field hypothesis

In order to simplify the system (11) we now apply the main hypothesis of many micromechanical methods, the so-called effective field hypothesis.

Hypothesis 1: each inclusion \( v_i \) has an ellipsoid form and is located in the field \( \bar{\sigma}_i \equiv \bar{\sigma}(\mathbf{x}) \) \( (\mathbf{x} \in v_i) \) which is homogeneous for the inclusion \( v_i \). The perturbation introduced by the inclusion \( v_i \) in the point \( \mathbf{y} \notin v_i \) is defined by the relation

\[
\int \Gamma(\mathbf{y} - \mathbf{x})V_i(\mathbf{x})\eta(\mathbf{x})d\mathbf{x} = \bar{v}_iT_i(\mathbf{y} - \mathbf{x}_i)\eta_i,
\]

(22)

where \( \eta_i \equiv \langle \eta(\mathbf{x})V_i(\mathbf{x}) \rangle_{(i)} \) is an average over the volume of the inclusion \( v_i \) (but not for the ensemble) and

\[
T_i(\mathbf{y} - \mathbf{x}_i) = (\bar{v}_i)^{-1} \int \Gamma(\mathbf{y} - \mathbf{x})V_i(\mathbf{x})d\mathbf{x}, \quad \mathbf{y} \notin v_i.
\]

(23)
Thermoelastic stress fluctuations in composites

Then in view of the linearity of the problem there exist constant fourth and second-rank tensors \( \mathbf{B}_i(x), \mathbf{R}_i(x) \) and \( \mathbf{C}_i(x), \mathbf{F}_i(x) \), such that

\[
\sigma(x) = \mathbf{B}_i(x)\hat{\sigma}(x) + \mathbf{C}_i(x), \quad \hat{\eta}(x) = \mathbf{R}_i(x)\hat{\sigma}(x) + \mathbf{F}_i(x), \quad x \in v_i,
\]

where

\[
\mathbf{R}_i(x) = \bar{v}_i \mathbf{M}_i(x)\mathbf{B}(x), \quad \mathbf{F}_i(x) = \bar{v}_i [\mathbf{M}_i(x)\mathbf{C}(x) + \beta_1(x)].
\]

According to Eshelby’s theorem (1961), the relations between the averaged tensors (24) are

\[
\mathbf{R}_i = \bar{v}_i \mathbf{Q}_i^{-1}(\mathbf{I} - \mathbf{B}_i), \quad \mathbf{F}_i = -\bar{v}_i \mathbf{Q}_i^{-1}\mathbf{C}_i,
\]

where \( \mathbf{f}_i = \langle \mathbf{f}(x) \rangle_{(i)} \) \( \mathbf{f} \) stands for \( \mathbf{B}, \mathbf{C}, \mathbf{R}, \mathbf{F} \) and the tensor \( \mathbf{Q}_i \) is associated with the well-known Eshelby tensor by

\[
\mathbf{S}_i = \mathbf{I} - \mathbf{M}^{(i)}\mathbf{Q}_i, \quad \mathbf{Q}_i \equiv -\langle \Gamma(x - y) \rangle_i = \text{const.}, \quad (x, y \in v_i).
\]

For example, for the homogeneous ellipsoid domain \( v_i \) with

\[
\mathbf{M}_i(x) = \mathbf{M}^{(i)}_1 = \text{const.}, \beta_1(x) = \beta^{(i)}_1 = \text{const} \quad \text{at} \quad x \in v_i,
\]

we obtain

\[
\mathbf{B}_i = \left( \mathbf{I} + \mathbf{Q}_i\mathbf{M}^{(i)}_1 \right)^{-1}, \quad \mathbf{C}_i = -\mathbf{B}_i\mathbf{Q}_i\beta^{(i)}_1,
\]

\[
\mathbf{R}_i = \bar{v}_i \mathbf{M}^{(i)}_1\mathbf{B}_i, \quad \mathbf{F}_i = \bar{v}_i(\mathbf{I} + \mathbf{M}^{(i)}_1\mathbf{Q}_i)^{-1}\beta^{(i)}_1.
\]

From a comparison between the relations (24) and (29) we see that the average thermoelastic response (i.e. the tensors \( \mathbf{B}_i, \mathbf{C}_i, \mathbf{R}_i, \mathbf{F}_i \)) of any coated inclusion \( v_i \) is the same as the response of some fictitious ellipsoid homogeneous inclusion with thermoelastic parameters

\[
\mathbf{M}^{(i)}_1 = \mathbf{Q}_i^{-1}(\mathbf{B}_i^{-1} - \mathbf{I}), \quad \beta^{(i)}_1 = -\mathbf{Q}_i^{-1}\mathbf{B}_i^{-1}\mathbf{C}_i,
\]

which can be expressed in terms of the tensors \( \mathbf{R}_i, \mathbf{F}_i \)

\[
\mathbf{M}^{(i)}_1 = (\mathbf{I} - \mathbf{Q}_i\mathbf{R}_i)^{-1}\mathbf{R}_i, \quad \beta^{(i)}_1 = \bar{v}_i^{-1}(\mathbf{M}^{(i)}_1\mathbf{Q}_i + \mathbf{I})\mathbf{F}_i.
\]

The parameters (30) and (31) of fictitious ellipsoid inclusions are simply a notational convenience. No restrictions are imposed on the microtopology of the coated inclusion or on the inhomogeneity of the stress state in the coated inclusions.

Using Hypothesis 1, the system (11) with fixed values \( \hat{\sigma}(x)_{1,\ldots,n} \ (x \in v_i) \) on the right-hand side of the equations becomes algebraic when the solution (24) for one inclusion in the field \( \hat{\sigma}(x_i) \ (i = 1, \ldots, n) \) is applied

\[
\mathbf{R}_i \hat{\sigma}(x_i) + \mathbf{F}_i = \sum_{j=1}^{n} Z_{ij} \left\{ \mathbf{R}_j \hat{\sigma}(x_j)_{1,\ldots,n} + \mathbf{F}_j \right\}.
\]

(32)
Here the matrix $Z^{-1}$ has the elements $(Z^{-1})_{ij}$

$$(Z^{-1})_{ij} = I \delta_{ij} - R_j T_{ij}(x_i - x_j), \quad (i, j = 1, \ldots, n),$$

which represent fourth-order tensors, with a Kronecker symbol $\delta_{ij}$ and

$$T_{ij}(x_i - x_j) = (\overline{v}_i \overline{v}_j)^{-1} \int \int \Gamma(x - y) V_i(x) V_j(y) dxdy.$$  

(34)

For the isotropic matrix and spherical inclusions the tensors $T_i(y - x_i)$ (3.2) and $T_{ij}(x_i - x_j)$ (34) are known (e.g. Buryachenko and Rammerstorfer, 1997).

3.2. Closing effective field hypothesis

**Hypothesis 2:** for a sufficiently large $n$, we close the system (11) with the assumption $(\bar{\sigma}(x)_{1,2,\ldots,n+1})_i = (\bar{\sigma}(x)_{1,2,\ldots,n})_i$, where the right-hand side of the equality does not contain the index $j \neq i$ ($1 < j < n, \quad x \in v_j$). System (11) can be solved by analytical methods if

$$\langle \bar{\sigma}(x)_{1,2} \rangle_i = \langle \bar{\sigma}(x) \rangle_i = \text{const.} \quad (i = 1, 2).$$

(35)

This independence of $\langle \bar{\sigma}(x)_{1,2} \rangle$ of the spacing between the inclusions $v_1$ and $v_2$ (35) occurs for the limiting case $|x_1 - x_2| \gg \max a_i^k$, where $a_i^k$ ($k = 1, 2, 3; \quad i = 1, 2$) are the semi-axes of the ellipsoidal inclusions $v_1$ and $v_2$, respectively. Then from (11), taking (24), (32), and (35) into account we obtain

$$\overline{v}_i(\eta_i) = \overline{v}_i \eta_i^0 + R_i \sum_{q=1}^N \left\{ \int T_{iq}(x_i - x_q) Z_{qi} \varphi(v_q, x_q; v_i, x_i) dx_q \overline{v}_i(\eta_i) \right.$$  

$$+ \int \left[ T_{iq}(x_i - x_q) Z_{qq} \varphi(v_q, x_q; v_i, x_i) - T_i(x_i - x_q) n^{(q)} \right] \overline{v}_q(\eta_q) dx_q \left\}, \right.$$

(36)

where $\overline{v}_i \eta_i^0 = R_i \sigma^0 + F_i$ is called the external strain polarization tensor in the component $i^{(i)}$, and the matrix elements $Z_{qi}$, $Z_{qq}$ are nondiagonal and diagonal elements respectively of the binary interaction matrix $Z$ (32) for the two inclusions $v_q$ and $v_i$. The algebraic system can be solved for $\langle \eta_i \rangle$

$$\overline{v}_i(\eta_i) = \sum_{j=1}^N Y_{ij} \overline{v}_j \eta_j^0$$

(37)

where the matrix $Y^{-1}$ has the following elements

$$(Y^{-1})_{ij} \quad (i, j = 1, 2, \ldots, N)$$

$$(Y^{-1})_{ij} = \delta_{ij} \left[ I - R_i \sum_{q=1}^N \int T_{iq}(x_i - x_q) Z_{qi} \varphi(v_q, x_q; v_i, x_i) dx_q \right]$$

$$- R_i \int \left[ T_{ij}(x_i - x_j) Z_{jj} \varphi(v_j, x_j; v_i, x_i) - T_i(x_i - x_j) n^{(j)} \right] dx_j.$$  

(38)
3.3. Average stresses in the components

The mean field of elastic stresses inside the inclusions \( \langle \sigma \rangle_i \) is obtained from (24) and (37)

\[
\langle \sigma \rangle_i(x) = B_i(x) R_i^{-1} \left\{ \sum_{j=1}^{N} Y_{ij} (R_j \sigma^0 + F_j) - F_i \right\} + C_i(x),
\]

and therefore

\[
\langle \sigma \rangle_i = B_i \left\{ D_i \sigma^0 + R_i^{-1} \sum_{j=1}^{N} (Y_{ij} - I \delta_{ij}) F_j \right\} + C_i, \quad D_i = R_i^{-1} \sum_{j=1}^{N} Y_{ij} R_j,
\]

where the tensor \( D_i \) \((i = 1, \ldots, N)\) has a simple physical meaning of the action of surrounding inclusions on the separate one: \( \langle \sigma \rangle_i = D_i \sigma^0 \) for \( \beta(x) \equiv 0 \). The mean matrix stress follows simply from the condition \( \langle \sigma \rangle = \sigma^0 \)

\[
\langle \sigma \rangle_0 = \frac{1}{c(0)} (\sigma^0 - (\sigma V)).
\]

3.4. Effective properties of the composite

After estimating average stresses within the inclusions [see (40)] the problem of calculating effective properties becomes trivial and leads, according to (18) and (20), to

\[
M^* = M^{(0)} + \sum_{i,j=1}^{N} Y_{ij} R_j n^{(i)}
\]

\[
\beta^* = \beta^{(0)} + \sum_{i,j=1}^{N} Y_{ij} F_j n^{(i)}
\]

\[
U^* = -\frac{1}{2} \sum_{i,j=1}^{N} \langle \beta_1^{(i)}(x) B_i(x) \rangle_i R_i^{-1} (Y_{ij} - I \delta_{ij}) F_j c^{(i)}
- \frac{1}{2} \sum_{i=1}^{N} \langle \beta_1^{(i)}(x) C_i(x) \rangle_i c^{(i)}.
\]

For homogeneous inclusions (28) the formulae (42)-(44) are equivalent to the results derived by Buryachenko (1993), and Buryachenko and Kreher (1995).

For identical inclusions with the same geometrical and mechanical properties (e.g. \( N = 1 \)) it is possible to prove that in the framework of Hypothesis 1, the relations (42)-(44) are exact and valid for any influence matrix \( Y_{11} \), although \( Y_{11} \) itself depends on the concrete averaging method of the purely elastic problem \( (\beta \equiv 0) \) using Hypothesis 1 [e.g. (38)]. Then (for \( N = 1, \ x \in v_i \subset v_{(1)} \)) one obtains

\[
\beta^* = \beta^{(0)} + (M^* - M^{(0)}) R_i^{-1} F_i,
\]

\[
U^{*II} = -\frac{1}{2} \langle \beta_1^{(1)}(x) B_i(x) \rangle_i \left[ M^* - M^{(0)} - R_i n^{(1)} \right] R_i^{-1} F_i v_i
- \frac{1}{2} \langle \beta_1^{(1)}(x) C_i(x) \rangle_i c^{(1)}.
\]
In particular, for homogeneous (i.e. noncoated) inclusions the constant tensors $B_i$, $C_i$, $R_i$, and $F_i$, according to Eq. (29), are determined by the Eshelby tensor $S_i$ and the jumps in the material property tensors $M_1^{(1)}$, $\beta^{(1)}_1$. Then from (45) and (46) the classical results for two-phase composites by Rosen and Hashin (1970) are derived. Effective properties, however, are not discussed in more detail in this paper.

3.5. Particular cases

Now we consider the limiting cases of relations (40) and (41). Eshelby’s model holds for dilute concentrations of the inclusions, i.e. $c \equiv \langle V \rangle \to 0$. Here we have $Y_{ij} \equiv I \delta_{ij}$ ($i, j = 1, \ldots, N$) and

$$
\langle \sigma \rangle_i = B_i \sigma^0 + C_i, \quad \langle \sigma \rangle_0 = \frac{1}{c_0} \left\{ \left[ I - \sum_{q=1}^{N} c^{(q)}_B B_q \right] \sigma^0 - \sum_{q=1}^{N} c^{(q)} C_q \right\},
$$

(47)

from which and from Eqs. (18) and (20) the effective thermoelastic properties can be found.

The known ‘quasi-crystalline’ approximation by Lax (1952) (see also Buryachenko and Parton, 1990; Ponte Castañeda and Willis, 1995) in our notations has the form

$$
\langle \bar{\sigma}_i(x) | v_i, x_i; v_j, x_j \rangle = \langle \bar{\sigma}_i \rangle, \quad x \in v_i
$$

(48)

and is equivalent to the equality

$$
Z_{ij} = I \delta_{ij}.
$$

(49)

Therefore, the matrix $Y^{-1}$ (Buryachenko and Parton, 1990) can be reduced to

$$
(Y^{-1})_{ij} = I \delta_{ij} - R_i \int [T_{ij}(x_i - x_j) \varphi(v_j, x_j ; v_i, x_i) - T_i(x_i - x_j) n^{(j)}] dx_j.
$$

(50)

The final results may be significantly simplified under the following additional assumptions

$$
\langle V_j(y) \eta_j | ; v_i, x_i \rangle = h_1(\langle \eta_j \rangle, \rho), \quad \rho \equiv \| a_i^{-1}(x_j - x_i) \|.
$$

(51)

Here the dependence of the function $h_1$ on the geometrical parameters of the inclusion $v_i$ is defined by the scalar value $\rho$; $a_i^{-1}$ identifies a matrix of affine transformation which transfers the ellipsoid $v_i$ into a unit sphere. According to relation (51) the conditional averaging properties of the composite have level surfaces which are obtained from the ellipsoid surfaces by the use of a homothetic transformation. Under the assumption of (48) the equality (51) is valid under the simplest conditional probability density

$$
\varphi(v_q, x_q ; v_i, x_i) = h_2(\rho)
$$

(52)

For spherical inclusions the relation (52) is realized for a statistical isotropy of the composite structure.
Thermoelastic stress fluctuations in composites

According to Eq. (27) and by virtue of the fact that the generalized function \( \Gamma'(x) \) is an even homogeneous function of order \(-3\), we obtain – under the assumption (51) – the following relation

\[
\sum_{q=1}^{N} \int \Gamma(x - x_q) |(V_q(x_q)\eta(x_q); v_i, x_i) - c^{(q)}(\eta_q)| dx_q = Q_i \sum_{q=1}^{N} \langle \eta_q \rangle c^{(q)}
\]

Taking assumptions (35) and (48) into account, Eq. (40) can be combined in a simple equation

\[
\langle \sigma \rangle_i = B_i \sigma^0 + C_i + B_i Q_i \left[ I - \sum_{q=1}^{N} n^{(q)} R_q Q_q \right]^{-1} \sum_{q=1}^{N} n^{(q)} (R_q \sigma^0 + F_q).
\]

Substitution of Eqs (24) and (54) in (18) and (20) gives

\[
M^* = M^{(0)} + \left[ I - \sum_{q=1}^{N} n^{(q)} R_q Q_q \right]^{-1} \sum_{q=1}^{N} n^{(q)} R_q,
\]

\[
\beta^* = \beta^{(0)} + \left( M^* - M^{(0)} \right) \left[ \sum_{q=1}^{N} n^{(q)} R_q \right]^{-1} \sum_{q=1}^{N} n^{(q)} F_q,
\]

\[
U^* = -\frac{1}{2} \sum_{i=1}^{N} c^{(i)} \left[ (\beta^{(i)}(x)B_i(x)), Q_i(\beta^* - \beta^{(0)}) + (\beta^{(i)}(x)C_i(x)) \right].
\]

Relations (56) and (57) are not the result of a generalization of Eqs. (45) and (46) to any number of components, but were obtained under the additional assumptions (48) and (52). For homogeneous inclusions (28) Eqs (55) and (56) reduce to some results found by Levin (1975, 1977) and Ponte Castañeda and Willis (1995) (see also Kunin, 1983). Thus, if the distribution of inclusions is taken to be the same for all inclusions pairs and, therefore, in Eq. (53) \( Q_i = \text{const.} \), for some ellipsoid domain \( v^{el} \), then Eq. (55) rewritten in terms of elastic moduli coincides with Eq. 3.20 from Ponte Castañeda and Willis (1995). If additionally one assumes that \( v^{el} \) has the form of a sphere, then Eqs (54) and (55) reduce to the results obtained by Levin (1977). For identical homogeneous ellipsoid inclusions randomly oriented in space, Eq. (55) coincides with the result of Levin (1975).

Equations (54)–(57) are in rather poor agreement with the multicomponent realization of the Mori–Tanaka scheme (Pettermann and Böhm, 1995), the inconsistency of which is well known (Benveniste et al., 1991). According to Eqs. (24) and (48) \( \langle \sigma \rangle \) is a function of the shape of the considered inclusion, whereas the Mori–Tanaka hypothesis \( \langle \sigma \rangle \equiv \langle \sigma \rangle_0 \) ignores such a dependence. On the other hand, for unidirectionally aligned and identical homogeneous inclusions, Eqs. (28) and (54) yield the Mori–Tanaka solution (for references, see Buryachenko and Kreher, 1995)

\[
\langle \sigma \rangle_1 = B_1 [I - Q_1 R_1 n^{(1)}]^{-1} \sigma^0 - c^{(0)} Q_1 B_1^{(1)},
\]

\[
M^* = M^{(0)} + c^{(1)} [c^{(0)} I + c^{(1)} B_1]^{-1} R_1,
\]

\[
\beta^* = \langle \beta \rangle + [M^* - M^{(1)}] [M^{(1)} - M^{(0)}]^{-1} \beta^{(1)}.
\]

Another trend of simplified assumptions concerns approximative solutions to the problem of binary interaction of the inclusions (Buryachenko and Rammerstorfer, 1997).
4. General integral equations for second stress moments within the components

To obtain the second stress moments in the component \(v^{(i)}\) of the inclusions \((i = 1, 2, \ldots)\) it is necessary to put the tensor product of (13) at \(n = 1\) into \(\overline{\sigma}(x)\)

\[
\overline{\sigma}(x) \otimes \overline{\sigma}(x) = \sigma^0 \otimes \sigma^0 + \sigma^0 \otimes \int \Gamma(x - x_p)[\eta(x_p)V_p(x_p) - \langle \eta \rangle]dx_p
\]

\[
+ \int \Gamma(x - x_q)[\eta(x_q)V_q(x_q) - \langle \eta \rangle]dx_q \otimes \sigma^0
\]

\[
+ \int \int \Gamma(x - x_p)[\eta(x_p)V_p(x_p) - \langle \eta \rangle]
\]

\[
\otimes \Gamma(x - x_q)[\eta(x_q)V_q(x_q) - \langle \eta \rangle]dx_pdx_q
\]

(61)

with \(x \in v_i\). The right-hand side of Eq.(61) is a random function of the arrangements of surrounding inclusions \(v_p, v_q \neq v_i\) \((p, q = 1, 2, \ldots)\). Averaging (61) over a realization ensemble leads to

\[
\langle \overline{\sigma} \otimes \overline{\sigma} \rangle(x) = \sigma^0 \otimes \sigma^0 + \sigma^0 \otimes \int \Gamma(x - x_p)[\langle \eta(x_p)V_p(x_p) \rangle|v_p, x_p; v_i, x_i\rangle - \langle \eta \rangle]dx_p
\]

\[
+ \int \Gamma(x - x_q)[\langle \eta(x_q)V_q(x_q) \rangle|v_q, x_q; v_i, x_i\rangle - \langle \eta \rangle]dx_q \otimes \sigma^0
\]

\[
+ \int \int \{ \langle \Gamma(x - x_p)\eta(x_p)V_p(x_p) \rangle \otimes \langle \Gamma(x - x_q)\eta(x_q)V_q(x_q) \rangle|v_p, x_p, v_q, x_q; v_i, x_i\rangle
\]

\[
- \langle \Gamma(x - x_p)\eta(x_p)V_p(x_p) \rangle|v_p, x_p, v_i, x_i\rangle \otimes \langle \Gamma(x - x_q)\eta(x_q)V_q(x_q) \rangle|v_q, x_q, v_i, x_i\rangle
\]

\[
+ \langle \Gamma(x - x_p)\eta(x_p) \rangle \otimes \langle \Gamma(x - x_q)\eta(x_q) \rangle \}
dx_pdx_q.
\]

(62)

The right-hand side of Eq. (62) includes double- and triple-point conditional probability densities in which the terms with \(x_p = x_q\) under the condition \(x_p \neq x_i, \quad x_q \neq x_i\) may be isolated with the aid of the equality

\[
\varphi(v_p, x_p, v_q, x_q; v_i, x_i) = \delta(x_p - x_q)\varphi(v_p, x_p; v_i, x_i)
\]

\[
+ \varphi(v_p, x_p; v_i, x_i)\varphi(v_q, x_q; v_p, x_p; v_i, x_i)
\]

(63)

Then Eq. (62) can be rewritten as

\[
\langle \overline{\sigma} \otimes \overline{\sigma} \rangle_i(x) = \langle \overline{\sigma} \rangle_i(x) \otimes \langle \overline{\sigma} \rangle_i(x) + \int \{ \langle \Gamma(x - x_p)\eta(x_p) \rangle
\]

\[
\otimes \langle \Gamma(x - x_q)\eta(x_q) \rangle|v_p, x_p; v_i, x_i\rangle \varphi(v_p, x_p; v_i, x_i)dx_p
\]

\[
+ \int \int \{ \langle \Gamma(x - x_p)\eta(x_p) \rangle \otimes \langle \Gamma(x - x_q)\eta(x_q) \rangle|v_p, x_p, v_q, x_q; v_i, x_i\rangle
\]

\[
\cdot \varphi(v_p, x_p; v_i, x_i)\varphi(v_q, x_q; v_p, x_p; v_i, x_i)
\]

\[
- \langle \Gamma(x - x_p)\eta(x_p) \rangle|v_p, x_p, v_i, x_i\rangle \otimes \langle \Gamma(x - x_q)\eta(x_q) |v_q, x_q, v_i, x_i\rangle
\]

\[
\cdot \varphi(v_p, x_p; v_i, x_i)\varphi(v_q, x_q; v_i, x_i) \}
dx_qdx_p
\]

(64)
with $x \in v_i$. Here the double integral was transformed in terms of average effective stresses.

The next simplification of (64) is connected with the use of the Hypothesis 1 assumption (22) and the known tensors (34). Then we obtain an approximative representation of the average of the second effective stress moment in the component

$$
\langle \sigma \otimes \bar{\sigma} \rangle_i = \langle \sigma \rangle_i \otimes \langle \sigma \rangle_i
$$

$$
+ \int \left\{ \left[ \mathbf{T}_{ip}(x_i - x_p) \eta_p \bar{v}_p \right] \otimes \left[ \mathbf{T}_{iq}(x_i - x_q) \eta_q \bar{v}_q \right] \right\} v_p, x_p; v_q, x_q; v_i, x_i
$$

$$
\cdot \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
+ \int \left\{ \left[ \mathbf{T}_{ip}(x_i - x_q) \eta_p \bar{v}_p \right] \otimes \left[ \mathbf{T}_{iq}(x_i - x_q) \eta_q \bar{v}_q \right] \right\} v_p, x_p; v_q, x_q; v_i, x_i
$$

$$
\cdot \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
- \left[ \mathbf{T}_{ip}(x_i - x_p) \eta_p \bar{v}_p \right] \otimes \left[ \mathbf{T}_{iq}(x_i - x_q) \eta_q \bar{v}_q \right] v_p, x_p; v_q, x_q; v_i, x_i
$$

$$
\cdot \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
= \int \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i) \bar{v}_p \bar{v}_q d(x_q d x_p)
$$

From Eq. (65) one can see that the effective stress dispersion

$$
\Delta \bar{\sigma}_{klmn}^{(i)2} \equiv \langle \bar{\sigma}_{kl} \bar{\sigma}_{mn} \rangle_i - \langle \bar{\sigma}_{kl} \rangle_i \langle \bar{\sigma}_{mn} \rangle_i
$$

presents a determined homogeneous function within the components $v_i, \quad (i = 1, \ldots, N)$. At the same time the stress dispersion

$$
\Delta \sigma^{(i)2}(x) \equiv \langle \sigma \otimes \sigma \rangle_i(x) - \langle \sigma \rangle_i(x) \otimes \langle \sigma \rangle_i(x)
$$

is an inhomogeneous function of the coordinates in the components $v^{(i)}$ which is connected with the effective stress dispersion (66) by

$$
\Delta \sigma^{(i)2}(x) = \mathbf{B}_i(x) \Delta \bar{\sigma}_{klmn}^{(i)2} \mathbf{B}_i^T(x).
$$

The formula (68) allows the estimation of the averaged stress dispersion in the inclusion volume

$$
\Delta \sigma^{(i)2} = \bar{v}_i^{-1} \int \Delta \sigma^{(i)2}(x) V_i(x) d x.
$$

By the use of the known properties of the second statistical moments, (68) and (65), a relation for the average second moment of stresses in the component $v_i$ is achieved

$$
\langle \sigma \otimes \bar{\sigma} \rangle_i(x) = \langle \sigma \rangle_i(x) \otimes \langle \sigma \rangle_i(x)
$$

$$
+ \int \left\{ \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_p) \eta_p \bar{v}_p \right] \otimes \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_p) \eta_q \bar{v}_q \right] \right\} v_p, x_p; v_i, x_i
$$

$$
\cdot \bar{v}_p^2 \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
+ \int \left\{ \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_p) \eta_p \bar{v}_p \right] \otimes \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_q) \eta_q \bar{v}_q \right] \right\} v_p, x_p; v_q, x_q; v_i, x_i
$$

$$
\cdot \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
- \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_p) \eta_p \right] \otimes \left[ \mathbf{B}_i(x) \mathbf{T}_{ip}(x_i - x_q) \eta_q \right] v_p, x_p; v_q, x_q; v_i, x_i
$$

$$
\cdot \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i)
$$

$$
= \int \varphi(v_p, x_p; v_i, x_i) \varphi(v_q, x_q; v_i, x_i) \bar{v}_p \bar{v}_q d(x_q d x_p)
$$
Relation (70) is derived by the use of triple interaction of the inclusions. As may be seen from Eq.(70), the neglect of binary interaction is tantamount to assuming homogeneity of the stresses within the component \( v_{(i)} \)

\[
\langle \sigma \otimes \sigma \rangle_i(x) = \langle \sigma \rangle_i(x) \otimes \langle \sigma \rangle_i(x).
\]  

(71)

The following approximation for the second moment can be obtained by taking only binary interaction of the inclusions into account

\[
\langle \sigma \otimes \sigma \rangle_i(x) = \langle \sigma \rangle_i(x) \otimes \langle \sigma \rangle_i(x) + \int \left[ \langle B_i(x)T_{ip}(x_i - x_p)\eta_p\bar{\nu}_p \rangle \otimes \langle B_i(x)T_{ip}(x_i - x_p)\eta_p\bar{\nu}_p \rangle \right] v_p, x_p; v_i, x_i \mid v_i, x_i \right] d\mathbf{x}_p.
\]  

(72)

Ignoring effective stress fluctuations in the inclusions \( v_p \) in the integral term (72) leads to

\[
\langle \sigma \otimes \sigma \rangle_i(x) = \langle \sigma \rangle_i(x) \otimes \langle \sigma \rangle_i(x) + \int \left[ \langle B_i(x)T_{ip}(x_i - x_p)\eta_p\bar{\nu}_p \rangle \otimes \langle B_i(x)T_{ip}(x_i - x_p)\eta_p\bar{\nu}_p \rangle \right] \cdot \varphi(v_p, x_p; v_i, x_i) d\mathbf{x}_p.
\]  

(73)

The relation (73) is known as ‘‘far field approximation’’.

In a similar manner, it is possible to obtain the estimation of the second stress moment averaged over a volume of the matrix. First we define the conditional probability density \( \varphi(v_p, x_p; v_q, x_q; \ldots; v_0, x_0) \) under the condition that the inclusions \( v_q, \ldots \) are located at points \( x_q, \ldots \), whereas the matrix material appears at point \( x_0 \). In analogy to the above considerations we obtain

\[
\langle \sigma \sigma \rangle_0 = \langle \sigma \rangle_0 \otimes \langle \sigma \rangle_0 + \int \left[ \langle T_p(x_0 - x_p)\eta_p\bar{\nu}_p \rangle \otimes \langle T_p(x_0 - x_p)\eta_p\bar{\nu}_p \rangle \right] v_p, x_p; v_0, x_0
\]

\[
\cdot \varphi(v_p, x_p; v_0, x_0) d\mathbf{x}_p
\]

(74)

\[
+ \int \left[ \langle T_p(x_0 - x_p)\eta_p \rangle \otimes \langle T_q(x_0 - x_q)\eta_q \rangle \right] v_p, x_p; v_q, x_q; v_0, x_0
\]

\[
\cdot \varphi(v_p, x_p; v_0, x_0) \varphi(v_q, x_q; v_0, x_0) d\mathbf{x}_p
\]

\[
- \langle T_p(x_0 - x_p)\eta_p \rangle \langle T_p(x_0 - x_q)\eta_q \rangle \varphi(v_q, x_q; v_0, x_0) d\mathbf{x}_p
\]

\[
\otimes \left[ \langle T_q(x_0 - x_q)\eta_q \rangle \varphi(v_q, x_q; v_0, x_0) \right] \bar{\nu}_p d\mathbf{x}_p
d\mathbf{x}_p.
\]

In contrast to (70) the second stress moment in the matrix, (74), does not depend on \( x_0 \in v_0 \). A more approximative estimation of the second moment \( \langle \sigma \otimes \sigma \rangle_0 \) can be obtained in direct analogy to (72) and (73) by replacing \( v_i, x_i \) by \( v_0, x_0 \).

5. Estimation of second stress moments within the components

A general logic pattern for the calculation of \( n \)-point elastic stress moments \( \langle \sigma(x_1) \otimes \cdots \otimes \sigma(x_n) \rangle \mid v_1, x_1; \ldots; v_n, x_n; v_{n+1}, x_{n+1}; \ldots; v_m, x_m \) has been obtained by Buryachenko (1987) via the MEFM.
Thermoelastic stress fluctuations in composites

The closing of the infinite system of integral equations in the two-particle approximation was performed with the help of the following assumptions

\[ \langle \tilde{\sigma}(x_{i,j}) \rangle = \langle \tilde{\sigma}(x_{i}) \rangle \otimes \langle \tilde{\sigma}(x_{j}) \rangle, \quad (i = 1, 2; \quad j = 3 - i). \]

(75)

\[ \langle \tilde{\sigma}(x_{i,j}) \rangle = \langle \tilde{\sigma} \rangle \otimes \langle \tilde{\sigma} \rangle, \quad (i = 1, 2; \quad j = 3 - i). \]

(76)

The assumption of no correlation between the values of the field \( \tilde{\sigma}(x_{i,j}) \) at different points (75) does not mean a lack of correlation between stresses \( \tilde{\sigma}(x_{i}) \) and \( \tilde{\sigma}(x_{j}) \). The average inclusion stresses \( \langle \tilde{\sigma} \rangle \) may be obtained by using formula (30). For the calculation of a second one-point effective stress moment \( \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle \) Buryachenko (1987) applied a cumbersome approximation method. For this purpose we now use the approach based on the approximative integral equation (65) as well as on the effective field approximation (22) of the binary interaction of the inclusions.

For two fixed inhomogeneities \( v_i, v_j \) we have, according to formulae (24) and (32),

\[ \tilde{\sigma}(x_i) = R_i^{-1} \left\{ \sum_{j=1}^{2} Z_{ij} (R_j \tilde{\sigma}(x_{j,1,2}) + F_j) - F_i \right\}. \]

(77)

Here \( \tilde{\sigma}(x_{i}) \) and \( \tilde{\sigma}(x_{j,1,2}) \) are homogeneous random fields in the inclusions \( v_i, v_j \). The statistical moment of the effective stresses inside the inhomogeneity \( v_i \) is obtained by taking the tensor product of (77)

\[ \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle = \langle \left\{ R_i^{-1} \left\{ \sum_{j=1}^{2} Z_{ij} (R_j \tilde{\sigma}(x_{j,1,2}) + F_j) - F_i \right\} \right\} \otimes \left\{ R_j^{-1} \left\{ \sum_{j=1}^{2} Z_{ij} (R_j \tilde{\sigma}(x_{j,1,2}) + F_j) - F_i \right\} \right\} \}

(78)

with \( (i, j = 1, 2) \). According to (22), (75) and (76) we use the following assumption to modify (65)

\[ \langle \eta_i \otimes \eta_j \rangle = \langle \eta_i \rangle \otimes \langle \eta_j \rangle, \quad i \neq j, \]

(79)

\[ \langle \eta_i \otimes \eta_i \rangle = \left\{ \sum_{j=1}^{2} \gamma_j \right\} \otimes \left\{ \sum_{j=1}^{2} \gamma_j \right\}, \]

(80)

where

\[ S_{ij} = Z_{ij} R_j, \quad \gamma_i = \sum_{j=1}^{2} Z_{ij} F_j. \]

(81)

Equations (79) and (80) are exact for the limiting case \( |x - y| \gg \max a_m \), where \( x \in v_i, \ y \in v_j \ (k = 1, 2, 3; \ m = i, j) \).

Then from (65) with (79) and (80) we find

\[ \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle = \langle \tilde{\sigma} \rangle \otimes \langle \tilde{\sigma} \rangle + \sum_{p=1}^{N} \left\{ H_{ip} \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle + H_{ip}^2 \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle + H_{ip}^3 \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle + H_{ip}^4 \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle + H_{ip}^5 \langle \tilde{\sigma} \otimes \tilde{\sigma} \rangle \right\}. \]

(82)
where the tensors $H^k_{i_p}$, $P^l_{i_p}$ ($k = 1, 2, 3, 4; l = 1, 2, 3$) depend on the binary probability density $\varphi(v_p, x_p; v_i, x_i)$ and are represented in Appendix A. The following tensor product notation has been applied

$$[(B_1 \otimes B_2)(b_1 \otimes b_2)]_{pqr} = B_{1pqkl}b_{1kl}B_{2rsmn}b_{2mn},$$
$$[(B_1 \otimes b_1)b_2]_{pqr} = B_{1pqkl}b_{2kl}b_{1rs},$$
$$[(b_1 \otimes B_1)b_2]_{pqr} = b_{1pq}B_{2rsmn}b_{2mn},$$

(83)

where $B_1$, $B_2$ and $b_1$, $b_2$ stand for any fourth and second order-tensor, respectively. Finally, from (82) we obtain the relation for second stress moments in the inclusions

$$\langle \sigma \otimes \sigma \rangle_i = \sum_{j=1}^{N} X_{ij} \sum_{p=1}^{N} \{ (\delta_{pj} + H^3_{jp}) \langle \sigma \rangle_p \otimes \langle \sigma \rangle_j$$
$$+ H^4_{jp} \langle \sigma \rangle_j \otimes \langle \sigma \rangle_p + P^1_{jp} \langle \sigma \rangle_p + P^2_{jp} \langle \sigma \rangle_j + P^3_{jp} \},$$

(84)

where the inverse matrix $X^{-1}$ of $X$ has the following elements

$$(X^{-1})_{ij} = \delta_{ij} \left\{ I \otimes I - \sum_{q=1}^{N} H^1_{iq} \right\} - H^2_{ij}, \quad (i, j = 1, \ldots, N),$$

(85)

On the right-hand side of Eq. (84) the values $\langle \sigma \rangle_i$ ($i = 1, \ldots, N$) are obtained from (40). An estimation of the stress dispersion (67) can be obtained from Eq. (68) by the use of Eq. (84). A simplification of Eq. (84) can be achieved by accepting the quasi-crystalline approximation (49), when

$$H^1_{ij}, H^3_{ij}, H^4_{ij}, P^2_{ij} \equiv 0.$$  

(86)

An estimation of the second stress moment averaged over the matrix can be found in analogy to the derivation of (68) and (84). If we restrict our consideration to binary interaction of the inclusions and use the assumption $\langle \sigma \otimes \sigma | v_i, x_i ; v_0, x_0 \rangle_i \equiv \langle \sigma \otimes \sigma \rangle_i$, we obtain

$$\langle \sigma \otimes \sigma \rangle_0 = \langle \sigma \rangle_0 \langle \sigma \rangle_0 + \sum_{p=1}^{N} \{ H^1_{p} \langle \sigma \otimes \sigma \rangle_p + P^1_{p} \langle \sigma \rangle_p + P^3_{p} \},$$

(87)

where the tensors $H^1_{p}$, $P^1_{p}$, $P^3_{p}$ are represented in Appendix A. The quantities $\langle \sigma \otimes \sigma \rangle_p$ and $\langle \sigma \rangle_p$ can be estimated by the use of Eqs. (84) and (37), respectively.

For homogeneous inclusions (28) without any transformation field $\beta \equiv 0$ the relations (84) and (87) reduce to the results found in a more cumbersome way by Buryachenko and Rammerstorfer (1997). A general scheme of an application of the formulae (84) and (87) under the assumption (28) was considered by Buryachenko and Rammerstorfer (1995) and applied to the analysis of elastoplastic deformations of two-phase composites.

It should be mentioned that in a number of practical cases some simplified methods (such as the Mori–Tanaka–Eshelby approaches (58)–(59)) lead to reasonably accurate estimations of the effective parameters $M^*$, $\beta^*$, $U^*$ (Buryachenko, 1996). However, in the case of estimating central stress moments inside the inclusions (14), (58) and (67), one obtains the trivial result $\Delta \sigma^{(1)}(x) \equiv 0, x \in v^{(1)} (N = 1)$. 

6. Application to the prediction of effective limiting properties

For strength predictions of composite tensor-polynomial strength criteria for each component (Buryachenko and Parton, 1992a; Buryachenko, 1993)

$$\Pi^{(k)}(\sigma) \equiv \Pi_{ij}^{(2(k))} \sigma_{ij} + \Pi_{ijmn}^{(4(k))} \sigma_{ij} \sigma_{mn} = 1, \quad (k = 0, 1, \ldots, N) \quad (88)$$

are well known. Here $\Pi^{(2(k))}$ and $\Pi^{(4(k))}$ are the second- and fourth-rank tensors of strength. Here the second- and fourth-rank tensors of strength $\Pi^{(2(k))}$, $\Pi^{(4(k))}$ can be expressed via technical strength parameters for different classes of symmetry (see e.g. Zhiging and Tennysin 1989; Theocaris 1991). Similarly to (88) the quadratic yield condition $\Pi^{(k)}(\sigma) \equiv \Pi_{ijmn}^{(4(k))} \sigma_{ij} \sigma_{mn} = 1$, $\Pi^{(2(k))} \equiv 0$ ($k = 0, 1, \ldots, N$) is widely used (Bergander, 1995) and generalizes some classical criteria, for example the von Mises criterion

$$ss = \frac{2}{3} r_k^2, \quad \text{(89)}$$

for which we have $\Pi^{(k)} = 3N_2/(2r_k^2)$, where $r_k$ stands for the onset of yielding of the component $v^{(k)}$ under uniaxial tension.

These limiting surfaces $\Pi^{(k)}(\sigma) = 1$ (88) are nonlinear functions of the stresses. Hence averaged stresses in the arguments do not necessarily lead the average limiting surface derived from more exact stress representations. For example, the effective limiting surface of the composite (Reifsnider and Gao, 1991; Dvorak, 1993) is often evaluated by

$$\Pi^{*}(\sigma) \equiv \max_k \left\{ \Pi_{ij}^{(2(k))} \langle \sigma \rangle_k + \Pi_{ijmn}^{(4(k))} \langle \sigma \rangle_k \otimes \langle \sigma \rangle_k \right\} = 1, \quad (90)$$

($k = 0, 1, \ldots, N$). This procedure is, however, in some sense inconsistent (see for details Buryachenko and Rammerstorfer, 1997).

The above-mentioned inconsistency can be avoided if averaging is performed according to Eqs (14), (84) and (87) (e.g. Buryachenko and Lipanov, 1989; Buryachenko, 1993)

$$\Pi^{*}(\sigma) \equiv \max_k \left\{ \Pi_{ij}^{(2(k))} \langle \sigma \rangle_k + \Pi_{ijmn}^{(4(k))} \langle \sigma \otimes \sigma \rangle_k \right\} = 1, \quad (91)$$

($k = 0, 1, \ldots, N$) for predicting the effective limiting surfaces. Equation (91) determines the second-order surface $\Pi^{*}(\sigma) = 1$ for onset of yielding within the six-dimensional stress space which, in the general case, depends on $\langle \sigma \rangle_0$.

Other methods of determining limiting surfaces (for example such as variational methods), are not discussed in this paper (references can be found in the reviews by Buryachenko, 1996; Ponte Castañeda, 1997; Suquet, 1997).

7. Simple probability model of composite strength

The proposed strength criteria (91) are based on the determination of the conditional average $\langle \Pi^{(k)}(\sigma) \rangle_k$, ($k = 0, 1, \ldots, N$). In fracture mechanics for random loading problems other approaches are known. Usually a further approach assumes that the damage density is sufficiently small so that the interaction of fractured elements can be neglected. For simplicity the stress distribution within each component is assumed to be a six-dimensional Gaussian one with a distribution function $\Phi_{\sigma}^{(k)}(\sigma)$, $\sigma = (\sigma_{11}, \ldots, \sigma_{12})^T$. Then a first-order estimation of the
fraction of fractured components (or fracture probability of the components) can be defined (Bolotin, 1993) by the relation

\[ f^{(k)} = 1 - f \Phi^{(k)}(\sigma), \quad (k = 0, 1, \ldots, N) \]  

(92)

where the integral domains are determined by the inequalities

\[ \langle \Pi^{(k)}(\sigma) \rangle_{\xi} \leq 1, \quad (k = 0, 1, \ldots, N) \]  

(93)

The boundary of the domain (93) is in simple cases of quadratic strength criteria (89) described by a surface of second order.

Very simple phenomenological ways of fracture probability calculation for composites (Sobczyk and Spencer, 1991) are based either on the total probability formula

\[ f = \sum_{k=0}^{N} c^{(k)} f^{(k)}, \]  

(94)

or on the extreme value distribution

\[ f = 1 - \prod_{k=0}^{N} (1 - f^{(k)}). \]  

(95)

Ortiz and Molinari (1988) considered a particular case of the above-described scheme with an application to random structure composites. By Fourier’s method they obtained an estimation for the second stress moment averaged over the whole volume of composites with elastically homogeneous properties, with, however statistically isotropic stress-free strain fluctuations. In such a case there are known exact representations for average stresses within the components, stored energy (20), and average stress fluctuations \( \langle \delta \sigma \otimes \delta \sigma \rangle \) (e.g Kreher and Pompe, 1989; Buryachenko and Rammerstorfer, 1996a). Since the residual stresses are self–equilibrated we have \( \delta \sigma \equiv \sigma - \langle \sigma \rangle = \sigma \). On this basis Ortiz and Molinari (1988) and Ma and Clarke (1994) assumed that the residual stresses are normally distributed at each point of the composite, and therefore the probability density function of \( \delta \sigma \) is entirely defined by the covariance matrix

\[ \mathbf{K}^{\sigma} = \langle \delta \sigma \otimes \delta \sigma \rangle \]  

(96)

and takes the form

\[ \varphi^{\sigma}(\sigma) = \frac{1}{(2\pi)^{3/2} \sqrt{\det(\mathbf{K}^{\sigma})}} \exp \left\{ -\frac{1}{2} (\delta \sigma)^{T} (\mathbf{K}^{\sigma})^{-1} \delta \sigma \right\}. \]  

(97)

where \( \det(\mathbf{K}^{\sigma}) \) is the determinant of the matrix \( \mathbf{K}^{\sigma} \) and \( d\Phi^{\sigma}(\sigma) \equiv \varphi^{\sigma}(\sigma) d\sigma \). The indicated approximation of the real process \( \delta \sigma \) by Gaussian distributions with zero average stress \( \langle \sigma \rangle = 0 \) can lead to overly crude estimations if conditional average stresses \( \langle \sigma \rangle_{i} \neq 0 \) \( (i = 0, \ldots, N) \). However, these average stresses can be estimated by using the exact formula. Then the definition of the conditional covariance matrix and the probability density function by

\[ \mathbf{K}^{\sigma(i)} = \Delta^{(i)} \]  

\[ \varphi^{(i)}(\sigma) = \frac{1}{(2\pi)^{3/2} \sqrt{\det(\mathbf{K}^{\sigma(i)})}} \exp \left\{ -\frac{1}{2} (\sigma - \langle \sigma \rangle_{i})^{T} (\mathbf{K}^{\sigma(i)})^{-1} (\sigma - \langle \sigma \rangle_{i}) \right\}, \]  

(98)
seem to be more correct. According to previously obtained estimations by Buryachenko and Kreher (1995), the random residual stress fluctuations within the components are not in excess of 10% of the average values \(|\langle \sigma_{11} \rangle_i|\). Thus, the probability that the stresses within the component \(\sigma^{(i)}\) exceed the value 1.3\(|\langle \sigma_{11} \rangle_i|\) is much smaller than predicted by using the formulae (96) and (97).

8. Numerical results

Let us consider as an example a composite consisting of isotropic homogeneous components and having identical spherical inclusions with yield properties according to (89). First we will use the integral equation method for the estimation of the second moments (84) and (87), accompanied by the determination of the yield surface (89) and (91). According to (84) and (87) all tensors used for the calculation of second stress moments (such as e.g. \(H^j_{ip}, P^l_{ip}, j = 1, \ldots, 4, l = 1, 2, 3, i, p = 1, \ldots, N\)) are isotropic ones. Then the yield surface (91) has ellipsoid form in the space of nondimensional coordinates

\[
X_{r1} = \frac{\langle \sigma \rangle_i \tau_i}{\bar{\tau}_i}, \quad Y_{r1} = \frac{\langle \sigma_0 \rangle_i \tau_i}{\bar{\tau}_i}, \quad Z_{r1} = \frac{\sigma_{11}^{hyd} \beta_{10}^{(i)}}{\bar{\tau}_i},
\]

for \(i = 0, 1, \ldots, N\) and, therefore, this yield surface is described by

\[
\frac{X^2}{a^2_r} + \frac{Y^2}{b^2_r} + \frac{Z^2}{c^2_r} - \frac{Y_r Z_r}{d^2_r} = 1,
\]

for the coordinates

\[
X_r = \frac{\sqrt{s^0 s^0}}{\bar{\tau}_i}, \quad Y_r = \frac{\sigma_0^0}{\bar{\tau}_i}, \quad Z_r = \frac{\sigma_{11}^{hyd}}{\bar{\tau}_i},
\]

because, according to Eqs (24) and (37), there is a linear dependence between the effective stresses \(\langle \sigma \rangle\), and the external loading \(\sigma_0, \beta_1^{(i)}\). Here the nondimensional coefficients \(a_r, b_r, c_r, d_r\) are expressed by means of the tensors \(H_{i10}, P_{i10}, P_{i0}, H_{ip}, P_{ip}, (j = 1, \ldots, 4; l = 1, 2, 3; i, p = 1, \ldots, N)\) and the quantity \(\sigma_{11}^{hyd} = -3B_i^k 3Q_{i10}^k \beta_{10}^{(i)}\) has the simple physical meaning of the hydrostatic component of the residual stresses within the single isolated inclusion in an infinite homogeneous matrix: \(\beta_{10}^{(i)} = \beta_{11}^{(i)} \delta / 3, B_i = (3B_i^k, 2B_i^p), Q_i = (3Q_{i10}^k, 2Q_{i10}^p)\). The half-ellipsoid surface (100) (with \(X_{r1} \geq 0\)) can be transformed into the canonical form by a rotation in the three-dimensional space \(\{X_r, Y_r, Z_r\}\) around the axis \(X_r\). It is interesting that, according to Eqs (72) and (74), the stress fluctuation \(\Delta \sigma^{(i)}_{12}\) (69) is identically zero for \(\eta_i \equiv 0\). The last equation defines the straight line \(Y_r = c_r Z_r, X_r = 0 (c_r = const.)\) in the coordinates (101). Therefore, in reality the ellipsoid surface is degenerated into an elliptical cylinder with a symmetry axis coinciding with the indicated straight line and \(d^2_r = a_r c_r \tau_r^2 / 2\). By consideration of Eq. (40) one can find that this symmetry axis does not depend on the concentration of the inclusions.

It should be mentioned that the widely-used yield surface described by (89) and (90)

\[
\frac{X^2}{a^2_r} = 1
\]

reduces to a plane parallel to the coordinate plane \(X_r \equiv 0\) with \(a_r c_r > a_r\). Therefore, the hydrostatic component due to the external loading \(\sigma_0^0\) as well as an eigenstrain \(\beta_1^{(i)}\) do not influence the composite yielding in traditional calculation methods.
Analogously we consider the exact relation for the calculation of the second invariant of stress deviator (17). Then the proposed yield surface (91) of the component \( v^{(i)}, (i = 0, 1, \ldots, N) \) is defined by the equation

\[
\frac{\partial \mathbf{M}^*}{\partial q^{(i)}} \sigma^0 \otimes \sigma^0 - 2 \frac{\partial \mathbf{U}^*}{\partial q^{(i)}} + 2 \frac{\partial \mathbf{\beta}^*}{\partial q^{(i)}} \sigma^0 = \frac{4}{3} c^{(i)} r_i^2.
\]

(103)

However, according to (42)-(44) the tensors used for the calculation of the effective parameters \( \mathbf{M}^*, \mathbf{U}^*, \mathbf{\beta}^* \) are isotropic and constant. Therefore, the yield surface has the form of an elliptical cylinder (8.2) with dimensionless coefficients \( a_{r1}, b_{r1}, c_{r1}, d_{r1} \) which are derived from the tensors \( \mathbf{Y}_{ij}, \mathbf{B}_i, \mathbf{R}_j \) \((i, j = 1, \ldots, N)\), defined by the solution of the purely elastic problem \( (\beta \equiv 0) \) by means of particular derivatives with respect to \( q^{(i)} \).

It should be mentioned that yield surfaces in the form of elliptical cylinders (100) are obtained if the average of the second stress moment in the component (91) is used. Buryachenko (1996) has shown that using the average second stress moment in the matrix in the vicinity of the inclusion leads to a new yield surface which is inserted in the old one.

The yield surface at \( \beta \equiv 0 \) (purely elastic loading) was studied comprehensively by Buryachenko (1996) and by Buryachenko and Rammerstorfer (1997). This case corresponds to the cross-section of the elliptical yield cylinder in the plane \( Z_r = 0 \). Because of this, in the following we consider only the other cross-section \( X_r = 0 \) of the yield surface.

To obtain concrete numerical results, we choose the Poisson’s ratio of the matrix \( \nu = 0.3 \), and the concentration of rigid identical spherical inclusions of the radius \( a \) is denoted by \( c^{(1)} \); \( \tau_0 = \tau_1 \). The conditional probability densities are the step functions

\[
\varphi(v_p, x_p; v_i, x_i) = H(|x_p - x_i| - 2a) n^{(1)},
\]

\[
\varphi(v_p, x_p; v_0, x_0) = H(|x_p - x_0| - a) n^{(1)},
\]

(104)

where \( H \) denotes the Heaviside function. Figure 2 gives the corresponding cross-sections \( \mathbf{X}_r = 0 \) of the yield surface of the matrix calculated by either the integral equation method (87), (100) or by the exact relation (103) for \( c^{(1)} = 0.2 \) as well as \( c^{(1)} = 0.4 \). One can see that methods (87), (100) and (103) lead to very similar results. The yield surface (103) has the form of an elliptical cylinder (100) with dimensionless coefficients \( a_{r1}, b_{r1}, c_{r1}, d_{r1} \). The surface (103) is inserted inside the other (87), (100): \( a_{r1} < a_r, b_{r1} < b_r, c_{r1} < c_r \). Obviously, for \( c^{(1)} \to 0 \) we have \( a_{r1}^2 \to a_r^2 \to 3/2, b_{r1}^2, b_r^2 \to 1/c^{(1)}, c_{r1}^2, c_r^2 \to 1/c^{(1)} \), and the yield surface converges to a plane.

As an example we consider a composite with the same elastic properties of the components; however, we now let the spherical coated inclusions \( v_i \) \((i = 1, 2, \ldots)\) of the radius \( a \) have a thin coating \( \nu_c \subset v_i \) of the thickness \( h \) \((h \ll 1)\) with elastic properties of the matrix and homogeneous mismatch properties:

\[
\mathbf{M}_1(x) = 0, \quad \mathbf{\beta}_1(x) = \mathbf{\beta}^{(i)}_1 = \beta^{(i)}_1 \delta \quad \text{at} \ x \in \nu_c,
\]

and the spherical cores \( v_i \equiv v_i \setminus \nu_c \) have thermoelastic parameters \( \mathbf{M}_1(x) = \mathbf{M}^{(i)}_1, \mathbf{\beta}_1(x) \equiv 0 \) \((x \in v_i)\). The solution for a single-coated ellipsoid inclusion in an infinite matrix is shown in Appendix B (more details are discussed by Buryachenko and Rammerstorfer, 1996b)

\[
\mathbf{B}(x) = \mathbf{B}^{(i)} \equiv (\mathbf{I} + \mathbf{Q}^{(i)} \mathbf{M}^{(i)}_1)^{-1}, \quad \mathbf{C}(x) = 0
\]

(105)

for the core \( x \in v_i \), and

\[
\mathbf{B}(x) = \mathbf{I} + \mathbf{v}^{(i)} \mathbf{T}^{(i)}(x_i - x) \mathbf{M}^{(i)}_1 \mathbf{B}^{(i)}, \quad \mathbf{C}(x) = -[\mathbf{Q}^{(i)} + \mathbf{v}^{(i)} \mathbf{T}^{(i)}(x_i - x)] \mathbf{\beta}_1^{(i)}
\]

(106)
for the coating $x \in v'$. Here the upper index $i$ for the tensors $Q^i$, $B^i$ and $T^i(x - x)$ stands on the calculation of these tensors by the use of the formulae (27), (29) and (23), respectively, by replacing $v_i$ by $v^i$. Then the averaged concentration tensors of the coated inclusion (24) can be obtained by the use of the analogous tensors for homogeneous inclusions

$$B_i = I + \frac{\bar{v}^i}{v_i}(B^i - I), \quad C_i = -\frac{\bar{v}^i}{v_i}Q^i\beta^i, \quad R_i = R^i = \bar{v}^i M^{(i)\ast}_i B^i, \quad F_i = \bar{v}^i \beta^i.$$  

(107)

After that we can find the dispersion of the effective field $\Delta \sigma^2_i$ (84) as well as an inhomogeneous stress dispersion $\Delta \sigma^2_i(x) = B(x) \Delta \sigma^2_i B^T(x)$ inside the coated inclusion $x \in v_i$; $\sigma^0_{ij} \equiv \sigma^0_{33} \delta_{ij} \delta_{33}$. Let us define the normalized fluctuation of the effective von Mises stress $\tau^{Mises}(x) \equiv |1.5 N_2 : \Delta \sigma^{(i)2}(x)|^{1/2} / |\sigma^0_{33}|$. Figure 3 shows the values $\tau^{Mises}(x)$ in a spherical coordinate system $x = r (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)^T$ as functions of the polar angle $\theta$ at $\varphi \equiv 0$ in the core $r < a(1 - h)$ (line 1) as well as in the coating for $r = a(1 - h)$ (line 2) and $r = a$ (line 3). A relative thickness $h = 0.1$ and a concentration of inclusions $c \equiv c(1) = \langle V^{(1)} \rangle = 0.4$ were assumed. In figure 3 two cases of loading, $\sigma^0_{33} \equiv 0$, $\beta^0_{10} \equiv 0$ (dashed lines) and $\sigma^0_{33} \neq 0$, $\beta^0_{10} = \sigma^0_{33} / (3h^3 Q_i)$ (solid lines), are represented. We see that $\tau^{Mises}(x)$ is a homogeneous function in the core $v'$, but that it is inhomogeneous in the coating $v^c$. At the same time for purely thermal loading, $\sigma^0 \equiv 0$, $\beta^0_{10} \neq 0$, the effective von Mises stress in the core $v'$, calculated by the use of statistical average stresses $\langle \sigma \rangle_v(x)$ (39), is identically zero.
9. Conclusion

Stress fluctuations in the components of random structure composites represent a measure of inhomogeneity of stress fields in the components. The fundamental roles of such inhomogeneities described by the stress fluctuations are discussed in detail by Buryachenko (1996), Ponte Castañeda (1997), and Suquet (1997) for a wide range of nonlinear problems in micromechanics. The principal advantages of the proposed method of integral representations for stress fluctuations (68), (84) and (87) in comparison with the perturbation method (14) and some others ones were shown by Buryachenko and Rammerstorfer (1997) for the purely isothermal elastic case ($\beta \equiv 0$) in a composite with uncoated inclusions. For the thermoelastic case, additional advantages and other interesting aspects when employing the proposed method should be mentioned. For elastoplastic analyses based on estimations of some nonlinear functions of local stresses, e.g. the yield condition, taking stress inhomogeneities in the components into account, it is common to use secant and tangent moduli concepts (see the references in the indicated papers). This way the nonlinear problem at each solution increment reduces to the averaging linear elastic problem with $\beta \equiv 0$. The use of the secant modulus concept creates the known complications, since generally the local stress state is not monotonic and proportional even with monotonic and proportional external loading. The tangent modulus concept, necessitates also consideration of the matrix material as being anisotropic at each solution step. This would not lead to any problem in the framework of the ‘quasi-crystalline’ approximation (49), but it leads to some computing difficulties at the realization of the MEFM for which advantages in comparison with some popular methods have been justified (Buryachenko, 1996; Buryachenko and Rammerstorfer, 1997). However, the integral representations for stress fluctuations (68), (84) and (87) permit the use of the incremental method with fixed elastic properties of the components and with accumulating plastic strains ($\beta^{(i)} \neq \text{const.}$). This method has been presented by Buryachenko and Rammerstorfer (1996a) for elastically homogeneous materials with a thermal mismatch of the components. The analysis of the
Thermoelastic stress fluctuations in composites

elastic mismatch will be pursued in a forthcoming study. Another improvement is connected with abandonment of the assumption of homogeneity of plastic strains in the matrix. In so doing the concentration of plastic strains in the vicinity of inclusions plays the role of a ‘coating’ exhibiting an inhomogeneous transformation field along the inclusion surfaces. This model was developed by Buryachenko et al. (1997) in the framework of the mean field method of Dvorak (1993), and can be generalized with regard to stress fluctuations in the components, as found in Eqs (68), (84) and (87).

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Appendix A: Notations

The notation in Eq. (82)

\[
H_{ip}^1 = \int [T_{ip}(x_i - x_p)S_{pi} \otimes T_{ip}(x_i - x_p)S_{pi}] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

\[
H_{ip}^2 = \int [T_{ip}(x_i - x_p)S_{pp} \otimes T_{ip}(x_i - x_p)S_{pp}] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

\[
H_{ip}^3 = \int [T_{ip}(x_i - x_p)S_{pp} \otimes T_{ip}(x_i - x_p)S_{pi}] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

\[
P_{ip}^1 = \int [T_{ip}(x_i - x_p)S_{pp} \otimes T_{ip}(x_i - x_p)\gamma_p
\]

\[+ T_{ip}(x_i - x_p)\gamma_p \otimes T_{ip}(x_i - x_p)S_{pp}] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

\[
P_{ip}^2 = \int [T_{ip}(x_i - x_p)S_{pi} \otimes T_{ip}(x_i - x_p)\gamma_p
\]

\[+ T_{ip}(x_i - x_p)\gamma_p \otimes T_{ip}(x_i - x_p)S_{pi}] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

\[
P_{ip}^3 = \int [T_{ip}(x_i - x_p)\gamma_p \otimes T_{ip}(x_i - x_p)\gamma_p] \phi(v_p, x_p; v_i, x_i) dx_p,
\]

where

\[S_{kl} = Z_{kl}R_{lk}, \quad \gamma_k = \sum_l Z_{kl}F_{lk}, \quad (k, l = i, p).
\]

The notations in Eq. (87):

\[
H_{ip}^{10} = \int [T_p(x_0 - x_p)R_p \otimes T_p(x_0 - x_p)R_p] \phi(v_p, x_p; v_0, x_0) dx_p,
\]

\[
P_{ip}^{10} = \int [T_p(x_0 - x_p)R_p \otimes T_p(x_0 - x_p)F_p
\]

\[+ T_p(x_0 - x_p)F_p \otimes T_p(x_0 - x_p)R_p] \phi(v_p, x_p; v_0, x_0) dx_p,
\]

\[
P_{ip}^{30} = \int [T_p(x_0 - x_p)F_p \otimes T_p(x_0 - x_p)F_p] \phi(v_p, x_p; v_0, x_0) dx_p.
\]
Appendix B: Single ellipsoid inclusion with thin coating

Let the single-coated inclusion \( v_1 \) with semiaxes \( a_j^c \) \((j = 1, 2, 3)\) consist of an ellipsoid core \( v^i \subset v_1 \) with semiaxes \( a_j^i \) \((j = 1, 2, 3)\) (same orientation) and a characteristic function \( V^i(x) \) and thermoelastic parameters \( \Gamma^i, \beta^i \equiv \text{const.} \), and a thin coating \( v^c \equiv v_1 \setminus v^i \) with a characteristic function \( V^c \equiv V_1 - V^i \) and thermoelastic inhomogeneous properties \( \Gamma^c(x), \beta^c(x) \neq \text{const.} \). In addition to (4), we define the jump in material properties \( \mathbf{f} = \mathbf{M}, \beta \) across the boundary \( s^i \) between the core and the coating as \( \mathbf{f}_2 \equiv \mathbf{f}^i - \mathbf{f}^c \).

We construct an approximate solution under the approximative assumption of a homogeneous stress state in the core, \( \sigma^i \), and at infinity, \( \sigma^0 \)

\[
\sigma(x) \equiv \sigma^i = \text{const.}, \quad x \in v^i, \quad \sigma^0 = \text{const.} \tag{B-1}
\]

and the use of the thin-layer hypothesis, which means that the characteristic function \( V^c(y) \) can be replaced by a surface \( \delta \)-function (with weighting function \( \rho \)) at the outer surface \( s^- \) of the boundary \( s = s^- \cup s^+ \). The weighting function \( \rho \) has the form

\[
\rho(s) = \left( \frac{s_1^2}{a_1^4} + \frac{s_2^2}{a_2^4} + \frac{s_3^2}{a_3^4} \right)^{-1/2} \sum_{j=1}^{3} \left( \frac{a_j^c - a_j}{a_j} \right) \frac{s_j^2}{a_j^c}, \tag{B-2}
\]

where \( s \equiv (s_1, s_2, s_3)^T \in s^- \).

Then the stresses in the coating, \( \sigma^c(s) \), as well as in the core, \( \sigma^i \), are found by the relations

\[
\sigma^c(s) = \sigma^0 + \left[ \mathbf{M}^i \sigma^i + \beta^i \right] - \frac{1}{v^i} \int_{S^i(s)} \left[ \mathbf{I} + \mathbf{Q} \mathbf{M}_1^c(s) \right]^{-1} \mathbf{Q} \mathbf{M}_1^c(s) \sigma^i + \beta^c_2(s) \] \[
= \sigma^0 + \left[ \frac{V^c}{v^i} \mathbf{Q} - \mathbf{Q}_1 \right] \mathbf{M}_1^i \sigma^i + \beta^i - \frac{1}{v^i} \int_{S^-} \mathbf{M}_1^c(s) \sigma^i + \beta^c_2(s) \rho(s) ds \tag{B-3}
\]

Here the characteristic function \( S^- \) of the boundary \( s^- \) indicates integration over the surface \( s^- \). The tensors \( \mathbf{Q}^i \) and \( \mathbf{Q}_1 \) for the ellipsoids \( v^i \) and \( v_1 \) are defined by the formula (3.6); for compliances \( \mathbf{M}^i \) and \( \mathbf{M}^c \), respectively; \( \mathbf{M}_2^i(s) \equiv \mathbf{M}^i - \mathbf{M}^c(s), \) \( \beta^c_2(s) \equiv \beta^i - \beta^c(s) \).

The interface operator \( \mathbf{I}(n, \mathbf{M}^c) \) is defined by

\[
\mathbf{I}(n, \mathbf{M}^c) = \mathbf{L}(n) - \mathbf{L}(n) \mathbf{U}(n) \mathbf{L}(n), \quad \mathbf{U}(n) = n \otimes \mathbf{G}(n) \otimes n, \quad \mathbf{G}(n) = \mathbf{L}(n) \mathbf{n}^{-1}, \quad \mathbf{L}(n) = \mathbf{L}(n) \mathbf{n} \otimes n, \quad \mathbf{L}(n) \equiv [\mathbf{M}(n)]^{-1}, \tag{B-4}
\]

applied with the compliance \( \mathbf{M}(n) \); \( n \) is the unit outward normal vector on \( s^- \) in the point \( s \). Estimations for the tensors \( \mathbf{B}(x), \mathbf{C}(x) \) (3.3) can be derived from Eq. (B-3). In this way the tensors \( \mathbf{B}(x), \mathbf{C}(x) \) are constant for \( x \in v^i \); however, \( \mathbf{B}(x), \mathbf{C}(x) \) are inhomogeneous functions of \( x \in v^c \).

A more specific case of Eq. (B-3) for elastically homogeneous coating was analyzed by Buryachenko and Rammerstorfer (1996b) \( (\mathbf{M}^c(x) = \text{const}) \) by the use of the generalization of the method proposed by Cherkaoui et al. (1995) for inclusions with homogeneous coating \( \mathbf{M}^c(x), \beta^c(x) = \text{const.} \).
Thermoeelastic stress fluctuations in composites

References


