A ‘warping’ theory of plate deformation

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Abstract — A theory of plate deformation is derived which accounts for the effects of transverse shear deformation, transverse normal strain and a non-linear distribution of the in-plane displacements with respect to the thickness coordinate. This theory uses the normal modes associated with the normal fibre (considered as a geometrical beam) as basic functions. Using only the rigid body modes, we find the classical theory and using the deformation normal modes, a high-order theory is constructed. Our theory is compared with lower-order plate theories, with higher-order theory and the exact solution through application to a particular problem involving a plate acted upon by a sinusoidal surface pressure. A circular plate submitted to in-plane forces is analysed. The present work involves homogeneous plates and can be extended to laminated plates. © Elsevier, Paris

high-order theory / plate deformation / ‘warping’ theory

1. Introduction

Sophisticated models compared to classical theories may be applied to problems where classical plate theory is simply inadequate to describe the behaviour in question. Examples of this are contact problems involving plates, laminated plates or high frequency analysis of plates. The present work concerns the derivation and evaluation of a particular ‘warping’ theory of plate behaviour.

Before describing the present theory, it is necessary to briefly review and comment on the ‘recent’ developments in the generalisation of classical plate theory. The studies of Reissner (1944, 1945), generalizing the classical plate theory, incorporate the effect of shear deformation. The derivation given by Reissner resulting in displacements of the form

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= u_1(x_1, x_2) + x_3\beta_1(x_1, x_2) \\
U_2(x_1, x_2, x_3) &= u_2(x_1, x_2) + x_3\beta_2(x_1, x_2) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2)
\end{align*}
\]

where \(x_1,x_2\) are the middle plane coordinates and \(x_3\) is the coordinate normal to the middle plane. Relation (1) predicts a uniform shear stress distribution through the thickness of the plate. It is then necessary to introduce a correction factor into the shear stress resultant, which is incorrect and in general would violate surface conditions.

The next higher-order theory involves displacement forms of the type

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= u_1(x_1, x_2) + x_3\beta_1(x_1, x_2) \\
U_2(x_1, x_2, x_3) &= u_2(x_1, x_2) + x_3\beta_2(x_1, x_2) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2) + x_3\beta_3(x_1, x_2) + (x_3)^2\zeta_3(x_1, x_2)
\end{align*}
\]

(2)
This theory includes the effect of transverse normal strain. Displacement assumption as in (2) has been used by Naghdi (1957) to derive a general shell theory and by Essenber (1975) to obtain the corresponding one-dimensional plate theory. They used a shear correction factor which is not appropriate for use with the displacement form of (2). This is because nonuniform shear stress is implied by (2) along with consequent possible satisfaction of upper and lower boundary conditions of shear tractions; thus the rationale for a correction factor is obviated.

Nelson and Lorch (1974) proposed the displacement forms

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= u_1(x_1, x_2) + x_3\beta_1(x_1, x_2) + (x_3)^2\zeta_1(x_1, x_2) \\
U_2(x_1, x_2, x_3) &= u_2(x_1, x_2) + x_3\beta_2(x_1, x_2) + (x_3)^2\zeta_2(x_1, x_2) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2) + x_3\beta_3(x_1, x_2) + (x_3)^2\zeta_3(x_1, x_2)
\end{align*}
\]

(3)

A shear correction factor was employed when it was inconsistent with the level of approximation in (3). Reissner (1975, 1985) has presented a theory which involves

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= x_3\beta_1(x_1, x_2) + (x_3)^3\phi_1(x_1, x_2) \\
U_2(x_1, x_2, x_3) &= x_3\beta_2(x_1, x_2) + (x_3)^3\phi_2(x_1, x_2) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2) + (x_3)^2\zeta_3(x_1, x_2)
\end{align*}
\]

(4)

Reissner (1975, 1985) has shown that the plate theory corresponding to (4) gives very accurate results compared with the elasticity solution for the bending of a plate with a circular hole. The theory regarding (4) neglects the contribution of in-plane modes of deformation; only out-of-plane effects are considered.

Lo et al. (1977) presented a high-order theory appropriate to the following displacement forms

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= u_1(x_1, x_2) + x_3\beta_1(x_1, x_2) + (x_3)^2\zeta_1(x_1, x_2) + (x_3)^3\phi_1(x_1, x_2) \\
U_2(x_1, x_2, x_3) &= u_2(x_1, x_2) + x_3\beta_2(x_1, x_2) + (x_3)^2\zeta_2(x_1, x_2) + (x_3)^3\phi_2(x_1, x_2) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2) + x_3\beta_3(x_1, x_2) + (x_3)^2\zeta_3(x_1, x_2)
\end{align*}
\]

(5)

2. Motivation: new proposition

Regarding upper and lower boundary conditions of shear traction, in the case of some loads (see Section 6) or in high dynamic analysis of plates, the classical theories except that defined by (5) do not give satisfaction concerning the nonuniform shear, the normal stress, and displacement distribution. Also, in the case of laminated plates, a high-order must be used because of the likely strongly nonlinear thickness distribution of stress and displacement. In Lo et al. (1977), it is seen that for plate bending problems where the loading characteristics possess a high degree of asymmetry with respect to the middle plane, a high-order theory of the type of (5) is required. The high-order theory defined by (5) cannot induce the same stress resultant for an in-plane equivalent loading. Lo et al.’s theory is an abridged version of Lévy’s theory, regarding which the author concluded that his solution was not of sufficient generality to allow satisfaction of three boundary conditions of flexural stress resultant.

Our theory is based on the nonuniform distribution of in-plane displacement: it will be called the ‘warping’ phenomenon. In the present high-order theory, the nonuniformity of in-plane plate displacement is considered by a linear combination of normal modes of the normal fibre to the mid-plane, regarded as a ‘geometrical structure’ (geometrical beam). When only the first six normal modes (rigid body modes) are considered, the Reissner-Mindlin lower-order theory is involved. In the present work, the transverse and longitudinal normal
deformation modes of the normal fibre are considered. This involves a high-order theory giving better results than that defined by (5).

3. Displacement field

Let \( \{ \phi_n \} \) and \( \{ \Phi_k \} \) denote respectively the \( n \)th transverse and \( k \)th longitudinal modes (see Appendix) inducing deformations in normal fibre which is considered as a geometrical beam. For each natural frequency associated with transverse normal modes, there are two corresponding transverse normal modes: one is in the \( x_1 \) direction and one is in the \( x_2 \) direction.

The present theory is appropriate for the following displacement forms

\[
\begin{align*}
U_1(x_1, x_2, x_3) &= u_1(x_1, x_2) + x_3 \beta_1(x_1, x_2) + n W_1^n(x_1, x_2) \phi_n(x_3) \\
U_2(x_1, x_2, x_3) &= u_2(x_1, x_2) + x_3 \beta_2(x_1, x_2) + n W_2^n(x_1, x_2) \phi_n(x_3) \\
U_3(x_1, x_2, x_3) &= u_3(x_1, x_2) + k W_3^k(x_1, x_2) \Phi_k(x_3)
\end{align*}
\]

(6)

The following notation will be used

\[ n W_1^n \phi_n \equiv W_1^n \phi_n; \quad \overline{u_\omega} = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right); \quad \overline{\beta} = \left( \begin{array}{c} \beta_1 \\ \beta_2 \end{array} \right); \quad \overline{W}^n = \left( \begin{array}{c} W_1^n \\ W_2^n \end{array} \right); \quad h : \text{ thickness of the plate} \]

(7)

This displacement field includes both in-plane and out-plane deformation modes. The vector \( \overline{W}^n \) and the function \( W_3^k \) represent the intensity of the participation of the deformation modes to the warping of normal fibre. The functions \( \phi_n \) and \( \Phi_k \) are called warping coordinates. Figure 1 shows the first two transverse and longitudinal modes. The number of modes used depends on the order of the theory needed.

![Figure 1](image-url)

Figure 1. a: The first two transverse modes; b: the first two longitudinal modes.
4. Deformation tensor–stress tensor: stress resultant

4.1. Deformation tensor

The deformation tensor associated with (7) is written as

\[
\begin{align*}
\varepsilon_{11} &= u_{1,1} + x_3 \beta_{1,1} + W_{1,1}^n \phi_n \\
\varepsilon_{22} &= u_{2,2} + x_3 \beta_{2,2} + W_{2,2}^n \phi_n \\
2\varepsilon_{12} &= u_{1,2} + u_{2,1} + x_3 (\beta_{1,2} + \beta_{2,1}) + (W_{1,2}^n + W_{2,1}^n) \phi_n \\
2\varepsilon_{13} &= \gamma_{13} = u_{3,1} + \beta_1 + W_{1,3}^n \phi_{n,3} + W_{3,1}^k \Phi_k \\
2\varepsilon_{23} &= \gamma_{23} = u_{3,2} + \beta_2 + W_{2,3}^n \phi_{n,3} + W_{3,2}^k \Phi_k \\
\varepsilon_{33} &= W_{3,3}^k \Phi_{k,3} \\
\end{align*}
\]

where \( f_i = \frac{f}{x_i} \)

4.2. Stress tensor

The normal stress components are

\[
\begin{align*}
\sigma_{11} &= \frac{E}{1 - \nu^2} [u_{1,1} + x_3 \beta_{1,1} + W_{1,1}^n \phi_n + \nu(u_{2,2} + x_3 \beta_{2,2} + W_{2,2}^n \phi_n + W_{3,3}^k \Phi_{k,3})] \\
\sigma_{22} &= \frac{E}{1 - \nu^2} [u_{2,2} + x_3 \beta_{2,2} + W_{2,2}^n \phi_n + \nu(u_{1,1} + x_3 \beta_{1,1} + W_{1,1}^n \phi_n + W_{3,3}^k \Phi_{k,3})] \\
\sigma_{33} &= \frac{E}{1 - \nu^2} [W_{3,3}^k \Phi_{k,3} + \nu(u_{1,1} + x_3 \beta_{1,1} + W_{1,1}^n \phi_n + u_{2,2} + x_3 \beta_{2,2} + W_{2,2}^n \phi_n)]
\end{align*}
\]

and the shear stress components are

\[
\begin{align*}
\sigma_{12} &= \frac{E}{2(1 + \nu)} [u_{1,2} + u_{2,1} + x_3 (\beta_{1,2} + \beta_{2,1}) + (W_{1,2}^n + W_{2,1}^n) \phi_n] \\
\sigma_{13} &= \frac{E}{2(1 + \nu)} [u_{3,1} + \beta_1 + W_{1,3}^n \phi_{n,3} + W_{3,1}^k \Phi_k] \\
\sigma_{23} &= \frac{E}{2(1 + \nu)} [u_{3,2} + \beta_2 + W_{2,3}^n \phi_{n,3} + W_{3,2}^k \Phi_k]
\end{align*}
\]

where \( E \) is Young’s modulus and \( \nu \) is Poisson’s ratio.

4.3. Stress resultants

The stress resultants are defined by

\[
\begin{align*}
[N] &= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} ; \quad [M] = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} ; \quad [P^n] = \begin{bmatrix} P_{11}^n & P_{12}^n \\ P_{21}^n & P_{22}^n \end{bmatrix} ; \\
\tilde{\mathbf{T}} &= \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} ; \quad \tilde{\mathbf{Q}}^n = \begin{pmatrix} Q_{1}^n \\ Q_{2}^n \end{pmatrix} ; \quad \tilde{\mathbf{S}}^k = \begin{pmatrix} S_{1}^k \\ S_{2}^k \end{pmatrix}
\end{align*}
\]
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\[
\begin{bmatrix}
N_{11} & N_{12} & N_{22} \\
M_{11} & M_{12} & M_{22} \\
P_{11}^n & P_{12}^n & P_{22}^n
\end{bmatrix} = \int_{\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix}
1 \\
x \phi_n \\
\sigma_{11}\sigma_{12}\sigma_{22}
\end{bmatrix} dx_3; \quad \begin{bmatrix}
T_1 \\
Q_1^n \\
S_1^k
\end{bmatrix} = \int_{\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix}
1 \\
\sigma_{13}\sigma_{23}
\end{bmatrix} dx_3
\]

\[R^k = \int_{\frac{h}{2}}^{\frac{h}{2}} \Phi_{k,3}\sigma_{33} dx_3\]

The stress resultants in terms of displacements are given by

\[
\bar{N} = D_2[(1 - \nu)\bar{e}(\bar{u}_\omega) + \nu \text{Tr}\{\bar{e}(\bar{u}_\omega)\} \bar{I}] + \nu D_2 \vartheta_k W_3^{k}\bar{I} \quad \text{with} \quad \bar{e}(\bar{u}_\omega) = \frac{1}{2} (\nabla \bar{u}_\omega + (\nabla \bar{u}_\omega)^T)
\]

(12)

\[
\bar{M} = D_1[(1 - \nu)\bar{\chi}(\bar{\beta}) + \nu \text{Tr}\{\bar{\chi}(\bar{\beta})\} \bar{I}] + \nu D_2 \psi_k W_3^{k}\bar{I} \quad \text{with} \quad \bar{\chi}(\bar{\beta}) = \frac{1}{2} (\nabla \bar{\beta} + (\nabla \bar{\beta})^T)
\]

(13)

\[
\bar{T} = D_3[\bar{\beta} + \nabla u_3 + \bar{W}^n \Theta_n]
\]

(14)

\[
\bar{P}^n = D_2 \mu[(1 - \nu)\bar{\Sigma}_n + \nu \text{Tr}\{\bar{\Sigma}_n\} \bar{I}] \quad \text{with} \quad \bar{\Sigma}_n = \bar{e}(\bar{W}^n)\Sigma_n
\]

(15)

\[
\bar{Q}^n = D_3[\bar{\beta} + \nabla u_3] \Theta_n + \Xi_n \bar{W}^n
\]

(16)

\[
\bar{S}^k = D_3[\nabla W_3 \xi_k]
\]

(17)

\[
R^k = D_2[\Lambda_k W_3^{k} + \nu \vartheta_k \text{div}(\bar{u}_\omega) + \nu \psi_k \text{div}(\bar{\beta})]
\]

(18)

where \(\bar{I}\) is the unit tensor. \(D_1, D_2\) and \(D_3\) are the classical rigidities defined by

\[
D_1 = \frac{Eh^3}{12(1 - \nu^2)}; \quad D_2 = \frac{Eh}{(1 - \nu^2)}; \quad D_3 = \frac{Eh}{2(1 - \nu^2)}
\]

(19)

The constants used in Eqs (12)–(18) are defined by

\[
\Theta_n = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\phi_n)_{,3} dx_3; \quad \Sigma_n = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\phi_n)^2 dx_3; \quad \Xi_n = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\phi_n)_{,3}(\phi_n)_{,3} dx_3
\]

(20)

\[
\vartheta_k = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\Phi_k)_{,3} dx_3; \quad \Psi_k = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\Phi_k)_{,3} dx_3; \quad \xi_k = \frac{1}{h} \int_{-\frac{h}{2}}^{\frac{h}{2}} (\Phi_k)^2 dx_3
\]

Remark: conditions on functions \(\phi_n\). It is well known that for two equivalent loads (having the same result and the same moment), the normal forces and the flexural moments must be the same; the stress tensor and displacement are obviously different. The displacement field in Lo et al.’s theory does not satisfy the above, but normal modes do satisfy it (see Section 6.1.).
5. Equilibrium equations

The principle of virtual work is used to derive the governing equilibrium equations. It is found that

\[
\begin{align*}
\bullet \text{ div } \bar{N} + \bar{f}_\omega &= \rho h \bar{\nu}_\omega \\
\bullet \text{ div } \bar{M} - \bar{T} + \bar{m}_\omega &= \frac{\rho h^3}{12} \beta \\
\bullet \text{ div } \bar{T} + f^3 &= \rho h \bar{u}_3 \\
\bullet \text{ div } \bar{P}^n + \bar{f}^n - \bar{Q}^n &= \rho h \Sigma_n \bar{W}^n \\
\bullet \text{ div } \bar{S}^k + f^k - R^k &= \rho h \xi_k \bar{W}_3^k
\end{align*}
\]

(21)

where: \( \bar{f}_\omega \) is the in-plane force vector, \( \bar{m}_\omega \) is the in-plane moments vector, \( \bar{f}^n \) is the projection of the in-plane force vector on the \( n \)th transverse normal mode, \( f^k \) is the projection of the out-plane force \( (f^3) \) on the \( k \)th longitudinal normal mode.

Boundary conditions are

\[
\begin{align*}
\bullet - \bar{N} \cdot \bar{v} + (\bar{f}_\omega)_s &= 0 \\
\bullet - \bar{M} \cdot \bar{v} + (\bar{m}_\omega)_s &= 0 \\
\bullet - \bar{T} \cdot \bar{v} + f^3_s &= 0 \\
\bullet - \bar{P}^n \cdot \bar{v} + (\bar{f}^n)_s &= 0 \\
\bullet - \bar{S}^k \cdot \bar{v} + (f^k)_s &= 0
\end{align*}
\]

(22)

where \( (\bar{f}_\omega)_s \) is the in-plane boundary force vector, \( (\bar{m}_\omega)_s \) is the in-plane boundary moments vector, \( (\bar{f}^n)_s \) is the projection of the in-plane boundary force vector on the \( n \)th transverse normal mode, \( (f^k)_s \) is the projection of the out-plane boundary force \( (f^3)_s \) on the \( k \)th longitudinal normal mode.

6. Evaluation of the present theory

6.1. Static calculation of a circular plate

Let a circular plate of radius \( R \) be submitted to two cases of force at \( r = R \)

\[
\text{case 1: } f(1 + \phi_1(x)) \bar{e}_r; \text{ case 2: } f(1 - \phi_1(x)) \bar{e}_r
\]

(23)

where \( \bar{e}_r \) is the radial vector and \( \phi_1 \) is the first transverse normal mode of the normal fibre. Using the orthonormality of normal modes, only the first mode participates in the displacement. Then the displacement field is

\[
\bar{U} = [u_1(r) + W_1^1(r) \phi_1(x)] \bar{e}_r
\]

(24)

The stress resultant tensor components are

\[
\begin{align*}
N_{11}(r) &= D_2 \left( u_{1,1} + \frac{u_1}{r} \right) \\
N_{22}(r) &= D_2 \left( \nu u_{1,1} + \frac{u_1}{r} \right) \\
P_{11}^1(r) &= D_2 \Sigma_1 \left( W_{1,1}^1 + \frac{W_1^1}{r} \right) \\
P_{22}^1(r) &= D_2 \Sigma_1 \left( \nu W_{1,1}^1 + \frac{W_1^1}{r} \right) \\
Q_1^1 &= D_2 W_1^1 \Sigma_1; \ Q_2^1 = 0
\end{align*}
\]

(25)
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The equations of equilibrium are of the following form

\[
\begin{align*}
\bullet & \quad \text{div } \overline{N} + \overline{f}_\omega = \bar{\delta} \\
\bullet & \quad \text{div } \overline{P}^1 + \overline{f}^1 - \overline{Q}^1 = \bar{\delta}
\end{align*}
\]  

(26)

with the boundary conditions

\[
\begin{align*}
\bullet & \quad \text{Case 1: } [N_{11}(R)] = fh; \quad \text{Case 2: } [N_{11}(R)] = fh \\
\bullet & \quad \text{Case 1: } [P^1_{11}(R)] = f^1 = fh\Sigma_1; \quad \text{Case 2: } [P^1_{11}(R)] = f^1 = -fh\Sigma_1
\end{align*}
\]  

(27)

The classical solution gives

\[
U_1 = u_1 = \frac{fr}{D_2(1 + \nu)}; \quad N_{11} = N_{22} = fh; \quad \sigma_{11} = \sigma_{22} = f 
\]  

(28)

For the present theory, \(W^1_{11}\) is the solution of the following equation

\[
r^2W^1_{1,11} + rW^1_{1,1} - (1 + r^2H_1^2)W^1_{11} = 0 \quad \text{where: } H_1^2 = \frac{D_3\Sigma_1}{D_2\Sigma_1} = \frac{(1 - \nu)\Sigma_1}{2\Sigma_1}
\]  

(29)

which is the first modified Bessel’s equation; the solution is

\[
W^1_{1}(r) = AI_1(H_1r)
\]  

(30)

where \(I_1\) is the first modified Bessel’s function. \(A\) is determined by using the boundary condition

\[
P^1_{11}|_{r=R} = f^1 \Rightarrow D_2\Sigma_1 \left[ W^1_{1,1} + \nu \frac{W^1_{1}}{r} \right] \bigg|_{r=R} = f^1
\]  

(31)

Eq. (31) gives

\[
W^1_{1}(r) = \frac{f^1}{\Sigma_1 D_2 \left[ I_1 \frac{I_0(H_1R) + I_2(H_1R)}{2} + \nu \frac{I_1(H_1R)}{R} \right]} I_1(H_1r)
\]  

(32)

where \(I_0, I_2\) are respectively the modified Bessel’s functions number zero and number two. For Cases 1 and 2, the present theory gives

\[
\begin{align*}
\text{Case 1: } U_1(r, x_3) &= \frac{fr}{D_2(1 + \nu)} \left\{ 1 + \frac{I_1(H_1r)(1 + \nu)\phi_1(x_3)}{r[H_1 \frac{I_0(H_1R) + I_2(H_1R)}{2} + \nu \frac{I_1(H_1R)}{R}]} \right\} \\
\text{Case 2: } U_1(r, x_3) &= \frac{fr}{D_2(1 + \nu)} \left\{ 1 - \frac{I_1(H_1r)(1 + \nu)\phi_1(x_3)}{r[H_1 \frac{I_0(H_1R) + I_2(H_1R)}{2} + \nu \frac{I_1(H_1R)}{R}]} \right\}
\end{align*}
\]  

(33)

For the present theory, the two cases of loading produce the same normal force tensor but neither the same displacement nor the same stress. The classical theory gives only the mean displacement and the mean stress. Figure 2 shows the evolution of the displacement at \(r = R\), as a function of the nondimensional thickness \(x_3/h\). For the stress component \(\sigma_{11}\) and for the example at \(r = R\), the present theory provides the exact solution and the classical theory gives only the mean stress, as shown in Figure 3.
Figure 2. Displacement distribution across the nondimensional thickness $x_3/h$ for $\nu = 0.25$, $R/h = 5$, at $r = R$.

Figure 3. Flexural stress distribution across the nondimensional thickness $x_3/h$ for $\nu = 0.25$, $R/h = 5$, at $r = R$.

6.2. Static calculation of an infinite plate

Let an infinite plate of thickness $h$ subjected to a pressure on the upper surface $x_3 = h/2$ of the form

$$q_0 \sin \frac{\pi x_1}{L}$$

(34)
where \( L \) is the half-wavelength of the sinusoidal loading pattern. The equations of equilibrium take the following special form for this problem:

\[
\begin{align*}
\text{div } \overline{N} = 0 & \Rightarrow D_2 u_{1,11} + \nu \vartheta_k D_2 W_{3,1}^k = 0 \\
\text{div } \overline{M} - \overline{T} = 0 & \Rightarrow D_1 \beta_{1,11} + \nu \psi_k D_2 W_{3,1}^k - D_3 \beta_1 + u_{3,1} + \theta_n W_1^n = 0 \\
\text{div } \overline{T} - q_0 \sin \left( \frac{\pi x_1}{L} \right) = 0 & \Rightarrow D_3 [\beta_{1,11} + u_{3,1} + \theta_n W_1^n] - q_0 \sin \left( \frac{\pi x_1}{L} \right) = 0 \\
\text{div } \overline{P} - \overline{Q} = 0 & \Rightarrow D_2 W_{1,11} - D_3 [\beta_{1,1} + u_{3,1}] \theta_n + \Xi_n W_1^n = 0 \\
\text{div } S^k + f^k - R^k = 0 & \Rightarrow D_3 \xi_k W_{3,11}^k - D_2 \Lambda_k W_{3,1}^k - \nu \vartheta_k D_2 u_{1,1} + \nu \psi_k D_2 \beta_{1,1} = -q_0 \sin \left( \frac{\pi x_1}{L} \right)
\end{align*}
\]

Equations (35-a)–(35-e) can be solved by analytical means. We obtain

\[
W_3^k = -\frac{q_0 - q_0 \nu \psi_k \left( \frac{L}{\pi} \right)^2 D_2^2}{[D_3 \xi_k \left( \frac{\pi}{L} \right)^2 + D_2 \Lambda_k - (\nu \vartheta_k)^2 D_2 - (\nu \psi_k)^2 D_2 \beta_{1,1}]} \sin \left( \frac{\pi x_1}{L} \right)
\]

Eq. (35-d) gives

\[
W_1^n = \frac{\theta_n \frac{L}{\pi}}{D_2 \Sigma_n \left( \frac{\pi}{L} \right)^2 + D_3 (\Xi_n - \theta_n^2) q_0 \sin \left( \frac{\pi x_1}{L} \right)}
\]

For the present theory, the stress component \( \sigma_{11} \) is given by

\[
\sigma_{11} = \left\{ \begin{array}{l}
\nu \vartheta_k \left[ q_0^k - 12q_0 \nu \psi_k \left( \frac{L}{\pi h} \right)^2 \right] - 12q_0 \nu \left( \frac{L}{\pi h} \right)^2 x_3 \\
+ \frac{12
\nu \psi_k [q_0^k - 12q_0 \nu \psi_k \left( \frac{L}{\pi h} \right)^2] x_3}{\left[ (1 - \nu) \xi_k \left( \frac{\pi h}{L} \right)^2 + \Lambda_k - (\nu \vartheta_k)^2 - 12(\nu \psi_k)^2 \right] h} \\
+ \frac{\nu \pi k [q_0^k - 12q_0 \nu \psi_k \left( \frac{L}{\pi h} \right)^2] \sin \left( \frac{\pi k (x_3 + h/2)}{h} \right)}{\Sigma_n \left( \frac{\pi h}{L} \right)^2 + \frac{1 - \nu}{2} \left( \Xi_n - \theta_n^2 \right) \phi_n} \sin \left( \frac{\pi x_1}{L} \right) \end{array} \right.
\]

This result can be compared with the exact solution taken from Little (1973), and with the higher-order approximate theory given by Lo et al. (1977). The classical theory, the shear deformation Reissner plate theory, the Essenberg and the ‘level 2’ theories all give the same results

\[
\sigma_{11} = 12q \left( \frac{L}{\pi h} \right)^2 \frac{x_3}{h} \sin \left( \frac{\pi x_1}{L} \right)
\]
For the present theory, the midplane displacement is given by

\[ u_3 = -\theta \frac{q_0}{D_1} \left( \frac{L}{\pi} \right)^4 \sin \left( \frac{\pi x_1}{L} \right) \quad \text{with} \quad \theta = \left[ 1 + \left( \frac{\pi h}{L} \right)^2 \right] \left[ 1 + \sum_{k=2}^{\infty} \frac{(1 - 12\nu^2)(\pi h / L)^2}{3(1 - \nu)(\pi h / L)^2 + 6\pi^2 / L^2 - 144\nu^2} \cos(k\pi) \right] \]

where \( \theta \) is a displacement coefficient. This result is to be compared with the exact solution taken from Little (1973), with other lower- and higher-order approximate theories. The classical theory sets \( \theta = 1 \). The shear deformation Reissner plate theory produces the results

\[ \theta = \left[ 1 + \frac{(2 - \nu)}{10(1 - \nu)} \left( \frac{\pi h}{L} \right)^2 \right] \]

Essenburg's theory gives

\[ \theta = \left[ 1 + \left( \frac{(2 - \nu) + \nu^2}{10(1 - \nu) + \nu} \right) \left( \frac{\pi h}{L} \right)^2 - \frac{3}{1120} \left( \frac{\pi h}{L} \right)^4 \right] \]

The Naghdi 'level 2' theory defined by Eq. (2) gives

\[ \theta = \left\{ 1 + \frac{20(\frac{\pi h}{L})^2}{(1 - \nu)[120 + (1 - \nu)(\frac{\pi h}{L})^2]} - \frac{[(4\nu^2 + 20\nu)(\frac{\pi h}{L})^2 + (1 - \nu)(\frac{\pi h}{L})^4]}{4(1 - \nu)[120 + (1 - \nu)(\frac{\pi h}{L})^2]} \right\} \]

The maximum displacements of the middle plane of the plate according to the various theories are compared with the exact result in Figure 4.

(1) Exact Solution
(2) Classical Theory
(3) Reissner level 1 Theory
(4) Level 2 Theory
(5) Lo Theory
(6) Present Theory

\[ u_3 = -\theta \frac{q_0}{D_1} \left( \frac{L}{\pi} \right)^4 \sin \left( \frac{\pi x_1}{L} \right) \]

**Figure 4.** Midplane displacement solution for \( \nu = 0.25 \).
7. Discussion and conclusion

For problems which involve rapidly fluctuating loads with the characteristic length of the order of the thickness or high-frequency analysis of plates, a high-order theory is required to give meaningful results. For the other problems, a lower-order simple theory is entirely satisfactory and sufficient.

For the infinite plate problem and for high h/L values, the high-order theory defined by Eq. (5) could not come close to reproducing the exact solution which deviates strongly from the mentioned solution. The high-order theory defined by Eq. (5) cannot induce the same stress resultants for an equivalent loading.

The present high-order theory can induce the same stress resultants for an equivalent loading. But due to the high order of the terms included in this theory, it is of course not convenient to use this for classical problems. The examples of infinite plate and circular plate would be helpful in determining guidelines by which one can ascertain when it is necessary to use a high-order theory and when a lower-order theory will suffice.

This high-order theory can be extended to laminated plate conditions. It is known that for laminates, distribution of in-plane displacements across the thickness may be strongly nonlinear.

Appendix

The transverse normal modes for a free-free beam are

\[ \varphi_n = \cos \left( \frac{\alpha_n x_3}{h} \right) + c h \left( \frac{\alpha_n x_3}{h} \right) - R_n \left[ \sin \left( \frac{\alpha_n x_3}{h} \right) + s h \left( \frac{\alpha_n x_3}{h} \right) \right] \]

The coefficients \( \alpha_n \) and \( R_n \) have the following values

\[ R_1 = 0.9825; \ R_2 = 1.0008; \ R_3 = 1.0000; \ R_4 = 1.0000. \]

\[ \alpha_1 = 4.730; \ \alpha_2 = 7.853; \ \alpha_3 = 10.996; \ \alpha_4 = 14.137. \]

For a free-free beam, the longitudinal modes are written as

\[ \Phi_k = \cos \left( k \pi \left( \frac{x_3}{h} + \frac{1}{2} \right) \right) \]

References


