The dependence between contour choice and numerical values of stress intensity factors computed from path independent integrals

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Abstract – A new method for calculating the approximations of stress intensity factors (SIF) in linear elastic fracture mechanics is presented. The method uses a least-squares approximation and is based on path independent integrals. The objective of the method is to overcome dependence between the SIF and contour choice and to use the contours near the crack tip without a loss of accuracy. In order to avoid the mode separation difficulty in mixed mode loading when the J-integral is used, we have based our procedure on the contour integral system. Numerical results are given to demonstrate the efficiency of this new approach to path independent integrals. © Elsevier, Paris

stress intensity factor / finite element method / superimposed mesh method / path independent integral / displacement correlation technique / quarter-point displacement technique

1. Introduction

Presently many methods for extracting stress intensity factors (SIF) from finite element solutions exist in fracture mechanics. Some of these methods use specific element displacement or stress functions (Tong et al., 1973; Henshell and Shaw, 1975; Barsoum, 1977; Staab and Sun, 1981; Ogen and Schiff, 1985) and others use path independent contour integrals (Rice, 1968; Stern et al., 1976; Chen, 1985; Dong, 1994) or virtual crack extension techniques (Parks, 1974).

Unfortunately, until now when the path independent contour integrals are used, two main disadvantages have been observed. Firstly, the values of the SIF deduced from path independent contour integrals depend numerically on the choice of integration contour. Secondly, the accuracy of the SIF values obtained from the innermost contours, i.e. near the crack tip, is very poor. For this second reason, it is recommended to use contours removed from the crack tip. With the method presented in this paper, we can practically overcome the first drawback and use the results obtained from the innermost contours without loss of accuracy.

Kpégba et al. (1996) have shown that the path independent integral introduced by Eshelby and Knowles (mentioned in Bui, 1978) and Xanthis et al. (1981) applied to antiplane problems, is a function of $d/l$, where $d$ is equal to the distance between the integration contour and crack tip and $l$ is the crack length. This permits a parameter proportional to the SIF to be extracted in mode III. This parameter is calculated as the value of this function of $d/l$ for $d$ equal to $c$, where $c$ represents the distance between the crack tip and the exterior contour of the problem domain.

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This approach is interesting because: i) it permits unique results of any grid to be obtained. Then the path independent contour integral becomes practically independent of contour selection; ii) although the results obtained from the innermost contours are always poor, their use can minimize the computational effort without degrading the obtained results and even ameliorate the SIF approximation. Moreover, this technique can be used in any finite element program with standard isoparametric or specific element subroutines.

The aim of this work is to generalize this technique of SIF extraction, described in Kpégba et al. (1996) for antiplane problems to Mode I, Mode II and mixed mode problems. As in Kpégba (1996) and Kpégba et al. (1996), we shall use the superimposed mesh method (SMM) to compute the solution of different problems presented in Section 5. In Section 3 we summarize the SMM basis for one-dimensional problems. A comprehensive survey of SMM is given in Ottavy (1989) and Kpégba (1996). Finally we choose to base our technique on the contour integral proposed by Stern et al. (1976) to distinguish contributions due to crack opening and those due to shear in mixed mode.

2. Theoretical considerations

The path independent contour integral introduced by Stern et al. [7] follows from Betti's reciprocal work theorem for plane elastic states with vanishing body forces

\[ \int_{\partial \Omega} (T\overline{u} - \overline{T}u)ds = 0 \]  

(1)

where \( u \) and \( T \) are the displacement field and the traction vector on the boundary \( \partial \Omega \) of a simply connected and bounded region \( \Omega \), corresponding to the solution of any particular equilibrium problem without body force. \( \overline{u} \) and \( \overline{T} \) are suitable displacements and tractions which also satisfy the equilibrium problem. We assume that \( u, T, \overline{u}, \) and \( \overline{T} \) are statically admissible.

It was shown in Stern et al. (1976) that Eq. (1) can be written in the following form

\[ I_r = - \int_{C_r} (T\overline{u} - \overline{T}u)ds = \int_{C_r} (T\overline{u} - \overline{T}u)ds \]  

(2)

with \( C_r \) designating the circular boundary of the deleted region constituted by a circle of radius \( r \) centered at the crack tip and \( C' \) the remaining boundary as indicated in figure 1. We assume, as in Stern et al. (1976), that, for the equilibrium stress state of interest the crack faces are free of traction. The traction components in the normal and tangential directions are prescribed so that we have a well-posed problem.

![Figure 1. Region \( \Omega \) and the two contours \( C_r \) and \( C' \).](image-url)
Concerning the numerical values of stress intensity factors

Using the classical stress and displacement fields in the neighborhood of the crack tip in Eq. (2), Stern et al. (1976) have represented the auxiliary elastic state defined by Eq. (3), such that the products $r^{1/2}|\bar{u}|$ and $r^{3/2}|\bar{T}|$ are finite on $C_r$

\[
\begin{align*}
\bar{u}_r &= \frac{1}{2(2\pi r)^{1/2}(1+k)} \left\{ (2k+1)\cos \frac{3\theta}{2} - 3\cos \frac{\theta}{2} \right\} c_1 + \left\{ (2k+1) \sin \frac{3\theta}{2} - 3\sin \frac{\theta}{2} \right\} c_2 \\
\bar{u}_\theta &= \frac{1}{2(2\pi r)^{1/2}(1+k)} \left\{ -(2k-1)\sin \frac{3\theta}{2} + 3\sin \frac{\theta}{2} \right\} c_1 + \left\{ (2k-1) \cos \frac{3\theta}{2} - \cos \frac{\theta}{2} \right\} c_2 \\
\bar{\sigma}_r &= -\frac{\mu}{2(2\pi r)^{1/2}(1+k)} \left\{ 7\cos \frac{3\theta}{2} - 3\cos \frac{\theta}{2} \right\} c_1 + \left\{ 7\sin \frac{3\theta}{2} - \sin \frac{\theta}{2} \right\} c_2 \\
\bar{\sigma}_\theta &= -\frac{\mu}{2(2\pi r)^{1/2}(1+k)} \left\{ \cos \frac{3\theta}{2} + 3\cos \frac{\theta}{2} \right\} c_1 + \left\{ \sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right\} c_2 \\
\bar{\sigma}_{r\theta} &= -\frac{\mu}{2(2\pi r)^{1/2}(1+k)} \left\{ 3\sin \frac{3\theta}{2} + \sin \frac{\theta}{2} \right\} c_1 - \left\{ 3\cos \frac{3\theta}{2} + \cos \frac{\theta}{2} \right\} c_2
\end{align*}
\]

where $c_1$ and $c_2$ are arbitrary constants, $\mu$ the shear modulus and $k = 3 - 4\nu$ (plane strain) $k = (3 - \nu)/(1 + \nu)$ (plane stress) with $\nu$ denoting Poisson’s ratio.

Thus, on the inner circular boundary $C_r$ the evaluation of the contour integral (in terms of traction and displacement components carried out in polar coordinates $r$ and $\theta$) leads to

\[
I_r = -\int_{C_r} (\bar{T}u - \bar{u}T)ds = \int_{-\pi}^{\pi} \left( \bar{\sigma}_r u_r + \bar{\sigma}_{r\theta} u_\theta - \sigma_r \bar{u}_r - \sigma_{r\theta} \bar{u}_\theta \right) r d\theta
\]

(4)

with $u_r$ and $u_\theta$ representing the radial and tangential components of the classical displacement vector in the vicinity of the crack tip in polar coordinates, while $\sigma_r$ and $\sigma_{r\theta}$ are the first and third stress components.

Introducing Eq. (3) and the classical expressions of $u_r$, $u_\theta$, $\sigma_r$ and $\sigma_{r\theta}$ in terms of SIF’s $K_I$ and $K_{II}$ into Eq. (4), we obtain

\[
I_r = c_1 K_1 - c_2 K_{II} + O(1)
\]

(5)

where the last term $O(1)$ goes to zero with $r$ as indicated. This establishes the following formula

\[
\int_{C_r} (\bar{T}u - \bar{u}T)ds = c_1 K_1 - c_2 K_{II}
\]

(6)

As we assume that $T$ and $\bar{T}$ are equal to zero on the crack faces, the contour $C’$ involves only the outer boundary. Thus, when $u$ and $T$ are obtained from prescribed data, the contour integral may be evaluated as a linear combination of $c_1$ and $c_2$. The coefficients of both these constants are the stress intensity factors $K_I$ and $K_{II}$.

To obtain relationship (5), a routine evaluation of the integral is carried out along a circular contour as indicated in Eq. (4). We are, however, going to calculate this integral on the contour defined by $[-a, a] \times [-b, b]$, with $a > 0$ and $b > 0$ (figure 2). Indeed, a detailed evaluation of (4) shows that to calculate $I_r$, it is sufficient to integrate $\cos^2 \frac{\theta}{2}$, $\cos^2 \frac{\theta}{2}$, $\sin^2 \frac{\theta}{2}$ and $\sin^2 \frac{3\theta}{2}$ between $-\pi$ and $\pi$. But we shall only carry out the $\cos^2 \frac{\theta}{2}$ integral between 0 and $\pi$ because the integrand is an even function of $\theta$. The remaining integrals are computed in the same manner. The parameter $\theta_1$ (figure 2) represents the polar angle between the $Ox$-axis and the first diagonal of the rectangular contour. We have

\[
\int_0^\pi \cos^2 \frac{\theta}{2} d\theta = \frac{1}{2} \left[ \int_0^{\theta_1} (1 + \cos \theta) d\theta + \int_{\pi-\theta_1}^\pi (1 + \cos \theta) d\theta + \int_{\pi-\theta_1}^{\pi-\theta_1} (1 + \cos \theta) d\theta \right]
\]
If $\theta$ belongs to $[0, \theta_1]$, we have $\cos \theta = \frac{a}{\sqrt{y^2+a^2}}$, and
\[
\int_{0}^{\theta_1} (1 + \cos \theta) d\theta = \int_{0}^{b} \frac{a}{y^2 + a^2} dy + \int_{0}^{b} \frac{a}{\sqrt{y^2 + \frac{a^2}{y^2}}} \frac{dy}{\sqrt{y^2 + a^2}}
\]
Between $\pi - \theta_1$ and $\pi$, $\cos \theta = -\frac{a}{\sqrt{y^2+a^2}}$; then
\[
\int_{\pi - \theta_1}^{\pi} (1 + \cos \theta) d\theta = -\int_{b}^{0} \frac{a}{y^2 + a^2} dy + \int_{0}^{b} \frac{a}{\sqrt{y^2 + \frac{a^2}{y^2}}} \frac{dy}{\sqrt{y^2 + a^2}}
\]
Finally, if $\theta$ belongs to $[\theta_1, \pi - \theta_1]$, $\sin \theta = -\frac{b}{\sqrt{x^2+b^2}}$. This means
\[
\int_{\theta_1}^{\pi - \theta_1} (1 + \cos \theta) d\theta = 2 \int_{0}^{a} \frac{b}{x^2 + b^2} dx
\]
By computing the remaining integrals in the same manner, we obtain the following equality
\[
I = \int_{C'} (T\bar{u} - \bar{T}u) ds = \frac{2}{\pi} \left[ \text{Arctg} \frac{a}{b} + \text{Arctg} \frac{b}{a} \right] (c_1 K_1 - c_2 K_{II})
\] (7)
Relation (7) totally agrees with Eq. (6), since $\text{Arctg} \frac{a}{b} + \text{Arctg} \frac{b}{a} = \frac{\pi}{2}$. Moreover, Eq. (7) is independent of $a$ and $b$; for this reason, we take $b = l$ (crack length) and $a = d$ (distance between the rectangular integration contour and the crack tip). We can write
\[
I = c_1 (H_0^1(d/l) + H_1^1(d/l)) - c_2 (H_0^{II}(d/l) + H_1^{II}(d/l))
\] (8)
where \( H_0^j(d/l) = \frac{2}{\pi} K_j \arctan(d/l) \) and \( H_1^j(d/l) = \frac{2}{\pi} K_j \arctan(l/d) \), with \( J = I, II \). If \( d \) goes to \( \infty \), the series expansion of \( H_0^j \) and \( H_1^j \) gives

\[
H_0^1(d/l) = \sum_{k=0}^{n} \frac{a_k}{(d/l)^k} + O((l/d)^n) \\
H_1^1(d/l) = \sum_{k=0}^{n} \frac{b_k}{(d/l)^k} + O((l/d)^n)
\]

(9)

Theoretically, \( a_0^1 = K_1, a_2^1 = 0, a_{2i-1}^1 = (-1)^{i+1} K_1/(2i-1), b_0^1 = 0 \) and \( b_i^1 = -a_i^1 \) for any positive integer values \( i \) and \( J = I, II \). But numerically these equalities are not satisfied. That is why, in our previous investigations in Kpégba (1996) and Kpégba et al. (1996) we have posed \( A_k^1 = a_k^1 + b_k^1 \). Thus we can write

\[
H_1(d/l) = H_0^1(d/l) + H_1^1(d/l) = \sum_{k=0}^{n} \frac{A_k}{(d/l)^k} + O((l/d)^n)
\]

(10)

\( H_0^1 \) and \( H_1^1 \) are the components of the integral along the edges of the contour which are parallel to the \( x \)-axis and \( y \)-axis respectively.

In our view, the fact that the coefficients \( A_k^1 \) are not equal to zero for any positive lower indices \( k \) explains why the values of the SIFs are not stable with regard to contour selection. Either Eq. (9) or (10) can be used to extract the numerical SIF values. We chose to use Eq. (10) as in Kpégba et al. (1996) because it requires fewer calculations than using Eq. (9). Therefore, the SIF values \( K_1 \) (\( J = I, II \)) are estimated by

\[
\overline{K}_1 = \lim_{d \rightarrow c} H_1(d/l)
\]

(11)

where \( c \) represents the distance between the crack tip \( O \) and the exterior contour of the problem domain.

3. Superimposed mesh method (SMM)

The superimposed mesh method (SMM) is a new procedure for the treatment of singularities with the finite element method, which has been used successfully by Kpégba (1996) and Kpégba et al. (1996) to compute stress intensity factors for two-dimensional problems. In conjunction with the technique for extraction of SIF described above, SMM is a reliable and computationally efficient scheme for the determination of SIF.

In this section we first summarize the fundamental basis of SMM. Since the SMM principle is more easy to understand for one-dimensional problems, we consider only this case in the next description. Obviously, SMM is also applicable to two-dimensional problems.

Let \( u \) be the solution of a classical elliptic problem \((P)\) in a suitable functional space \( V \) defined by

\[
a(u, \nu) = l(\nu) \quad \nu \in V
\]

(12)

with \( a \) representing a bilinear form defined on \( V \times V \) and \( l \) a linear form defined on \( V \). The solution domain is the real interval \( \Omega_1 = [a, b] \). We assume that the problem \((P)\) presents the following features: 1) only an approximation of the solution \( u \) is accessible. The numerical process is the finite element method; 2) there is a part \( \Omega_2 = [a', b'] \) of \( \Omega_1 \) where the approximation must be as correct as possible. In two-dimensional problems \( \Omega_2 \) may be the vicinity of the crack tip. Conveniently, we assume that \( a < a' \) and \( b' < b \), but this is not essential.

We suppose that finite element grids are associated with domains \( \Omega_1 \) and \( \Omega_2 \) such that the grid on \( \Omega_2 \) is fine enough in order to have a good approximation on this part. The SMM procedure consists of introducing
a transition strip \( \Lambda = [a', a' + w] \cup [b' - w, b'] \) (attachment strip) along the boundary of \( \Omega_2 \), where \( w \) is an appropriate real parameter chosen with respect to the properties of the trial functions on \( \Omega_2 \). In practice, \( w \) is chosen equal to a fraction of the minimum of the distance between the nodes on \( \Omega_2 \) and its boundary.

We denote \( u_1 \) and \( u_2 \) as the interpolation functions which represent the solution of the problem (P) in \( \Omega_1 \) and \( \Omega_2 \). The extension of \( u_2 \), to zero in \( \Omega_1 \setminus \Omega_2 \), still denoted by \( u_2 \), is defined throughout \( \Omega_1 \). Then the SMM solution of (P) is defined by

\[
\begin{align*}
    u &= \lambda^+ u_2 + \lambda^- u_1, \\
    \lambda^- &= 1 - \lambda^+
\end{align*}
\] (13)

where the function \( \lambda^+ \) is defined over \( \Omega_1 \) by

\[
\lambda^+(x) = \begin{cases} 
1, & x \in \Omega_2 \setminus \Lambda \\ 
0, & x \in \Omega_1 \setminus \Omega_2 \\ 
g(x), & x \in \Lambda
\end{cases}
\]

\( g \) being a suitable function taking into account the functional properties of space \( V \). For example, if \( V \) is a subspace of continuous function space \( C^0 \), \( g \) is piecewise linear (figure 3). To guarantee the independence of both approximation structures, the function \( \lambda^+ \) is defined on \( \Omega_2 \) and extended to zero in \( \Omega_1 \setminus \Omega_2 \).

![Figure 3. Grids on domains \( \Omega_1 \) and \( \Omega_2 \) with the graph of function \( \lambda^+ \).](image)

So if we denote \((\nu_k^{(1)})\) and \((\nu_m^{(2)})\) as the nodal basis of approximation on \( \Omega_1 \) and \( \Omega_2 \), with \( 1 \leq k \leq n_1 \) and \( 1 \leq m \leq n_2 \), \( n_1 \) and \( n_2 \) being the number of nodes on \( \Omega_1 \) and \( \Omega_2 \), the solution of (P) can be written as

\[
u = \sum_{k=1}^{n_1} \lambda^- \chi_k^{(1)} \nu_k^{(1)} + \sum_{m=1}^{n_2} \lambda^+ \chi_m^{(2)} \nu_m^{(2)}
\] (14)
where \((\chi_k^{(i)})\) are the components of \(u_i\) on the basis \((\nu_k^{(i)})\). Thus \(u\) is the solution of the following linear algebraic system

\[
\begin{align*}
\sum_{i=1}^{n_1} a(\lambda^+\nu_k^{(1)},\lambda^-\nu_m^{(1)})\chi_i^{(1)} + \sum_{j=1}^{n_2} a(\lambda^+\nu_j^{(2)},\lambda^-\nu_m^{(1)})\chi_j^{(2)} &= I(\lambda^-\nu_m^{(1)}) \\
\sum_{i=1}^{n_1} a(\lambda^+\nu_i^{(1)},\lambda^+\nu_k^{(2)})\chi_i^{(1)} + \sum_{j=1}^{n_2} a(\lambda^+\nu_j^{(2)},\lambda^+\nu_k^{(2)})\chi_j^{(2)} &= I(\lambda^+\nu_k^{(2)})
\end{align*}
\]

(15)

The properties of this system explain the efficiency of the SMM procedure, namely: 1) the possibility to have parallel computations on the two meshes; 2) the unknown \((x_j^{(2)})\) can be eliminated by the inversion of the matrix \(\{a(\lambda^+\nu_j^{(2)},\lambda^+\nu_k^{(2)})\}\). This matrix is invertible because the restriction of the problem upon \(\Omega_2\) may be viewed as a Dirichlet problem with homogeneous conditions on the boundary. Indeed the algebraic system (16) may be written in the next compact form

\[
(S) \quad \begin{cases}
A_{11}\tilde{X}_1 + L\tilde{X}_2 = \tilde{F}_1 \\
L^T\tilde{X}_1 + A_{22}\tilde{X}_2 = \tilde{F}_2
\end{cases}
\]

As the matrix \(A_{22}\) is invertible, the system \((S)\) is equivalent to the following system \((S')\)

\[
(S') \quad \begin{cases}
A_{11}'\tilde{X}_1 = \tilde{F}_1' \\
\tilde{X}_2 = A_{22}^{-1}(\tilde{F}_2 - L^T\tilde{X}_1)
\end{cases}
\]

where

\[
A_{11}' = A_{11} - LA_{22}^{-1}L^T, \\
\tilde{F}_1' = \tilde{F}_1 - LA_{22}^{-1}\tilde{F}_2, \\
\tilde{X}_1 = (x_i^{(1)}) \text{ and } \tilde{X}_2 = (x_j^{(2)}) \{1 \leq i \leq n_1, 1 \leq j \leq n_2\}
\]

The shape of system \((S)\), and its equivalence to \((S')\) demonstrate the advantages of SMM. It allows a simple manipulation of sub-domains associated with each singularity in situations where there are many singularities (Ottavy, 1989; Kpégba, 1996).

What distinguishes SMM from other sub-structuring methods such as the s-version finite element method [16] is the manner in which the continuity is ensured at the interface between different sub-domains. Furthermore, the SMM method ensures \(C^n\) continuity by simply choosing the function \(\lambda^+\) to be \(C^n\) continuous.

4. Mesh generation and partial derivatives estimation

To prevent the appearance of too squashed elements around the crack tip during mesh generation when the relevant domain is not symmetric with regard to the crack tip \(O\), we construct the grid on the domain as follows.

In the first stage, we define the real \(s\) by min \((L, B - L, A)\) in the case wherein the domain is symmetric about the crack line (figure 4a) and by min \((L, C - L, A, B)\) in the case where there is no symmetry with respect to the crack line (figure 4b).

The next step consists of constructing the grid on the domain in order to have no squashed elements in the mesh on the square region \(R\) whose edge is equal to \(s\). In practice, we use a uniform grid or a quadratic grid (figure 5) on the square region \(R\).
Finally, to eliminate the presence of discontinuities in the partial derivatives, we estimate the partial derivatives by the following expressions

\[
\begin{align*}
\frac{\partial u}{\partial x}(x_0, y_0) &= \frac{u(x_0 + h, y_0) - u(x_0 - h, y_0)}{2h}, \\
\frac{\partial u}{\partial y}(x_0, y_0) &= \frac{u(x_0, y_0 + h) - u(x_0, y_0 - h)}{2h}, \\
\frac{\partial u}{\partial x}(x_0, y_0 + h) &= \frac{u(x_0 + h, y_0 + h) - u(x_0 - h, y_0 + h)}{2h}, \\
\frac{\partial u}{\partial y}(x_0, y_0 + h) &= \frac{u(x_0, y_0 + h) - u(x_0, y_0)}{h}
\end{align*}
\]

where \((x_0, y_0)\) is a nodal point of the grid on \(\Omega_1\) (figure 6). In all the numerical results presented below, we chose \(l = L\) and \(c = s/2\) and the integrals are computed by the standard trapezium method.
Concerning the numerical values of stress intensity factors

Domain boundary

\[ \Omega_1 \]

\[ (x_0, y_0) \]

\[ h \text{ (grid size)} \]

Figure 6. Partial grid on domain \( \Omega_1 \).

5. Numerical examples

To illustrate the performance of SMM combined with the SIF extraction technique described above, two mode I, one mode II and one mixed mode crack problems are chosen for computation of stress intensity factors in this section. The examples quoted below are all calculated on a VAX 3100 m38 computer with Lagrangian bilinear elements. As mentioned in Section two, the first step in the \( K_1 \) determination consists of computing the coefficients of \( c_1 \) and (or) \( c_2 \) in Eq. (4). It is not necessary to use the values obtained for these coefficients on all existing contours, because this increases the computational time without improving the accuracy of the SIF approximation. To minimize the computational time and to take into account the singularity which is localized around the crack tip, we use the results on the first five contours from the crack tip. So, by using five points in the least-squares approximation, the degree \( n \) of our polynomial approximation must satisfy \( 1 \leq n \leq 4 \). The coefficients \( A_i^j \) in Eq. (10) and SIF values for \( n = 2, 3 \) are listed in tables I–V. In the next section, we denote

<table>
<thead>
<tr>
<th>( n )</th>
<th>( A_0^1 )</th>
<th>( A_1^1 )</th>
<th>( A_2^1 )</th>
<th>( A_3^1 )</th>
<th>( K_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM (( N_1 = 325 )) with uniform grid on ( \Omega_1 (c = 1) )</td>
<td>2</td>
<td>2.38133393</td>
<td>0.00272150</td>
<td>0.00050996</td>
<td>2.33845654</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.39009177</td>
<td>-0.00277156</td>
<td>0.00148449</td>
<td>-0.00004816</td>
</tr>
<tr>
<td>SMM (( N_1 = 45, N_2 = 276 )) with uniform grid on ( \Omega_1 ) and ( \Omega_2 (c = 0.5) )</td>
<td>2</td>
<td>2.46525490</td>
<td>0.00164632</td>
<td>0.00019268</td>
<td>2.46931827</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.47437583</td>
<td>-0.00178616</td>
<td>0.00055806</td>
<td>-0.00001083</td>
</tr>
<tr>
<td>FEM (( N_1 = 325 )) with quadratic grid on ( \Omega_1 (c = 1) )</td>
<td>2</td>
<td>2.48503463</td>
<td>-0.00087348</td>
<td>0.00000040</td>
<td>2.48328927</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.48360529</td>
<td>-0.00065600</td>
<td>-0.00000594</td>
<td>0.00000003</td>
</tr>
<tr>
<td>SMM (( N_1 = 45, N_2 = 276 )) with uniform grid on ( \Omega_1 ) and quadratic grid on ( \Omega_2 (c = 0.5) )</td>
<td>2</td>
<td>2.50729293</td>
<td>-0.00061639</td>
<td>0.00000007</td>
<td>2.50606042</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>2.50595506</td>
<td>-0.00046983</td>
<td>-0.00000301</td>
<td>0.00000001</td>
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Table II. Values of $A_k^l$ and the SIF approximation for example 2 in Mode 1 ($l = 0.5$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_0^l$</th>
<th>$A_1^l$</th>
<th>$A_2^l$</th>
<th>$A_3^l$</th>
<th>$K_1^l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM ($N_1 = 650$) with uniform grid on $\Omega_i$ ($c = 0.5$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.4528826</td>
<td>-0.00053604</td>
<td>0.00067618</td>
<td>1.45302239</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.45428268</td>
<td>-0.00141441</td>
<td>0.00083201</td>
<td>-0.00000770</td>
<td>1.45369258</td>
</tr>
<tr>
<td>SMM ($N_1 = 126$, $N_2 = 276$) with uniform grid on $\Omega_i$ and $\Omega_2$ ($c = 0.25$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.46229573</td>
<td>-0.00126445</td>
<td>0.00028903</td>
<td>1.46092296</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.46040177</td>
<td>-0.00061649</td>
<td>0.00022633</td>
<td>0.0000169</td>
<td>1.46008762</td>
</tr>
<tr>
<td>FEM ($N_1 = 600$) with quadratic grid on $\Omega_i$ ($c = 0.5$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.45642056</td>
<td>0.00023057</td>
<td>0.00000392</td>
<td>1.45665504</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.46123788</td>
<td>-0.00050241</td>
<td>0.00002529</td>
<td>-0.0000012</td>
<td>1.46076065</td>
</tr>
<tr>
<td>SMM ($N_1 = 126$, $N_2 = 276$) with uniform grid on $\Omega_1$ and quadratic grid on $\Omega_2$ ($c = 0.25$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1.45283436</td>
<td>0.00033411</td>
<td>0.00000160</td>
<td>1.45350898</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1.45876027</td>
<td>-0.00020241</td>
<td>0.00001091</td>
<td>-0.0000003</td>
<td>1.45839885</td>
</tr>
</tbody>
</table>

the number of nodes on $\Omega_i$ ($i = 1, 2$) as $N_i$. To compare SMM to FEM results the numbers of nodes on $\Omega_1$ and $\Omega_2$, when SMM is used, are computed by dividing the number of nodes on $\Omega_1$ in the FEM case between $\Omega_1$ and $\Omega_2$ and by conserving at least the global density of unknowns on $\Omega_2$, i.e. $N_1$(FEM) $\approx N_1$(SMM) $+ N_2$(SMM) and $N_2$(SMM)/area($\Omega_2$) $\geq N_1$(FEM)/area($\Omega_1$).

5.1. Mode 1

The first Mode I problem considered here is a region near the tip of a crack, as shown in figure 7. It is loaded by traction’s given by an analytic solution corresponding to the first symmetric mode of stress intensity factor solution

$$\begin{align*}
\sigma_x &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\
\sigma_y &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left( 1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \\
\tau_{xy} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}
\end{align*}$$

(16)

Plane strain conditions are assumed. Poisson’s ratio $\nu$ is taken as equal to 0.3 and Young’s modulus $E$ is chosen as equal to 1.0. Only half of the square region is analyzed because of symmetry. Likewise, as in Lo and Lee (1992) we set $K_I = \sqrt{2\pi} = 2.50663$ as the exact value.

The results summarized in table I agree with the exact value. For comparison, the results reported by Lo and Lee (1992) are 2.49189 using quarter-point displacement technique (QPDT) and 2.49092 with the displacement correlation technique (DCT). In addition, the accuracy of the SMM results is very satisfactory compared with conventional finite element methods. For example, in the case of a uniform grid, the value of SIF obtained by FEM is within 4.7% for $n = 3$, whilst the SMM gives a value which is within 1.3%.

The second Mode I problem is a double-edge crack (figure 12). The theoretical SIF value in this case is given in Tada et al. (1973)

$$K_I = T(\pi a)^{\frac{1}{2}} \left(1 + 0.122 \left(\cos \frac{\pi a}{2b}\right)^4 \left(\frac{2b}{\pi a} \tan \frac{\pi a}{2b}\right)^{\frac{1}{2}}\right)$$

(17)
Concerning the numerical values of stress intensity factors

Table III. Values of $A_k^{11}$ and the SIF approximation for example 1 in Mode II ($l = 1$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_0^{11}$</th>
<th>$A_1^{11}$</th>
<th>$A_1^{11}$</th>
<th>$A_1^{11}$</th>
<th>$K_1^{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM ($N_1 = 325$) with uniform grid on $\Omega_1 (c = 1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.43973357</td>
<td>-0.01899395</td>
<td>-0.00621103</td>
<td>2.37690156</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.47798984</td>
<td>-0.04298994</td>
<td>-0.00195405</td>
<td>2.38251278</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 45, N_2 = 276$) with uniform grid on $\Omega_1$ and $\Omega_2 (c = 0.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.48547650</td>
<td>0.00433960</td>
<td>0.00172080</td>
<td>2.49484400</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.52125601</td>
<td>-0.00912532</td>
<td>0.00160537</td>
<td>2.50908688</td>
<td></td>
</tr>
<tr>
<td>FEM ($N_1 = 325$) with quadratic grid on $\Omega_1 (c = 1)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<td>-0.00264769</td>
<td>0.00000685</td>
<td>2.47574657</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.48403973</td>
<td>-0.00310798</td>
<td>0.00002027</td>
<td>2.47790429</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 45, N_2 = 276$) with uniform grid on $\Omega_1$ and quadratic grid on $\Omega_2 (c = 0.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.50102398</td>
<td>-0.00190209</td>
<td>0.00000553</td>
<td>2.49900769</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2.50231888</td>
<td>-0.00151506</td>
<td>0.00000351</td>
<td>2.50003069</td>
<td></td>
</tr>
</tbody>
</table>

Table IV. Values of $A_k^1$ and the SIF approximation for the mixed-mode problem of Section 5.3, uniform grid case ($l = 3.5$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_0^1$</th>
<th>$A_1^1$</th>
<th>$A_2^1$</th>
<th>$A_3^1$</th>
<th>$K_1^{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM ($N_1 = 735$) with uniform grid on $\Omega_1 (c = 3.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30.5788807</td>
<td>0.9843985</td>
<td>-0.1799847</td>
<td>31.3832946</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>32.2828912</td>
<td>-0.2918429</td>
<td>0.0930510</td>
<td>32.0616160</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 135, N_2 = 529$) with uniform grid on $\Omega_1$ and $\Omega_2 (c = 1.75)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30.2500572</td>
<td>0.6443734</td>
<td>-0.0546125</td>
<td>31.3293540</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>32.6465628</td>
<td>-0.17426440</td>
<td>0.0246595</td>
<td>32.3795738</td>
<td></td>
</tr>
<tr>
<td>FEM ($N_1 = 735$) with uniform grid on $\Omega_1 (c = 3.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.9867766</td>
<td>0.3158545</td>
<td>-0.0636939</td>
<td>4.2389371</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.5245932</td>
<td>-0.0889389</td>
<td>0.0224840</td>
<td>4.4530277</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 135, N_2 = 529$) with uniform grid on $\Omega_1$ and $\Omega_2 (c = 1.75)$</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>3.7351506</td>
<td>0.2194670</td>
<td>-0.0198433</td>
<td>4.0947111</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>4.5496973</td>
<td>-0.0598282</td>
<td>0.0072019</td>
<td>4.4530153</td>
<td></td>
</tr>
</tbody>
</table>

Table V. Values of $A_k^1$ and the SIF approximation for the mixed-mode problem of Section 5.3, quadratic grid case ($l = 3.5$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$A_0^1$</th>
<th>$A_1^1$</th>
<th>$A_2^1$</th>
<th>$A_3^1$</th>
<th>$K_1^{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM ($N_1 = 651$) with quadratic grid on $\Omega_1 (c = 3.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>34.1662086</td>
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<td>-0.0011319</td>
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</tr>
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<td>3</td>
<td>33.16722594</td>
<td>-0.0289430</td>
<td>-0.0032346</td>
<td>33.1280329</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 135, N_2 = 529$) with uniform grid on $\Omega_1$ and quadratic grid on $\Omega_2 (c = 1.75)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>34.2859850</td>
<td>-0.1556701</td>
<td>-0.002845</td>
<td>33.9735065</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>32.8344520</td>
<td>-0.0242515</td>
<td>0.0002563</td>
<td>32.7757459</td>
<td></td>
</tr>
<tr>
<td>FEM ($N_1 = 651$) with quadratic grid on $\Omega_1 (c = 3.5)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.7968715</td>
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<td>-0.0003742</td>
<td>4.7030499</td>
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</tr>
<tr>
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<td>4.4525128</td>
<td>-0.00179979</td>
<td>-0.0035430</td>
<td>4.4399334</td>
<td></td>
</tr>
<tr>
<td>SMM ($N_1 = 135, N_2 = 529$) with uniform grid on $\Omega_1$ and quadratic grid on $\Omega_2 (c = 1.75)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>-0.0001002</td>
<td>4.8210900</td>
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</tr>
<tr>
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<td>4.4154688</td>
<td>-0.0102750</td>
<td>0.0009186</td>
<td>4.3912651</td>
<td></td>
</tr>
</tbody>
</table>
Figure 7. Problem 1 in Mode I and the associated domains.

Figure 8. FEM results for the first example in Mode I.
As in Lo and Lee [17], plane strain conditions are assumed and we take $E = 1$, $\nu = 0.3$, $T = 1$ and $a = 0.5$. Thus we can take $K_I = 1.45735$ as the exact value. Taking into account the symmetry of the problem, only one quarter of the domain is analyzed.

As in first example, the results listed in table II show the good accuracy of the present technique for extracting the SIF. Although we have considered only the values obtained on the first five contours from the crack tip, the estimation obtained for the SIF is very satisfactory. Lo and Lee [17] reported 1.44243 with the DCT and 1.49467 with the quarter-point displacement technique (QPDT). The percentage errors for the quadratic grid (QG) used in this example are 0.23% for FEM and 0.07% for SMM, with $n = 3$.

The results of these two problems are plotted in figures 8, 9, 10 and 11. Although we only use the SIF values obtained on the first five contours from the crack tip in our polynomial approximation for $n = 3$, we observed a good correlation between all computed and interpolation values.

5.2. Mode II

For Mode II problems, we consider the same region and the same elastic parameters as in the first Mode I example. The region is loaded by tractions given by an analytical solution corresponding to the shearing Mode
Figure 10. FEM results for the second example in Mode I.

Figure 11. SMM results for the second example in Mode I.
Concerning the numerical values of stress intensity factors

(a) Double edge crack

(b) Domain $\Omega_1$ with the half area of square $R$

(c) SMM analysis domains $\Omega_1$, $\Omega_2$ and $\Lambda$ for second problem in mode I

Figure 12. Problem 2 in Mode I and the associated domains.
of the stress intensity factor solution

\[
\begin{align*}
\sigma_x &= -\frac{K_{II}}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \left( 2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\
\sigma_y &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\theta}{2} \cos \frac{\theta}{2} \right) \\
\tau_{xy} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \left( \frac{\theta}{2} \right) \left( 1 - \sin \frac{\theta}{2} \sin \frac{\theta}{2} \right)
\end{align*}
\]
Concerning the numerical values of stress intensity factors

We take $K_{II} = \sqrt{2\pi} = 2.50663$ as the exact value. In the same manner, one can observe that the results in table III are in good accordance with the exact value of the SIF. The results of this problem are plotted in figures 13 and 14.

5.3. Mixed mode

We consider here the problem studied in Stern et al. (1976) and Ogen and Schiff (1985) shown in figure 15. A rectangular sheet containing a symmetrically placed edge crack is subjected to mixed-mode loading. Tables IV and V show that our results agree with those reported by Stern et al. (1976) ($K_I = 33.2$ and $K_{II} = 4.50$), who have used the contour integral from which our approach is derived, and those given by Ogen and Schiff

![Figure 15](image)

Figure 15. Cracked plate with mixed-mode loading and associated domains.

![Figure 16](image)

Figure 16. FEM results for the mixed-mode problem of Section 5.3.
(1985) ($\overline{K}_I = 33.1$ and $\overline{K}_{II} = 4.36$) by using constrained finite elements. In figures 16 and 17 we plot the results of this problem and the graphs of $H_I$ and $H_{II}$ for $n = 3$.

6. Conclusion

A new approach for the use of path-independent integrals in the determination of SIF has been presented. It has allowed us to explain the numerical dependence between path independent integrals and SIF. Taking into account this dependence, the estimation of stress intensity factors can be made by using contours near the crack tip without high resolution in this region. The numerical examples presented here show that this approach combined with the SMM is an efficient tool for stress intensity determination. The additional computations necessary to use this approach in existing finite element codes are negligible compared to those using specialized finite elements. Furthermore, if a quadratic grid is used, good accuracy can be obtained for the numerical results.

Finally, we notice that the main goals of our further investigation are the extension of SMM to three-dimensional problems and the introduction of other trial functions in the SMM approximation. This work is in progress.

References

Concerning the numerical values of stress intensity factors


