Lateral vibrations of an axially compressed beam on an elastic half-space due to a moving lateral load

A.V. Metrikine *, H.A. Dieterman †

Faculty of Civil Engineering, Delft University of Technology, Stevinweg 1, 2628 CN Delft, The Netherlands
(Received 26 September 1997; revised and accepted 19 December 1997)

Abstract – The steady-state response of an axially compressed Euler–Bernoulli beam on an elastic half-space due to a uniformly moving lateral load has been investigated. It is assumed that the beam has a finite width and that the half-space and beam deflections are equal along the center line of the beam. To analyze the problem, firstly the equivalent lateral stiffness of the half-space is derived as a function of the phase velocity of waves in the beam. Then using the expressions for the equivalent stiffness, a dispersion relation is obtained for the lateral waves in the beam. Analyzing this equation, it is shown that lateral waves can propagate in the beam only when the axial force in the beam is larger than a 'cut-off compressional force'. The critical (resonance) velocities of a uniformly moving constant and harmonically varying load are determined as functions of the axial compressional force in the beam. It is shown that the critical velocity of the harmonically varying load is always smaller than that of the constant load. A comparison is made between the critical velocity of a vertical and lateral constant load showing that the lateral constant load is smaller. © Elsevier, Paris

lateral vibration / axially compressed beam / elastic half-space / moving lateral load / railroad track

1. Introduction

The response of an axially-compressed railroad track due to a uniformly moving constant load has been derived firstly by Kerr (1972) using the model of a beam on a Winkler foundation and extended by Labra (1975) who considered the model of a beam on an elastic half-space. The investigations were initiated by the introduction of continuously welded track where compressional axial forces are introduced due to an increase in temperature. It was shown in the papers that the compressional stresses may reduce the critical velocity of the track to the range of operational velocities of modern high-speed trains. Both investigations were related to the vertical track vibrations under a vertical constant load.

The purpose of this paper is to analyze vibrations of the track under a lateral moving load. It is of practical interest, since the lateral vibrations may serve as an misalignment and substantially reduce the critical buckling temperature (Samavedam et al., 1993; Van, 1996). There are two types of moving lateral loads. The first one is the so-called Klingel lateral motion of a train (Esveld, 1989) arising due to conical profiles of the train wheels. Since Klingel motion is periodic, the load given by the motion varies harmonically. The second type of loading originates if two parallel tracks are located near each other and a train moves along one track. Then the deformation field on the track excited by the moving train will serve as a constant lateral load for the other track. Both loads are comparatively small, but if they move with resonance velocities, the dynamic amplification of the track vibration can be substantial.

* Correspondence and reprints
† Deceased June 1998
To determine the critical velocities of the loads, we investigated a uniform motion of a harmonically varying lateral load along a beam on an elastic half-space (when the load frequency is equal to zero there is the constant load as a special case). A three-dimensional model of the track subsoil was chosen since it gives more realistic results, especially if the train moves with a velocity close to the that of Rayleigh waves (Filippov, 1961; Labra, 1975; Dieterman and Metrikine, 1996). It is assumed that the beam has a finite width, that stresses are uniformly distributed over the beam width and that the half-space and beam displacements are equal along the center line of the beam. In the frame of the model it can be shown that lateral beam vibrations are uncoupled from the vertical and longitudinal vibrations, and that we can therefore consider lateral beam vibrations independently (Metrikine and Dieterman, 1997).

In analyzing the model, we have used the concept of the half-space equivalent stiffness (Dieterman and Metrikine, 1996). The lateral stiffness of the half-space as a function of phase velocity of waves in the beam has been obtained and discussed. With the help of expressions for the equivalent stiffness the initial 3-D problem has been reduced to a one-dimensional issue of beam vibrations on a 1-D elastic foundation with frequency and wave-length dependent stiffness. Then the steady-state displacement of the beam under the moving load is found in the form of a single integral with respect to wave number in the beam. The velocities of the load at which the integral diverges are the critical (resonance) velocities. It is shown that resonance occurs if the load velocity is equal to the group velocity of the waves in the beam, so the necessary condition for resonance is that waves have to be able to propagate in the beam. The investigation shows that lateral waves can propagate in the beam only when the compressional force is larger than a 'cut-off compressional force'. Thus resonance can occur when the compressional force is larger than this 'cut-off force'.

The critical velocities are determined as functions of the compressional force in the beam. The critical velocities related to Klingel motion (harmonically varying load) are found to be smaller than the critical velocities of a constant load. A comparison of the critical velocity of the lateral and the vertical constant load further shows that the lateral motion results in smaller critical velocities than the vertical motion. This difference grows as the compressional force increases.

2. Model

We consider a harmonically varying load \( P \exp(i\Omega t) \) moving uniformly along an axially compressed Euler–Bernoulli beam with a width \( 2a \) on an elastic half-space as depicted in figure 1. It is assumed that stresses at the interface are uniformly distributed along the beam width and the displacements of the half-space surface are equal to the beam displacements along the beam center line. In the frame of the model a lateral force acting on the beam cannot generate either vertical or longitudinal beam vibrations. Indeed, for the Euler–Bernoulli model free vibrations of the beam in vertical, longitudinal and lateral directions are uncoupled, so the coupling can take place only due to the beam interaction with the half-space. Further, in our model the half-space can cause beam vibrations only along the line \( x = 0, z = 0 \) (the beam center line). Suppose, that there is an \( x \)- or a \( z \)-deflection of the beam center line under the lateral force; then due to the symmetry, this deflection will be in the same direction both for positive \( P \) and negative \( (-P) \) lateral forces. This means that if we apply the force \( F = P + (-P) = 0 \) we would have a nonzero deflection for the linear system. So, we can conclude by contradiction that lateral beam vibrations are uncoupled from the vertical and longitudinal vibrations and therefore longitudinal (\( x \)-direction) and vertical (\( z \)-direction) stresses at the half-space interface can be taken as being equal to zero when a lateral load acts on the beam only.
Lateral load along beam on half-space

Then the governing equations for the problem can be written as follows, i.e. the half-space motion using scalar $\Phi(x, y, z, t)$ and vector $\vec{\Psi}(x, y, z, t)$ potentials is determined by the equations

$$
\Delta \Phi = \frac{1}{c_L^2} \frac{\partial^2}{\partial t^2} \Phi, \quad \Delta \vec{\Psi} = \frac{1}{c_T^2} \frac{\partial^2}{\partial t^2} \vec{\Psi}, \quad \text{div}(\vec{\Psi}) = 0
$$

(1)

where $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_T = \sqrt{\mu/\rho}$ are the velocities of compressional and shear waves respectively, $\lambda$ and $\mu$ are Lamé's constants for the elastic half-space and $\rho$ is its mass density. The balance of stresses at the surface of the half-space reads as follows

$$
2a\tau_{yz}(x, y, 0, t) = \left( m \frac{\partial^2 V^0}{\partial t^2} + 2EI \frac{\partial^4 V^0}{\partial x^4} + \tilde{N} \frac{\partial^2 V^0}{\partial x^2} - P \exp(-i\Omega t)\delta(x - \alpha t) \right) H(a - |y|)
$$

$$
\tau_{xz}(x, y, 0, t) = \sigma_{zz}(x, y, 0, t) = 0
$$

(2)

Here $V^0$ is the lateral beam displacement, $m, E, 2I$ is the beam mass per unit length. Young's modulus and lateral moment of the cross-sectional inertia respectively, $\tilde{N}$ is the compressional axial force; $H(\cdot), \delta(\cdot)$ is the unit step function and the Dirac delta-function. We denote the compressional force as $\tilde{N}$ to underline that if we apply our model to a railroad track then the force is not equal to a temperature force in the rails, but also depends on the stiffness of the rail-tie structure (Kerr and El-Sibaie, 1987).

![Figure 1. Study model and reference system.](image)

The compatibility condition (equality of the half-space and beam lateral deflection along the center line of the beam) is

$$
V(x, 0, 0, t) = V^0(x, t)
$$

(3)

where $V(x, y, z, t)$ is the half-space displacement in the $y$ direction.

3. Lateral steady-state beam response

To obtain an expression for the steady-state lateral vibrations of the beam under the load, we apply the following Fourier transforms with respect to time and horizontal spatial coordinates

$$
\{ f(k_1, k_2, z, \omega), \vec{g} \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ \Phi(x, y, z, t), \vec{\Psi}(x, y, z, t) \} \exp(\omega t - k_1 x - k_2 y) dt dx dy
$$

$$
h_y(k_1, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V^0(x, t) \exp(i(\omega t - k_1 x)) dt dx
$$

(4)
This gives the following system of equations in the Fourier domain: for the half-space motion [from Eq. (1)]
\[
\frac{\partial^2 f}{\partial z^2} + \left(\frac{\omega^2}{c_L^2} - k_1^2 - k_2^2\right) f = 0, \quad \frac{\partial^2 g}{\partial z^2} + \left(\frac{\omega^2}{c_T^2} - k_1^2 - k_2^2\right) g = 0, \quad \frac{\partial g_z}{\partial z} + i k_1 g_x + i k_2 g_y = 0
\]  
(5)
and for the stress balance at \( z = 0 \) [from Eq. (2) using the expression for \( \tau_{gz} \) given in Achenbach (1973)]
\[
\left\{ \begin{array}{l}
2ik_1 \frac{\partial f}{\partial z} + \frac{\partial}{\partial z} \left( ik_2 g_z - \frac{\partial g_y}{\partial z} \right) + ik_1 (ik_1 g_y - ik_2 g_r) \\
2ik_2 \frac{\partial f}{\partial z} - \frac{\partial}{\partial z} \left( ik_1 g_z - \frac{\partial g_x}{\partial z} \right) + ik_2 (ik_1 g_y - ik_2 g_r)
\end{array} \right\}
\bigg|_{z=0} = 0
\]
\[
\left\{ \begin{array}{l}
-\frac{\lambda \omega^2}{c_L^2} f + 2\mu \left( \frac{\partial^2 f}{\partial z^2} + i \frac{\partial}{\partial z} (k_1 g_y - k_2 g_r) \right)
\end{array} \right\}
\bigg|_{z=0} = 0
\]
(6)
where
\[
D_y(k_1, \omega) = -m \omega^2 + 2EI_L k_1^4 - \tilde{N} k_1^2
\]
(7)
is the dispersion relation for the lateral vibrations of a compressed free beam. The following integral representation has been used to obtain the delta function (Korn and Korn, 1961)
\[
2\pi \delta(s) = \int_{-\infty}^{\infty} \exp(\pm isq) dq
\]
for the compatibility condition (from Eq. 3)
\[
h_y(k_1, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \nu(k_1, k_2, 0, \omega) dk_2
\]
(8)
where
\[
\nu(k_1, k_2, z, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} V(x, y, z, t) \exp(i(\omega t - k_1 x - k_2 y)) dt dk_1 dk_2
\]
is the Fourier image of the lateral displacement of the half-space.

The general solutions of the wave equations in (5), accounting for the proper behavior for large positive values of \( z \), are
\[
f = A \exp(-zR_L), \quad \{g_x, g_y, g_z\} = \{B_x, B_y, B_z\} \exp(-zR_T), \quad R_{L,T} = \sqrt{k_1^2 + k_2^2 - \omega^2/c_L^2}
\]
(9)
provided that the branches in the complex domain are chosen such that the radicals have a positive real part, i.e. \( \text{Re}(R_L) > 0, \text{Re}(R_T) > 0 \).

Substitution of (9) into the last equation of (5) and the balance of stresses given by (6) yields the following set of linear algebraic equations
\[
(ik_1)B_x + (ik_2)B_y + (-R_T)B_z = 0
\]
\[
(-2ik_1 R_L)A + (k_1 k_2)B_x + (-k_1^2 - R_T^2)B_y + (-ik_2 R_T)B_z = 0
\]
\[
(-2ik_2 R_L)A + (k_2^2 + R_T^2)B_x + (-k_1 k_2)B_y + (ik_1 R_T)B_z = H_y
\]
\[
(2(k_1^2 + k_2^2) - \omega^2/c_T^2)A + (2ik_2 R_T)B_x + (-2ik_1 R_T)B_y = 0
\]
(10)
where
\[
H_y = \frac{1}{\mu} \left( h_y D_y - 2\pi P \delta(\omega - \Omega - \alpha k_1) \right) \frac{\sin(ak_2)}{ak_2}
\]
When solved for \( A \) and \( \{B_x, B_y, B_z\} \), system (10) gives
\[
A = \Delta / \Delta, \quad \{B_x, B_y, B_z\} = \{\Delta_{B_x}, \Delta_{B_y}, \Delta_{B_z}\} / \Delta
\]
\[
\Delta = -R_T \frac{\omega^2}{c_T^2} \left( \left( 2k_1^2 + 2k_2^2 - \frac{\omega^2}{c_T^2} \right)^2 - 4(k_1^2 + k_2^2)R_L R_T \right), \quad \Delta_A = -2ik_2 R_T^2 \frac{\omega^2}{c_T^2} H_y
\]
\[
\Delta_{B_x} = -R_T \left( 4k_1^2 R_L R_T + \left( \frac{\omega^2}{c_T^2} - 2k_1^2 \right) \left( 2k_1^2 + 2k_2^2 - \frac{\omega^2}{c_T^2} \right) \right) H_y
\]
\[
\Delta_{B_y} = 2k_1 k_2 R_T \left( 2k_1^2 + 2k_2^2 - \frac{\omega^2}{c_T^2} - 2R_L R_T \right) H_y
\]
\[
\Delta_{B_z} = ik_1 \left( \left( 2k_1^2 + 2k_2^2 - \frac{\omega^2}{c_T^2} \right)^2 - 4R_L R_T (k_1^2 + k_2^2) \right) H_y
\]

(11)

In order to obtain an equation for the beam lateral vibrations on the half-space we have to apply the compatibility condition (8). The Fourier image of the lateral half-space displacement at the surface has the form (the expression for the elastic space displacement in the \( y \)-direction as function of the scalar and vector potentials is given in Achenbach (1973))
\[
\nu(k_1, k_2, 0, \omega) = ik_2 A - ik_1 B_z - R_T B_x
\]

(12)

Now employing (11) we find
\[
\nu(k_1, k_2, 0, \omega) = -\frac{1}{R_T} \left( 1 + \frac{k_2^2}{\Delta_0} \left( 4R_L R_T + 3 \frac{\omega^2}{c_T^2} - 4(k_1^2 + k_2^2) \right) \right) H_y
\]

(13)

where
\[
\Delta_0 = \left( 2k_1^2 + 2k_2^2 - \frac{\omega^2}{c_T^2} \right)^2 - 4(k_1^2 + k_2^2)R_L R_T
\]

Finally, substituting (13) into the compatibility conditions (8) and using the expression for \( H_y \) [see Eq. (10)], we obtain the following equation (describing the steady-state lateral vibrations of the beam on the elastic half-space under the moving load in Fourier domain)
\[
h_y = -\frac{I_{22}}{2\pi \mu}(h_y D_y - 2\pi P \delta(\omega - \Omega - \alpha k_1))
\]

(14)

where
\[
I_{22}(k_1, \omega) = \int_{-\infty}^{\infty} \frac{1}{R_T} \left( 1 + \frac{k_2^2}{\Delta_0} \left( 4R_L R_T + 3 \frac{\omega^2}{c_T^2} - 4(k_1^2 + k_2^2) \right) \right) \frac{\sin(ak_2)}{ak_2} dk_2
\]

(15)

Introducing the lateral equivalent stiffness of the half-space as (see Dieterman and Metrikine, 1996)
\[
\chi_L(k_1, \omega) = \frac{2\pi \mu}{I_{22}(k_1, \omega)}
\]

and substituting the expression for \( D_y \), Eq. (14) can be rewritten in the form
\[
h_y(-m \omega^2 + 2EI_L k_1^4 - \bar{N} k_1^2 + \chi_L(k_1, \omega)) = 2\pi P \delta(\omega - \Omega - \alpha k_1)
\]

(16)
Applying finally the inverse Fourier transform with respect to $\omega$ and $k_1$ to (16), we obtain the following expression for steady-state lateral beam displacement

$$V^0(x, t) = \frac{P}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta(\omega - \Omega - \alpha k_1) \exp(-i(\omega t - k_1 x))}{w^2 + 2E I L k_1^4 - \tilde{N} k_1^2 + \chi_L(k_1, \Omega) + \exp(i k_1(x - \alpha t))} \, dk_1 \, d\omega$$

(17)

4. Equivalent lateral stiffness of the half-space

To analyze the lateral beam displacement given by (17) we have to elaborate the lateral equivalent stiffness $\chi_L(k_1, \omega)$ of the half-space which is inversely proportional to $I_{22}(k_1, \omega)$. Introducing a new variable of integration $\xi = k_2/k_1$, denoting $V_{ph} = \omega/k_1, \beta_{L,T} = V_{ph}/c_{L,T}, K = ak_1V_{ph}$ is the phase velocity of waves in the beam, $K$ is a dimensionless wave number of waves in the beam), we rewrite the integral $I_{22}(k_1, \omega)$ in the form

$$I_{22} = \frac{1}{K} \int_{-\infty}^{\infty} \frac{1}{R_T(\xi)} \left(1 + \frac{\xi^2}{\Delta_0(\xi)} \left(4R_L(\xi)R_T(\xi) + 3\beta_T^2 - 4(1 + \xi^2)\right)\right) \sin(K\xi) \frac{d\xi}{\xi}$$

(18)

The integral (18) is the sum of the two following integrals

$$I_{22} = I_{22}^{(1)} + I_{22}^{(2)}, \quad I_{22}^{(1)} = \frac{1}{K} \int_{-\infty}^{\infty} \frac{1}{R_T(\xi)} \sin(K\xi) \frac{d\xi}{\xi}$$

$$I_{22}^{(2)} = \frac{1}{K} \int_{-\infty}^{\infty} \frac{1}{R_T(\xi)} \left(1 + \frac{\xi^2}{\Delta_0(\xi)} \left(4R_L(\xi)R_T(\xi) + 3\beta_T^2 - 4(1 + \xi^2)\right)\right) \sin(K\xi) \frac{d\xi}{\xi}$$

The appropriate form of $I_{22}^{(2)}$ for numerical calculations is given in the Appendix. We do not pay attention to the process of evaluation of this integral since it is completely analogous to that given in Dieterman and Metrikine (1996) for the vertical equivalent stiffness of the half-space. Let us not only that this integral is finite for an arbitrary phase velocity of waves in the beam, as can be seen from the results given in the Appendix.

The appropriate expression for $I_{22}^{(1)}$ is also given in the Appendix, but it is important to show separately that this integral diverges when $\beta_T \to 1$ (phase velocity of waves in the beam tends to the velocity of shear waves in the half-space). Indeed, using the even character of the integrand and substituting $R_T$ we can rewrite $I_{22}^{(1)}$ in the form

$$I_{22}^{(1)} = \frac{2}{K} \int_{0}^{\infty} \frac{1}{\sqrt{1 + \xi^2 - \beta_T^2}} \sin(K\xi) \frac{d\xi}{\xi}$$

Evidently this integral converges if $\beta_T \neq 1$. Indeed, the singularity $\xi = \sqrt{\beta_T^2 - 1}$ is integrable in this case and for $\xi \to \infty$ the integrand decreases fast enough (proportionally to $\xi^{-2}$) to be convergent.

For $\beta_T \to 1$ we represent $I_{22}^{(1)}$ as follows

$$I_{22}^{(1)} = 2 \int_{0}^{\varepsilon} \frac{d\xi}{\sqrt{1 + \xi^2 - \beta_T^2}} + \frac{2}{K} \int_{\varepsilon}^{\infty} \frac{1}{\sqrt{1 + \xi^2 - \beta_T^2}} \sin(K\xi) \frac{d\xi}{\xi}$$

(19)

where $\varepsilon \to 0^+$, but $\sqrt{|1 - \beta_T^2|} < \varepsilon$. The second integral in (19) converges since $\sqrt{1 + \xi^2 - \beta_T^2} \neq 0$ for $\xi \in [\varepsilon; \infty]$ and for $\xi \to \infty$ its integrand decreases fast enough. The first integral in (19) can be taken analytically...
(Gradshteyn and Ryzhik, 1994), it yields

$$2 \int_{\xi}^{\epsilon} \frac{d\xi}{\sqrt{1 + \xi^2 - \beta_T^2}} = 2 \ln(\epsilon + \sqrt{1 + \epsilon^2 - \beta_T^2}) - \ln(1 - \beta_T^2) \quad (20)$$

Obviously the expression at the right-hand side of Eq. (20) tends to infinity as $\beta_T \to 1$ ($\epsilon$ is not equal to zero). Thus

$$\lim_{\beta_T \to 1} I_{22} = \lim_{\beta_T \to 1} I_{22}^{(1)} = \infty \Rightarrow \lim_{\beta_T \to 1} \chi_L = 0$$

which implies that the lateral stiffness of the half-space is equal to zero when the phase velocity of waves in the beam is equal to the velocity of shear wave in the half-space.

In figure 2 the dependency of the lateral equivalent stiffness versus the ratio of the beam-wave phase velocity and the shear wave velocity in the half-space is plotted for $K = 0.5$

![Figure 2. Equivalent lateral stiffness of the half-space for $K = 0.5$.](image)

Figure 2 shows that when the phase velocity of waves in the beam is smaller than the Rayleigh wave velocity ($V_{ph} < c_R$), the imaginary part of the equivalent stiffness is equal to zero. This is due to the fact that such ‘slow’ waves in the beam do not generate waves in the half-space and the half-space reaction is purely elastic. As $V_{ph} > c_R$ the equivalent stiffness has a nonzero imaginary part, since the beam vibrations excite waves in the half-space. First, only a Rayleigh wave is generated ($c_R < V_{ph} < c_T$); then the Rayleigh and shear waves ($c_T < V_{ph} < c_L$), and lastly the Rayleigh, shear and longitudinal waves ($c_L < V_{ph}$). Therefore for $V_{ph} < c_R$ the reaction of the half space has a visco-elastic character, since waves radiated into the half-space waves play the role of an equivalent viscosity. It is important to point out again that the lateral equivalent stiffness is equal to zero when $V_{ph} = c_T$, in contrast with the vertical equivalent stiffness which tends to zero as $V_{ph} \to c_R$.

5. Critical velocities of lateral loads

Now we come back to expression (17) which describes the steady-state lateral displacement of the beam on a half-space. Our goal is to determine the critical velocities of the load at which the steady-state beam displacement $V^0(x, t)$ is infinite. According to (17), $V^0(x, t)$ is determined by the following integral

$$V^0(x, t) = \frac{P}{2\pi} \exp(-i\Omega t) \int_{-\infty}^{\infty} \frac{\exp(ik_1(x - \alpha t))}{-m(\Omega + \alpha k_1)^2 + 2EI_Lk_1^4 - \bar{N}k_1^2 + \chi_L(k_1, \Omega + \alpha k_1)} dk_1$$

(21)
and the load velocities related to the divergence of the integral are the critical velocities.

The integrand in (21) decreases proportionally to $1/k_1^4$ as $k_1 \to \pm \infty$ so the divergence of the integral cannot be related to integration at infinity. Therefore the only possibility for the divergence is that the denominator of the integrand has some real finite zero $k_1^*$. Moreover this real zero has to be of the second or higher order, since for the simple zero the integral converges in the Cauchy principle sense (Fichtenholz, 1964). Physically, the existence of a simple real zero of the denominator in (21) implies that the moving load generates waves in the beam.

Thus, to find out the critical velocities of the load we have to determine the load velocities $\alpha(\Omega)$ for which the equation

$$F(k_1, \alpha(\Omega)) = -\omega(\Omega + \alpha k_1)^2 + 2EI_L k_1^4 - \tilde{N} k_1^2 + \chi_L(k_1, \Omega + \alpha k_1) = 0 \tag{22}$$

has a real root $k_1^*$ of at least the second order. This condition has the clear physical background. Indeed, Eq. (22) is equivalent to the following system of equations

$$\begin{cases} D(k_1, \omega) = -m \omega^2 + 2EI_L k_1^4 - \tilde{N} k_1^2 + \chi_L(k_1, \omega) = 0 \\ \omega = \Omega + \alpha k_1 \end{cases} \tag{23}$$

The first equation in (23) is the dispersion equation for lateral waves in the beam on the half-space. The second equation gives the relation between the frequency $\Omega$ of the moving load and the frequency $\omega$ of a wave generated by this load. These frequencies are not equal due to the Doppler effect (Ginzburg, 1979). Each real pair of roots $(\omega^*, k_1^*)$ of the system (23) determines the frequency $\omega^*$ and the wave number $k_1^*$ of a wave generated by the moving load. Geometrically, $\omega^*$ and $k_1^*$ are the coordinates [in the plane $(\omega, k_1)$] of the crossing point of the dispersion curve $D(k_1, \omega) = 0$ and the straight line $\omega = \Omega + \alpha k_1$ which is usually called the ‘kinematic invariant’ (Vesnitskii, 1991). Evidently a root $(\omega^*, k_1^*)$ will be multiple and consequently the integral (21) is divergent if the dispersion curve and the ‘kinematic invariant’ are tangential. The condition of the tangential behavior can be written in the following form

$$\frac{\partial \omega}{\partial k_1} \bigg|_{\omega = \Omega + \alpha k_1} = \alpha \tag{24}$$

where $\omega(k_1)$ is the solution of the dispersion equation $D(k_1, \omega) = 0$. Equation (24) determines the critical (resonance) velocities of the load. Physically it implies that resonance takes place if the group velocity $\frac{\partial \omega}{\partial k_1}$ of a wave radiated by the load is equal to the velocity of the load $\alpha$.

In figure 3 the dispersion curves $\omega(k_1)$ (the real solution of the equation $-m \omega^2 + 2EI_L k_1^4 - \tilde{N} k_1^2 + \chi_L(k_1, \omega) = 0$) are depicted for different values of $N$. The parameters of the beam and the half-space are taken as follows: Poisson’s ratio of the half-space $\nu = 0.3$; Lamé’s coefficient of the half-space $\mu = 3.27 \times 10^7 N/m^2$; half-space density $\rho = 1.96 \times 10^3 kg/m^3$; the beam mass per unit length $m = 108.5 kg/m$; the beam Young’s modulus $E = 2.06 \times 10^{11} N/m^2$; the beam lateral moment of inertia $2I_L = 7.18 \times 10^{-6} m^4$; the beam width $2a = 1.6 m$. The beam parameters represent a practical railroad track (rails and wooden sleepers).

Analyzing the figure, the following conclusions can be drawn.

1) Waves propagate in the beam if its phase velocities are smaller than the Rayleigh wave velocity in the half-space. If the phase velocity of a beam-wave is higher than the Rayleigh wave velocity, the equivalent stiffness of the half-space possesses an imaginary part (see the previous section) and the beam waves decay.

2) For the chosen parameters the lateral waves can propagate in the beam only if the compressional force is larger than a ‘cut-off compressional force’ $N^*$ which is approximately equal to $0.6N_{cr}$. The critical compressional force $N_{cr}$ is the force related to the static lateral buckling of the beam. The dispersion curve drawn for $\tilde{N} = N_{cr}$ is tangential to the axis $\omega = 0$.

Now we will determine the critical (resonance) velocities of a constant and harmonically varying lateral load which moves uniformly along the beam. As it is mentioned earlier, the resonance takes place when the ‘kinematic
Figure 3. Dispersion curves of lateral waves in the beam on the half-space for different compressional forces.

invariant’ is tangential to the dispersion curve. For a constant load, the ‘kinematic invariant’ is given as \( \omega = \alpha k_1 \) [see Eq. (23) for \( \Omega = 0 \)] and for harmonically varying as \( \omega = \Omega + \alpha k_1 \). We consider in our paper a special type of a harmonically varying load which is related to the Klingel lateral motion of a train. In this case the frequency of the load is proportional to the train velocity and the ‘kinematic invariant’ can be written as \( \omega = (q + k_1)\alpha \), where \( q = \sqrt{2\gamma/r_s} \), \( \gamma \) is the conicity of the wheel tread (inclination), \( r \) is the radius of the wheels at the center, \( s \) the distance between rails [a detailed description of the Klingel effect is given in Eseveld (1989)]. In figure 4 the geometrical solution of the system (23) in the resonance cases are depicted. Figure 4a is related to a constant lateral load and figure 4b to the harmonically varying (Klingel) load. The parameters of the beam and the half-space are the same as for figure 3, \( \bar{N} = 0.9N_{cr}, \gamma = 0.05, r = 0.45m, s = 1.435m \) (c = 0.39).

Comparing figure 3 and figure 4, we can conclude that the critical velocity of the lateral load (both constant and harmonically varying) is decreasing when the compressional force increases. Further, since the lateral waves can propagate in the beam only when the axial compressional force is larger than the ‘cut-off compressional force’ \( N^* \), resonance in the system can occur only if \( \bar{N} > N^* \). The dependencies of the critical velocity \( \alpha_{cr} \) upon the compressional force \( \bar{N} \) for the harmonically varying Klingel load (line 1) and the constant load (line 2) are depicted in figure 5.

The figure shows that the critical velocity of the harmonically varying Klingel load is always slightly smaller than that of the constant load. It is seen also that line 1 (harmonically varying load) starts not immediately at the ‘cut-off compressional force’ \( N^* \). The reason for this can be understood from figure 4b. Indeed, the ‘kinematic invariant’ \( \omega = (q + k_1)\alpha \) and the dispersion curve can not be tangential for an arbitrary shape of the dispersion curve. The slope of the dispersion curve (for some frequency) has to be small enough in order to allow the tangential line to cross the \( k_1 \)-axes at a point \( k_1^* \leq -q \). This condition in our case is satisfied not directly for \( \bar{N} = N^* \), but for a slightly larger compressional force.

In figure 5 the dependency \( \alpha_{cr}(\bar{N}) \) of the critical velocity of a vertical constant load moving uniformly along the beam on the half-space (line 3) is also plotted. For line 3 we have used the results of Dieterman and Metrikine (1996) in which the vertical equivalent stiffness of the half-space was derived (the results of Labra
Figure 4. Location of the dispersion curve and the ‘kinematic invariant’ for the
critical case (a) is related to a constant load; (b) to the harmonically varying load.

Figure 5. Dependencies of the critical velocity on the compressional force. Lines 1 and 2 refer to the constant and harmonically varying lateral
loads. Line 3 is related to the critical velocity of the vertical constant load (from Labra, 1975; Kerr and Dieterman and Metrikine, 1996).

(1975) can also be used). The parameters of the beam and the half-space are the same as for figure 3, the
beam vertical moment of inertia is $2I_y = 4.146 \times 10^{-5}$ m$^4$ (this is a practical value for a vertical displacement
of a railroad track). In the interval $N \in [N^*; N_{cr}]$, where the lateral waves can cause resonance, it is seen
that the critical velocity of the lateral load can be substantially smaller than that of the vertical load. Note
that the vertical load gives resonance starting with a zero value of the compressional force in contrast with
the lateral load. This is because vertical waves can propagate in the beam even without axial compression (see Dieterman and Metrikine, 1996).

6. Conclusions

In this paper the steady-state lateral vibrations of an axially compressed beam on an elastic half-space due to a uniformly moving lateral force have been investigated. To analyze the beam vibrations the equivalent lateral stiffness of the half-space has been derived. This equivalent stiffness is equal to zero when the phase velocity of waves in the beam is equal to the shear wave velocity in the half-space. Using the expression for the equivalent stiffness the dispersion equation was obtained for the lateral waves in the beam. Analysis of this equation has shown that the lateral waves can propagate in the beam only when the axial force in the beam is larger than a ‘cut-off compressional force’.

The critical (resonance) velocities of a uniformly moving constant and a harmonically varying load have been determined as functions of an axial compressional force in the beam. Resonance takes place only if the compressional force in the beam is larger than the ‘cut-off compressional force’. The condition for resonance is that the load velocity is equal to the group velocity of waves in the beam. It has been shown that the critical velocity for the harmonically varying load is always smaller than that of the constant load.

A comparison of the critical velocities of the vertical and lateral constant loads has been made. In the interval of compressional forces where the lateral load can cause resonance it has been shown that the critical velocity of the lateral load can be substantially smaller than that of the vertical load.

Appendix

\[
I_{22}^{(1)} = \frac{1}{K} \begin{cases} \int_0^\infty \frac{\exp(-K \eta) - 1}{\eta Q_T^2} d\eta, & V_{ph} < c_T \\ -\int_0^\sqrt{\frac{1}{\beta_R^2-1}} \exp(iK\xi) - 1 \xi P_T^- d\xi, & V_{ph} < c_T \end{cases}
\]

\[
I_{22}^{(2)} = \frac{1}{K} \begin{cases} \text{Int}_1, & V_{ph} < c_R \\ \text{Int}_2, & c_R < V_{ph} < c_T \\ \text{Int}_3, & c_T < V_{ph} < c_L \\ \text{Int}_4, & c_L < V_{ph} \end{cases}
\]

\[
\begin{align*}
\text{Int}_1 &= S_1 - 2G_1 - 2G_2, \\
\text{Int}_2 &= S_2 - 2G_1 - 2G_2 + iS_3, \\
\text{Int}_3 &= S_2 - 2G_3 - 2G_4 + iS_3, \\
\text{Int}_4 &= S_2 - 2G_5 - 2G_6 - 2G_7 + iS_3 \end{align*}
\]

\[
S_1 = S_0(\exp(-K \sqrt{1 - \beta_R^2}) - 1), \quad S_2 = S_0(\cos(K \sqrt{\beta_R^2} - 1) - 1), \quad S_2 = S_0 \sin(K \sqrt{\beta_R^2} - 1)
\]

\[
S_0 = \frac{\pi}{2} \frac{4 \tilde{R}_T \tilde{R}_L + 3\beta_T^2 - 4\beta_R^2}{(4\beta_R^2 - 2\beta_T^2 - \beta_T^2 \tilde{R}_L / \tilde{R}_T - \beta_R^2 \tilde{R}_L / \tilde{R}_T - 2\tilde{R}_T \tilde{R}_L)}
\]

\[
G_1 = \int_0^\infty F_1(\eta) d\eta, \quad G_2 = \int_0^\sqrt{1 - \beta_L^2} F_2(\eta) d\eta, \quad G_3 = \int_0^{\sqrt{1 - \beta_L^2}} F_2(\eta) d\eta, \quad G_4 = -\int_0^{\sqrt{\beta_L^2 - 1}} F_3(\xi) d\xi,
\]

\[
G_5 = \int_0^\infty F_1(\eta) d\eta, \quad G_6 = -\int_0^{\sqrt{\beta_L^2 - 1}} F_3(\xi) d\xi, \quad G_7 = -\int_0^{\sqrt{\beta_L^2 - 1}} F_4(\xi) d\xi,
\]
\[ F_1 = -\frac{3\beta_T^2 - 4(1 - \eta^2) - 4Q_T^+Q_L^+}{(2(1 - \eta^2) - \beta_T^2)^2 + 4Q_T^+Q_L^+(1 - \eta^2)} \exp(-Kn) - 1 \eta \]

\[ F_2 = -\frac{(4(Q_T^+)^2 - \beta_T^2)(2(Q_T^+)^2 - \beta_T^2) - 16(1 - \eta^2)(Q_T^+)^2(Q_T^-)^2}{(2(1 - \eta^2) - \beta_T^2)^2 + 16(Q_T^+)^2(Q_T^-)^2(1 - \eta^2)^2} \exp(-Kn) - 1 \eta \]

\[ F_3 = \frac{(4(P_T^-)^2 - \beta_T^2)(2(P_T^-)^2 - \beta_T^2) - 16(1 + \xi^2)(P_T^-)^2(P_T^+)^2}{(2(1 + \xi^2) - \beta_T^2)^2 + 16(P_T^-)^2(P_T^+)^2(1 + \xi^2)^2} \exp(iK\xi) - 1 \xi \]

\[ F_4 = \frac{3\beta_T^2 - 4(1 + \xi^2) - 4P_T^-P_T^+}{(2(1 + \xi^2) - \beta_T^2)^2 + 4P_T^-P_T^+(1 + \xi^2)^2} \exp(iK\xi) - 1 \xi \]

\[ \beta_{L,T,R}^2 = \beta_{L,T,R}^2 \]

\[ Q_{L,T}^\pm = \sqrt{\pm\beta_{L,T}^2 \pm \eta^2 + 1} \quad P_{L,T}^\pm = \sqrt{\pm\beta_{L,T}^2 + \xi^2 \pm 1} \]

In the quasi-static case when \( V_{ph} \ll c_R \), the following expression is adequate to calculate \( \text{Int}_1 \)

\[ \text{Int}_1 = 2 \int_0^\infty \left(1 + \frac{\xi^2(c_T^2/c_T^2 - 2)}{2(1 + \xi^2)(1 - c_T^2/c_T^2)}\right) \sin(K\xi) \sqrt{1 + \xi^2} d\xi \]

References


Esveld C., Modern Railway Track. MRT Productions, Germany, 1989, p. 8.


