Wave propagation in elastic media with cracks.
Part II: Transient nonlinear response of a cracked matrix

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Abstract – The transient dynamic response of an elastic medium containing a random array of aligned penny-shaped cracks is studied. Nonlinearity in the response due to contact between crack faces during the motion is taken into account. The mean value of wave fields is considered and an effective constitutive relation is adopted which contains an internal variable defining the mean opening of the cracks. An integral representation of the effective displacement field is given using the Green function of the uncracked body. One-dimensional nonlinear waves are studied using the solution of the single scattering problem proposed by the authors in a previous paper. Numerical examples showing the role of the characteristic crack dimension on the overall nonlinear response of the cracked medium are presented. © Elsevier, Paris

crack / wave propagation / microstructure / damage

1. Introduction

Wave propagation in an elastic medium in the presence of distributed cracks is of interest in areas of engineering, rock mechanics and geophysics. A theoretical description of the scattering process, accompanied by experimental observations of velocity variation and attenuation of ultrasonic waves, provides a nondestructive means for damage characterisation in the material. A theoretical model can also be useful for estimating the extent of fracturing in the context of oil extraction, or for analysing the seismic behaviour of rock layers.

As for the static response of an isotropic homogeneous solid permeated by a random distribution of circular or elliptic flat cracks, in recent years analytic estimations of the overall elastic moduli have been proposed. The effective moduli were determined either neglecting crack interaction (Garbin and Knopoff, 1973, 1975) in the case of dilute crack concentrations, or by approximating the crack interaction on the basis of a self-consistent approach (Budiansky and O’Connell, 1976), or by introducing a differential scheme (Hashin, 1988) for higher crack densities. In any case, locations and orientations of the cracks were assumed to be sufficiently random so as to render the cracked body isotropic and homogeneous on a scale large compared with the crack dimensions. The extension to the case of preferred orientations which render the body anisotropic in the large has been carried out by Hoenig (1979).

Nevertheless, when the time-harmonic dynamic behaviour is considered, the effects of wave scattering have to be taken into account. In fact, due to the presence of cracks (scatterers), phase velocity and attenuation of the coherent (average) wave become frequency dependent and the effective medium turns out to be dispersive. For dilute concentrations of circular cracks (aligned or randomly oriented), estimates of the attenuation of waves have been given by Piau (1979, 1980), Hudson (1981) and Crampin (1984) for long wavelengths (Rayleigh

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limit) and by Gross and Zhang (1992) for shorter wavelengths. For moderately higher crack densities, Zhang and Achenbach (1991) have approximated the effects of neighbouring cracks on a reference crack as the effects of dipoles located at the geometrical centres of the cracks. Still for non-dilute crack arrays, a self-consistent scheme has been developed by Smyshlyaev et al. (1993).

In all the aforementioned papers, cracks are supposed to be open and their faces never come into contact. Nevertheless, real situations may involve contact and load transfer between crack faces. When crack closure effects are involved, the medium response turns out to be nonlinear. In the static case, Kachanov (1982) developed a model of inelasticity due to sliding friction between crack faces, whereas Hori and Nemat-Nasser (1983) proposed a self-consistent method for estimating the overall instantaneous moduli in the presence of crack closure and frictional sliding.

For the time-dependent mean response of a solid containing closing cracks, a nonlinear formulation was developed by Smyshlyaev and Willis (1996). They proposed an effective constitutive relation containing, as internal variable, the mean opening of the cracks. For a dilute array of cracks, this internal variable is governed by a nonlocal evolution law which describes the response of a single crack to the ambient field. For the numerical implementation of the model, they also developed a Galerkin-type approximation using for the crack-opening displacement, the shape function being provided by the quasi-static solution. As is underlined by the same authors, when the incident wave is not harmonic, a reasonable approximation is expected as long as harmonics whose wavelengths are long compared with the crack dimension, play a significant role in the locally incident field.

In a previous paper (Capuani and Willis, 1997; here referred to as Part I), the authors presented a formulation for the transient response of a flat crack to general time-dependent incident waves. A unilateral constraint was introduced to describe the interaction between frictionless crack faces during the motion.

In the present paper, the transient nonlinear response of an elastic medium containing a random distribution of aligned penny-shaped cracks is studied. Cracks are traction-free when open but they transmit normal stress when closed. The problem is formulated in terms of the ensemble averages of displacement, strain and stress fields and the effective constitutive relation proposed by Smyshlyaev and Willis (1996) is adopted. An integral representation of the effective displacement field is proposed using the Green function of the uncracked body. As a sample problem, the one-dimensional propagation of nonlinear waves along the direction normal to the crack surfaces is developed in detail. A discrete model is proposed coupling a finite difference scheme with the nonlinear solution of the single scattering problem proposed in Part I. This model allows contact to occur on different portions of the crack surfaces at different times.

In the numerical examples, body force and initial value problems have been analysed, showing the role that the characteristic crack dimension plays on the overall nonlinear response of the cracked medium.

References to equations in Part I are given the prefix I.

2. Interaction of distributed cracks

In Part I the transient dynamic response of a single crack to an incident wave in an infinite elastic medium was studied. The problem was formulated for a planar crack of arbitrary shape and explicit formulae were given for the case of a penny-shaped crack excited by a normally incident longitudinal wave. Here, the formulation is extended to the case of a medium containing a given distribution of cracks.

The medium is assumed to be homogeneous, isotropic and linearly elastic with Lamé constants $\lambda$, $\mu$ and density $\rho$. The cracks are taken to be parallel and penny-shaped, with radius $R$. A Cartesian system of
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coordinates \( \mathbf{x} = (x_1, x_2, x_3) \) is chosen in such a way that normals to the cracks lie along the \( x_3 \)-axis (figure 1). Denoting by \( S_0 \) the set

\[
S_0 = \{ \mathbf{x} : x_1^2 + x_2^2 \leq R^2, x_3 = 0 \}
\]

the set of points occupied by the crack centred at \( \mathbf{x}_A \) is

\[
S_A = \{ \mathbf{x} : \mathbf{x} - \mathbf{x}_A \in S_0 \}
\]

Outside the cracks, the displacement field \( \mathbf{u} \) satisfies the equation of motion

\[
\rho (\alpha^2 - \beta^2) \nabla \nabla^T \mathbf{u} + \rho \beta^2 \Delta \mathbf{u} + \mathbf{g} = \rho \partial_t^2 \mathbf{u}
\]

where \( \mathbf{g} \) is the body force and \( \alpha = [(\lambda + 2\mu)/\rho]^{1/2}, \beta = [\mu/\rho]^{1/2} \) are longitudinal (\( P \)) and shear (\( S \)) wave velocities.

Across the crack faces, the displacement field \( \mathbf{u} \) need not be continuous and the displacement jump across the surface \( S_A \) is denoted by

\[
\mathbf{b}_A^A(\mathbf{x}, t) = [\mathbf{u}(\mathbf{x}, t)], \quad \mathbf{x} \in S_A
\]

On the crack surface the following boundary conditions are adopted

\[
\sigma_{33} \leq 0, \quad b_3^A \geq 0, \quad \sigma_{33} \cdot b_3^A = 0
\]

\[
\sigma_{13} = \sigma_{23} = 0
\]

Figure 1. Distributed penny-shaped cracks.
stating that the frictionless faces are traction free when the crack is open and transmit normal compressive traction when it is closed.

The components of the stress vector \( \sigma_3 = (\sigma_{13}, \sigma_{23}, \sigma_{33})^T \) appearing in the conditions (5), (6) are given by the stress-displacement relation

\[
\sigma_3 = C(\nabla)u
\]

where

\[
C(\nabla) = \rho \begin{bmatrix}
\beta^2 \partial_3 & 0 & \beta^2 \partial_1 \\
0 & \beta^2 \partial_3 & \beta^2 \partial_2 \\
(\alpha^2 - 2\beta^2) \partial_1 & (\alpha^2 - 2\beta^2) \partial_2 & \alpha^2 \partial_3
\end{bmatrix}
\]

and \( \partial_i \) stands for \( \partial/\partial x_i \).

Let \( u^0 \) be the displacement field in the uncracked body. Then, considering the contribution of each single crack to the scattered field as given by Eq. (I, 3-1), the displacement field at any point \( y \), outside the cracks can be given the integral representation

\[
u(y, t) = u^0(y, t) + \sum_A \int_{S_A} (C(\nabla)G(x, y, y))^T \ast b_A(x, t)dS(x)
\]

where the symbol * denotes the convolution with respect to time \( t \) and \( G \) is the infinite-body elastodynamic Green tensor defined according to Eqs (I, 3.2–3.4). If the number of cracks in a bounded region is \( N \), the summation in Eq. (9) extends to \( A = 1, 2, \ldots, N \).

Hence, using Eqs (7), (9), and recalling that \( \partial G/\partial y_i = -\partial G/\partial x_i \), the expression for the stress vector \( \sigma_3 \) becomes

\[
\sigma_3(y, t) = \sigma_3^0(y, t) - \sum_A \int_{S_A} K(x, y, t) \ast b_A(x, t)dS(x)
\]

where

\[
K(x, y, t) = C(\nabla)(C(\nabla)G(x, y, t))^T
\]

Equation (10), as it stands, applies only where the stress is defined, that is, in the matrix. The representation (9) for the displacement incorporates the prescribed displacement jumps across the crack faces; correspondingly, the associated representation for the strain contains delta function contributions on the crack surfaces. Direct allowance can be made for these by replacing (10) with

\[
\sigma_3(y, t) = \sigma_3^0(y, t) - \sum_A \int_{S_A} K(x, y, t) \ast b_A(x, t)dS(x) - \sum_A \int_{S_A} C(n)\delta(y - x)b_A(x, t)dS(x)
\]

where \( n = (0, 0, 1)^T \) is the normal to the crack surfaces (Kunin, 1983).

Equations (9), (10) together with conditions (5), (6) represent the boundary integral formulation for the problem at hand. In principle, on the basis of these equations, a numerical procedure like the one outlined in Part I for the single crack problem could be developed. In fact, by a proper discretization of the crack surfaces and by means of a collocation technique, the displacement jumps over all the cracks can be determined by making use of Eq. (10) and conditions (5), (6). Obviously, in the presence of a large number of cracks, this procedure would become extremely cumbersome and a reasonable objective is to obtain an ‘averaged’ solution.

In the next section, attention will be focused upon a matrix containing a random distribution of cracks and the problem is formulated in terms of the expectation value, or ensemble average, of the wave field.
3. The constitutive model for the average wave field

Equation (9) refers to a particular configuration where the matrix contains a collection of \( N \) cracks and the position is known for each of them. Nevertheless, when there is not sufficient information to establish the configuration of the collection, the ensemble of possible configurations may be considered and the problem can be formulated in terms of average values of fields. Indicating the ensemble of configurations by a probability distribution function \( p_A = p(x_A) \), the configurational average \( \langle f \rangle \) of a function \( f \) is given by

\[
\langle f \rangle = \int f(x_A)p(x_A)dx_A
\]  
(13)

According to definition (13) and to Eq. (9), the ensemble average \( \langle u \rangle \) of the displacement field is given by

\[
\langle u \rangle (y, t) = u^0(y, t) + \int dx_Ap_A \int_{S_A} (C(\nabla)G(x, y, t))^T \ast \langle b^4 \rangle_A(x, t) dS(x)
\]  
(14)

where \( \langle b^4 \rangle_A \) stands for the mean value of \( b^4 \) conditional upon finding a crack centred at \( x_A \). It is assumed that the cracks are randomly distributed, so that all the positions of cracks are equally probable, and the probability density \( p_A \) coincides with the number density of cracks (number of cracks per unit volume) \( \nu \), i.e. \( p_A = \nu \).

Here the problem is formulated in terms of the ensemble averages of displacement \( \langle u \rangle \), strain \( \langle e \rangle \) and stress \( \langle \sigma \rangle \).

The average displacement and stress fields generated by a body force \( g \) satisfy the equation of motion

\[
\text{div} \langle \sigma \rangle + g = \rho \langle \ddot{u} \rangle
\]  
(15)

For the average stress field, the following constitutive relation is adopted

\[
\langle \sigma \rangle = L[\langle e \rangle] - \nu L[\mathcal{L}(\langle b^4 \rangle_A)]
\]  
(16)

where \( L \) is the elasticity tensor of the matrix, \( \langle e \rangle \) is the average strain field given by the strain-displacement relation

\[
\langle e \rangle = \frac{1}{2} (\nabla \langle u \rangle + \nabla \langle u \rangle^T)
\]  
(17)

and \( \mathcal{L} \) is the kinematic operator defined as follows

\[
\mathcal{L}(\langle b^4 \rangle_A)(x, t) := \frac{1}{2} \int dx_A[\langle b^4 \rangle_A(x, t) \otimes e_3 + e_3 \otimes \langle b^4 \rangle_A(x, t)] \cdot H[R^2 - (x_1 - x_{1,A})^2 - (x_2 - x_{2,A})^2] \delta(x_3 - x_{3,A})
\]  
(18)

with \( H(x) \) and \( \delta(x) \) being the Heaviside function and the Dirac function respectively, and \( e_3 \) the unit vector along the \( x_3 \)-axis. It accounts for the unwanted contribution to the mean strain \( \langle e \rangle \) from the opening of the cracks [cf. the last term in (12)].

Equation (16) is the ‘effective’ constitutive relation for a cracked solid proposed by Smyshlyaev and Willis (1996). For the isotropic matrix, the elasticity tensor is given by

\[
L_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]  
(19)

and therefore Eq. (16) takes the form

\[
\langle \sigma \rangle = \lambda \text{tr} \langle e \rangle I + 2\mu \langle e \rangle - \nu \int dx_A[\lambda \langle b^4 \rangle_A I + \mu (\langle b^4 \rangle_A \otimes e_3 + e_3 \otimes \langle b^4 \rangle_A)]
\cdot H[R^2 - (x_1 - x_{1,A})^2 - (x_2 - x_{2,A})^2] \delta(x_3 - x_{3,A})
\]  
(20)
In this equation, the field \( \langle b^A \rangle_A \) plays the role of an internal variable which can be determined by making use of conditions (5), (6) on the crack faces. According to Smyshlyaev and Willis (1996), under the assumption of small number density, the traction on the crack \( A \) can be written as

\[
\sigma_3(y,t) = \langle \sigma_3 \rangle(y,t) - \int_{S_A} K(x,y,t) \ast b^A(x,t) dS(x)
\]

(21)

where the kernel \( K \) is given by Eq. (11). In Eq. (21) the cracks interact only through the mean stress vector \( \langle \sigma_3 \rangle \) and, in this lowest order approximation, \( b^A \) coincides with its conditional mean value \( \langle b^A \rangle_A \).

Hence, Eq. (21) together with boundary conditions (5), (6) represents the nonlinear nonlocal evolution law required for the determination of the internal variables \( b^1, b^2, b^3 \).

It is worth noting that, when point \( y \) approaches \( S_A \), according to Eqs (I, 3.12, 3.13), Eq. (21) in the limit can be split into the following equations

\[
\sigma_3^{(2)}(y,t) = \langle \sigma_3^{(2)} \rangle(y,t) - \int_{S_A} K^{(2)}(x,y,t) \ast b^{A(2)}(x,t) dS(x)
\]

(22)

\[
\sigma_{33}(y,t) = \langle \sigma_{33} \rangle(y,t) - \int_{S_A} K_{33}(x,y,t) \ast b^3(x,t) dS(x)
\]

(23)

where \( \sigma_3^{(2)} = (\sigma_{13}, \sigma_{23})^T, b^{A(2)} = (b^1, b^2)^T \) and the kernels \( K^{(2)} \) and \( K_{33} \) are given by Eqs (I, 3.14), (I, 3.15). The kernels \( K^{(2)} \) and \( K_{33} \) are hypersingular, of order \( r^{-3} \), as \( r = |x-y| \) tends to zero. Therefore, the integrals on the right-hand side of (22), (23) are specified in the finite-part Hadamard sense. A regularization of the singular integral equations involving derivatives, in the crack plane, of the crack-opening displacement was proposed in Part I.

4. The integral representation of the effective field

By making use of the effective constitutive relation (20) and of the strain-displacement relation (17), the equation of motion (15) can be written in terms of displacements as follows

\[
(\lambda + \mu)\nabla \text{div}(u) + \mu \Delta \langle u \rangle + g + g^* = \rho \langle \ddot{u} \rangle
\]

(24)

where, noting that \( b^A \) vanishes along the crack edge \( \partial S_A \), \( g^* \) is given by

\[
g^* = -\nu \int d\mathbf{x}_A \left[ \lambda \nabla b^A_3 + \mu \left( \frac{\partial b^A}{\partial x_3} + e_3 \text{div} b^A \right) \right] \cdot H[R^2 - (x_1 - x_{1A})^2 - (x_2 - x_{2A})^2] \delta(x_3 - x_{3A})
\]

\[
-\nu \int d\mathbf{x}_A [\lambda b^A_3 e_3 + \mu (\mathbf{b}^A + b^3_3 e_3)] \cdot H[R^2 - (x_1 - x_{1A})^2 - (x_2 - x_{2A})^2] \delta'(x_3 - x_{3A})
\]

(25)

and the prime stands for derivative with respect to the argument.

The vector \( g^* \) appearing in Eq. (24) can be regarded as an extra body force in the uncracked matrix. Hence, when the initial conditions

\[
\langle u \rangle(x,0) = u_0(x), \quad \langle \dot{u} \rangle(x,0) = v_0(x)
\]

(26)

are prescribed, according to Love’s integral identity, the displacement \( \langle u \rangle \) can be given the following integral representation

\[
\langle u \rangle(y,t) = \int d\mathbf{x} G(x,y,t) * (g(x,t) + g^*(x,t)) + \rho \int d\mathbf{x} [G(x,y,t)v_0(x) + \dot{G}(x,y,t)u_0(x)]
\]

(27)
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where the convolution integral involving the extra body force is given by

\[
\int dxG(x, y, t) \ast g^*(x, t) = -\nu \int dx_A \int_{S_A} dS(x) G(x, y, t) \ast \left[ \lambda \nabla b_3^A + \mu \left( \frac{\partial b_3^A}{\partial x_3} + e_3 \text{div} b^A \right) \right] \\
+ \nu \int dx_A \int_{S_A} dS(x) \left[ \lambda \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) + \mu \left( \frac{\partial}{\partial x_3} (G \ast b^A) + \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) \right) \right]
\]  

(28)

Using the following identities

\[
G \nabla b_3 = \text{div}(b_3 G) - b_3 \text{div}G \\
Ge_3 \text{div}b = \text{div}(Ge_3 \otimes b) - \nabla(Ge_3)b \\
G \frac{\partial b}{\partial x_3} = \frac{\partial}{\partial x_3} (Gb) - \frac{\partial G}{\partial x_3} b
\]  

(29)

in the first integral at the RHS of Eq. (28) yields

\[
\int dxG(x, y, t) \ast g^*(x, t) = \nu \int dx_A \int_{S_A} dS(x) \left[ \lambda b_3^A(x, t) \ast \text{div}G(x, y, t) \\
+ \mu \left( \frac{\partial G(x, y, t)}{\partial x_3} \ast b^A(x, t) + \nabla(G(x, y, t)e_3) \ast b^A(x, t) \right) \right] dS(x) \\
- \nu \int dx_A \int_{S_A} dS(x) \left\{ \lambda \left[ \text{div}(b_3^A \ast G) - \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) \right] \\
+ \mu \left[ \text{div}(G \ast e_3 \otimes b^A) - \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) \right] \right\}
\]  

(30)

In the last integral of Eq. (30), the terms involving the derivatives with respect to \(x_3\) within square brackets cancel out. Then, applying the divergence theorem and bearing in mind the vanishing of \(b^A\) along the crack edge \(\partial S_A\), the identity is obtained

\[
\int_{S_A} dS(x) \left\{ \lambda \left[ \text{div}(b_3^A \ast G) - \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) \right] + \mu \left[ \text{div}(G \ast e_3 \otimes b^A) - \frac{\partial}{\partial x_3} (b_3^A \ast Ge_3) \right] \right\} \\
= \int_{\partial S_A} ds(x) \{ \lambda b_3^A \ast G n + \mu (G \ast e_3 \otimes b^A) n \} = 0
\]  

(31)

Hence, due to Eqs (30), (31), the integral representation (27) of the mean displacement field takes the form

\[
(u)(y, t) = \int dxG(x, y, t) \ast g(x, t) + \nu \int dx_A \int_{S_A} dS(x) \left[ \lambda b_3^A(x, t) \ast \text{div}G(x, y, t) \\
+ \mu \left( \frac{\partial G(x, y, t)}{\partial x_3} \ast b^A(x, t) + \nabla(G(x, y, t)e_3) \ast b^A(x, t) \right) \right] dS(x) \\
+ \rho \int dx [G(x, y, t)v_0(x) + \dot{G}(x, y, t)u_0(x)]
\]  

(32)

The representation (32) together with the evolution law (21, 5, 6) for the internal variables can be employed in a numerical formulation to study the three-dimensional propagation of waves in the cracked medium. In what follows, attention is focused on the sample problem of one-dimensional nonlinear waves along the \(x_3\)-direction.
5. One-dimensional nonlinear wave motion

The characteristic features of the effective wave motion in the cracked medium can be brought out by an analysis of the one-dimensional propagation along the direction normal to the crack surfaces. Consider the problem in which the effective displacement is a function of the space variable \( x_3 \)

\[
\langle u \rangle = \langle u \rangle(x_3, t)
\]

and the value of the internal variable \( b^A(x, t) \) at any point \( x \) of the reference crack \( S_A \) only depends upon the coordinate \( x_3, A \) of the crack centre and on the relative position \( x - x_A \)

\[
b^A = b^{x_A}(x - x_A, t) \quad x - x_A \in S_0
\]

In this case, using the strain-displacement relation (17), the effective stress tensor (20) takes the form

\[
\langle \sigma \rangle = \lambda \left( \frac{\partial (u_3)}{\partial x_3} - \nu \pi R^2 \hat{b}_3 \right) I + \mu \left( \frac{\partial (u)}{\partial x_3} - \nu \pi R^2 \hat{b} \right) \times e_3 + \mu e_3 \times \left( \frac{\partial (u)}{\partial x_3} - \nu \pi R^2 \hat{b} \right)
\]

where

\[
\hat{b}(x_3, t) = \frac{1}{\pi R^2} \int_{x-x_A \in S_0} dS(x_4) b^{x_3}(x - x_A, t) = \frac{1}{\pi R^2} \int_{S_0} dS(z) b^{x_3}(z, t)
\]

and, correspondingly, the equations of motion are given by

\[
\mu \frac{\partial^2 (u_k)}{\partial x_3^2} - \mu \nu \pi R^2 \frac{\partial \hat{b}_3}{\partial x_3} + g_k = \rho \langle \ddot{u}_k \rangle \quad (k = 1, 2)
\]

\[
(\lambda + 2\mu) \frac{\partial^2 (u_3)}{\partial x_3^2} - (\lambda + 2\mu) \nu \pi R^2 \frac{\partial \hat{b}_3}{\partial x_3} + g_3 = \rho \langle \ddot{u}_3 \rangle
\]

Equations (35)–(38) together with the evolution laws (22, 6), (23, 5) clearly indicate that every component of the effective displacement vector is solved from a set of equations involving that particular component only. In particular, the problem of determining the shear components \( \langle u_1 \rangle, \langle u_2 \rangle \) through Eqs (37), (22) and boundary conditions (6), turns out to be linear. On the contrary, for the longitudinal component \( \langle u_3 \rangle \), the problem is nonlinear since Eq. (38) is associated with the nonlocal law (23) and the unilateral constraints (5).

In the following, the nonlinear case in \( \langle u_3 \rangle, \hat{b}_3 \), will be treated and unnecessary suffixes 3’s will be dropped. To this purpose, Eq. (38) is rewritten as

\[
\frac{\partial^2 \langle u \rangle}{\partial t^2} - \alpha^2 \frac{\partial^2 \langle u \rangle}{\partial x^2} = \frac{g}{\rho} - \alpha^2 \nu \pi R^2 \frac{\partial \hat{b}}{\partial x}
\]

and the solution is given the integral representation

\[
\langle u \rangle(x, t) = \frac{1}{\rho} \int \Gamma(x - x', t) * g(x', t) dx' - \alpha^2 \nu \pi R^2 \int \Gamma(x - x', t) * \hat{b}_{x}(x', t) dx'
\]

where \( \Gamma(x, t) \) is the one-dimensional Green’s function satisfying the equation

\[
\Gamma_{,tt} - \alpha^2 \Gamma_{,xx} = \delta(x) \delta(t)
\]

having the expression

\[
\Gamma(x, t) = \frac{1}{2\alpha} H(t)[H(x + \alpha t) - H(x - \alpha t)]
\]

In Eq. (40) initial displacement and velocity are assumed to be equal to zero. Finally, relations (23, 5) governing the evolution of the internal variable can be written as

\[
\sigma^x(x', t) = \langle \sigma \rangle(x, t) - \int_{S_0} K(z, x', t) * b^x(z, t) dS(z), \quad z' \in S_0
\]
where the superscript on \( \sigma \) and on \( b \) indicates the dependence upon the space coordinate \( x \), and, according to Eqs (40), (35), the mean stress \( \langle \sigma \rangle \) is given by

\[
\langle \sigma \rangle(x, t) = \rho \alpha^2 \left\{ \frac{1}{\rho} \int_{\Gamma} \gamma(x - x') \ast g(x', t) dx' - \alpha^2 \nu \pi R^2 \int_{\Gamma} \gamma(x - x') \ast \hat{b}_r(x', t) dx' - \nu \pi R^2 \hat{b} \right\}
\]

(45)

6. The discrete model

The problem of determining the internal variable \( b^r \) through the evolution law (43, 44) can be solved by adopting the approach proposed in Part I for the single scattering problem. According to this approach, the integral equation is regularized by isolating the hypersingular terms and transforming them into regular line integrals along the crack edge. Hence, the time interval \( (0, t_N) \) is divided into \( N \) time steps \( \Delta t \) and time-convoluted kernels for each time step are explicitly obtained. Moreover, a system of polar coordinates \((\zeta, \theta)\) with the origin at the centre of the disk \( S_0 (0 \leq \zeta \leq R) \) is introduced where the crack opening displacement and the contact stress can be written as

\[
b^r = b^r(\zeta, t), \quad \sigma^r = \sigma^r(\zeta, t)
\]

(46)

Finally, according to Eq. (I, 6-15), the integral Eq. (43) can be given the following form

\[
\sigma^r(\xi, t_N) + \int_0^R \{ A^{11}(b^r(\zeta, t_N) - b^r(\xi, t_N)) - \xi B^{11}(\xi A^{11}) b^r(\xi, t_N) \} \zeta d\zeta
+ 2 \frac{\rho}{\pi} \left[ - \left( 1 - \frac{\beta^2}{\alpha^2} \right) \frac{\beta^2}{\alpha^2} + \frac{1}{4} \frac{\alpha}{\Delta t} \int_0^{\Omega(t, \xi, t_N)} b^r(\xi(t_1, \omega), t_N) d\omega \right]
+ 2 \frac{\rho}{\pi} \left( 1 - \frac{\beta^2}{\alpha^2} \right) \frac{\beta^2}{\alpha^2} \frac{\alpha}{\Delta t} b^r(\xi, t_N) = \langle \sigma \rangle(x, t_N) - R^r N^{-1}
\]

(47)

where \( 0 < \xi < R \); moreover, the kernels \( A^{11}(\xi, \zeta), B^{11}(\xi, \zeta), \) the angle \( \Omega(R, \xi, t) \) and the term \( R^r N^{-1} \) collecting the contributions of all the previous \( N - 1 \) time steps, are defined by Eqs (I, 6.6), (I, 6.7), (I, 6.9) and (I, 6.13) respectively.

Equation (47) provides a time-marching scheme for the determination of the crack-opening displacement and of the contact stress at any step. To this end, the crack radius is divided into \( Q \) intervals \([\zeta_q, \zeta_{q+1}]\) \((q = 1, ..., Q; \zeta_1 = 0; \zeta_{Q+1} = R)\) and a linear variation of the crack-opening displacement is assumed within each interval

\[
b^r(\zeta_q + \lambda \Delta \zeta, t_n) = b^r_q + \frac{\lambda}{2} b^r_{q+1}
\]

(48)

with \( t_n = n \Delta t \), \( \Delta \zeta = \zeta_{q+1} - \zeta_q \) and \( \lambda \in [0, 1] \). Hence, spatial integrations in Eq. (47) can be carried out following the procedure described in Part I. Moreover, due to the representation (48), the function \( \hat{b} \) introduced in Eq. (36) can be evaluated as follows

\[
\hat{b}(x, t_n) = \frac{1}{R^2} \left\{ \frac{1}{3} [\Delta q^2 b^r_q + \sum_{q=2}^Q [\Delta q - 1/3 \Delta q_{q-1} + \Delta q_{q+1}/3] b^r_q] \right\}
\]

(49)

Finally, using a collocation method, i.e. taking \( \xi = \zeta_p \) \((p = 1, ..., Q)\), a system of \( Q \) equations at each time step can be obtained which can be written in matrix form as follows

\[
\{ \sigma^r \}_{x, N} + [C^1] \{ b^r \}_{x, N} = \langle \sigma \rangle \{ 1 \} - \{ R^r \}_{x, N-1}
\]

(50)
where \( \{1\} \) is a unit vector. In Eq. (50) the matrix \([C^{1}]\) is evaluated at the first time step and the vector \( \{R^{1,N-1}\} \) collects the contributions corresponding to the previous \( N-1 \) time steps.

Along the \( x \)-direction a finite difference scheme is adopted, assuming a constant step \( \Delta x = \alpha \Delta t \). Hence, setting for any function \( f(x,t) \)

\[
f^{i,n} = f(i\Delta x, n\Delta t)
\]

Equation (50) can be rewritten at \( x = i\Delta x \) as

\[
\{\sigma^{i,N}\} + [C^{1}]\{b^{i,N}\} = \langle \sigma \rangle^{i,N}\{1\} - \{R^{i,N-1}\}
\]

where, according to Eq. (45), \( \langle \sigma \rangle^{i,N} \) is approximated as

\[
\langle \sigma \rangle^{i,N} = \rho\alpha^2 \left[ \varepsilon^{i,N} - \frac{\nu\pi R^2}{2} \frac{\alpha \Delta t}{\Delta x} \sum_{n=1}^{N} \left( \hat{\theta}^{i+n,N-n} - \hat{\theta}^{i+n-1,N-n} - \hat{\theta}^{i-n+1,N-n} + \hat{\theta}^{i-n,N-n} \right) \right]
\]

with \( \hat{\theta}^{i,n} \) given by Eq. (49) and

\[
\varepsilon^{i,N} = \frac{\Delta t}{2\rho\alpha} \sum_{n=1}^{N} (g^{i+n,N-n} - g^{i-n,N-n})
\]

When an initial value problem is considered, with \( \langle u \rangle(x,0) = u_0(x) \) and \( \langle \dot{u} \rangle(x,0) = 0 \), Eq. (54) is to be substituted by

\[
\varepsilon^{i,N} = \frac{1}{2\Delta x} (u_0^{i+N} - u_0^{i+N-1} + u_0^{i-N+1} - u_0^{i-N})
\]

At each time instant \( t_N \), the system (52) is solved, for all \( x = i\Delta x \), by means of an iterative procedure requiring the following conditions to be satisfied \( \forall p \ (p = 1, ..., Q) \)

\[
\sigma_p^{i,N} \leq 0, \quad b_p^{i,N} \geq 0, \quad \sigma_p^{i,N} \cdot b_p^{i,N} = 0
\]

Once vectors \( \{b^{i,n}\} \) are determined for all \( i \) and for all \( n \leq N \), the displacement given by Eq. (40) can be analogously computed by finite differences as follows

\[
\langle u \rangle^{i,N} = \overline{u}^{i,N} - \frac{\nu\pi R^2}{2} \frac{\alpha \Delta t}{\Delta x} \sum_{n=1}^{N} (\hat{\theta}^{i+n,N-n} - \hat{\theta}^{i-n,N-n})
\]

where

\[
\overline{u}^{i,N} = \frac{\Delta t \Delta x}{2\rho\alpha} \sum_{n=1}^{N} \sum_{k=i-n}^{i+n} g^{k,N-n}
\]

for a body force problem, and

\[
\overline{u}^{i,N} = \frac{1}{2} (u_0^{i+N} + u_0^{i-N})
\]

for an initial value problem with \( \langle u \rangle(x,0) = u_0(x) \) and \( \langle \dot{u} \rangle(x,0) = 0 \).

### 7. Examples

An elastic isotropic matrix with Poisson's ratio equal to 0.25, corresponding to \( \alpha/\beta = \sqrt{3} \), is considered. The matrix contains a random distribution of aligned penny-shaped cracks whose number density is assumed
Figure 2. (a) Body force problem: displacement field against $x/R$ at different time instants ($t' = \alpha t / R$). The undistorted displacement waveform in the medium without cracks is also reported at time $t' = 20$. (b) Body force problem: crack-opening field against $x/R$ at different time instants ($t' = \alpha t / R$).

to be $\nu = 0.2/R^3$. The matrix response is analysed for body force and initial value problems involving one-dimensional mean wave motion along the direction of normals to the cracks. In all cases, the crack radius is divided into 20 elements and a time step $\Delta t = 0.05 R/\alpha$ (i.e. $\Delta x = 0.05 R$) is chosen.

In the first example the medium is subjected to the body force

$$g(x,t) = \begin{cases} g_0 H(1 - x/R) \cdot H(1 + x/R), & 0 < t \leq 2R/\alpha \\ 0 & \text{otherwise} \end{cases}$$
Figure 3. (a) Initial displacement triangular distribution ($\lambda/R = 2$): displacement field against $x/R$ at different time instants ($t' = \alpha t/R$).
(b) Initial displacement triangular distribution ($\lambda/R = 2$): crack-opening field against $x/R$ at different time instants ($t' = \alpha t/R$).

where $g_0$ is a reference amplitude. The mean-displacement waveforms caused by the applied force are reported in figure 2a at different instants of time. The corresponding variation of the internal variable $\hat{b}$, describing the crack opening in the medium, is reported in figure 2b. In figure 2a the undistorted displacement waveform produced by the force in the medium without cracks is also reported at time $\alpha t/R = 20$. The applied force tends to generate a compressive disturbance propagating in the positive $x$ direction and a tensile disturbance propagating in the negative $x$ direction. Hence, cracks open for $x < 0$ and tend to remain closed for $x > 0$ as is shown in figure 2b. Correspondingly, in figure 2a the displacement wave moves with the same velocity.
Figure 4. (a) Initial displacement triangular distribution ($\lambda/R = 2$): comparison of displacement and crack-opening fields given by the present analysis (solid lines) at time $\alpha t/R = 4$ with those given by the Smyshlyaev and Willis (1996) solution (dashed lines). (b) Magnification of the shock discontinuity around $x/R = -4$ in figure 4a.

of the uncracked medium in the positive $x$ direction and with a lower velocity in the negative $x$ direction. The same figure shows that the presence of cracks increases the maximum value of displacement as is to be expected due to the augmented compliance of the medium.

In the second example no body forces are present and the following initial displacement (with amplitude $u_0$) is imposed to the medium

$$
\langle u \rangle (x, 0) = u_0 \left\{ \left(1 + \frac{2x}{\lambda}\right) \left[ H \left( x + \frac{\lambda}{2} \right) - H(x) \right] + \left(1 - \frac{2x}{\lambda}\right) \left[ H(x) - H \left( x - \frac{\lambda}{2} \right) \right] \right\}
$$
with null initial velocity. The waveforms of displacement and of the crack opening parameter are plotted in figures 3a and 3b for a ratio of the characteristic length $\lambda$ to the crack radius $\lambda/R = 2$. The initial displacement distribution gives rise to right-moving and left-moving stress pulses having both tensile and compressive components. It can be observed that over a length $\lambda/2$ behind the right-moving wavefront, where the medium undergoes compression, the displacement wave propagates undistorted (figure 3a) and cracks remain closed (figure 3b). On the contrary, considering the leftward wavefront, as time passes the left-moving compressive disturbance overtakes the left-moving tensile disturbance causing first a gradual reduction of the positive
Figure 6. (a) Multiple-sine initial displacement: displacement waveforms for different $\lambda/R$ ratios at time $\alpha t/\lambda = 2$.
(b) Multiple-sine initial displacement: displacement waveforms for different $\lambda/R$ ratios at time $\alpha t/\lambda = 5$.

displacement and then a reversal of the displacement sign (figure 3a). In figure 3b the tendency of cracks to remain closed in the region of the displacement reversal can be noticed.

Figure 4a compares the mean displacement and the crack opening parameter given by the present analysis at a fixed instant of time ($\alpha t/R = 4$) with those obtained using the solution proposed by Smyshlyaev and Willis (1996). These authors developed a Galerkin-type approximation using for the crack-opening displacement the shape function provided by the quasi-static solution. The differences between the two solutions become evident
around \( x/R = -4 \) where the shock discontinuity associated with the initial displacement field is noticed. This part of the graph is magnified in figure 4b.

Finally, figure 5a shows the mean-displacement waveform at the time \( \alpha t/\lambda = 2 \) for three different ratios of the characteristic length \( \lambda \) to the crack radius (\( \lambda/R = 1, \lambda/R = 2, \lambda/R = 4 \)). The waveform is magnified around \( x/\lambda = -2 \) in figure 5b, where it can be seen that the shock discontinuity is still evident for \( \lambda/R = 1 \) and \( \lambda/R = 2 \) but it is damped down for \( \lambda/R = 4 \).

In the last example the initial displacement imposed on the medium is a multiple sine given by the expression

\[
\langle u \rangle(x, 0) = u_0 \sin \left( \frac{2\pi x}{\lambda} \right) [H(2\lambda - x) \cdot H(2\lambda + x)]
\]

Figures 6a and 6b show the mean-displacement as a function of the dimensionless coordinate \( x/\lambda \) for three different ratios of the wavelength \( \lambda \) to the crack radius (\( \lambda/R = 1, \lambda/R = 2, \lambda/R = 4 \)), at two different instants of time (\( \alpha t/\lambda = 2 \) and \( \alpha t/\lambda = 5 \)). At \( \alpha t/\lambda = 2 \) (figure 6a), a phase difference between the curves can be observed which becomes more pronounced with increasing \( \lambda/R \) ratios. Wave distortion is accentuated at \( \alpha t/\lambda = 5 \) (figure 6b), where the increased value of the maximum mean displacement can be noticed for \( \lambda/R = 4 \).

8. Conclusion

The effective dynamic response of an elastic medium containing aligned cracks which may open under tension and close under compression has been analysed. An integral representation of the effective displacement has been proposed which can be employed in a numerical formulation to study the three-dimensional propagation of waves in the cracked medium. The nonlinear case of plane-wave disturbances travelling along the direction normal to the crack surfaces has been developed in detail. In this case, a discrete model has been proposed in which a finite difference scheme along the wave-propagation direction is coupled with a boundary element method on the crack surfaces.

The model makes allowance for partial crack closure and can deal with general time-dependent loadings. The numerical examples have shown the dependence of shape distortion and velocity variation of waves on their tensile and compressive components. Moreover, the influence on wave distortion of shape and frequency content of pulses and of their relation with the characteristic crack dimension has been put in evidence.

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