In this paper, axisymmetric bending of through-the-thickness functionally graded circular plates is studied using the Mindlin plate theory, i.e., account for the transverse shear strains. Due to nonsymmetric grading of the material through the thickness, the bending–stretching coupling exists. A general solution to the Mindlin plate problem for arbitrary variation of the constituents is derived in terms of the isotropic Kirchhoff plate solution. Particular solutions are developed for a number of boundary conditions. The effect of material distribution through the thickness and boundary conditions on deflections and stresses is presented. The following literature review provides a background for the present study.

Noda (1991) presented an extensive review that covers a wide range of topics from thermoelastic to thermoinelastic problems. He discussed the effect of temperature-dependent mechanical properties on stresses and suggested that temperature-dependent properties of the material be taken into account in order to perform a more accurate analysis. Tanigawa (1995) compiled a comprehensive list of papers on the analytical models of thermoelastic behavior of functionally gradient materials. Fukui and Yamanaka (1992) examined the effects of the gradation of components on the strength and deformation of thick-walled functionally gradient material tubes under internal pressure. Additionally, they conducted numerical analyses to study the effectiveness of the functionally gradient material tubes for a structural member. Fukui et al. (1993) further extended their previous work by considering a thick-walled FGM tube under uniform thermal loading. In the paper, they investigated the effect of graded components on a residual stress of the tube and generated an optimum composition of the FGM tube by minimizing the compressive circumferential stress at the inner surface. Fuchiyama et al. (1993) used an eight-node quadrilateral axisymmetric element to study transient thermal stresses and stress intensity factors of functionally gradient materials with cracks. In their analysis, they concluded that temperature-dependent properties should be considered in order to obtain more realistic results.

Tanigawa (1992) used a layerwise model to solve a one-dimensional transient heat conduction problem and the associated thermal stress problem of a nonhomogeneous plate. He further formulated the optimization problem of the material composition to reduce the thermal stress distribution. Tanaka et al. (1993a,b) designed FGM property profiles using sensitivity and optimization methods based on the reduction of thermal stresses. Jin and Noda (1993) used the minimization of thermal stress intensity factor for a crack in a metal/ceramic functionally gradient material as a criterion for optimizing material property variation. In the same context, they also studied both the steady-state (Noda and Jin, 1993) and the transient (Jin and Noda, 1994) heat conduction problems, but neglected the thermomechanical coupling.

The in-plane thermal stress distributions due to a temperature distribution in the thickness of a plate was studied by Obata et al. (1992). Further, they made use of an appropriate law of mixture for the material properties and discussed the relationship between the volume fraction of composed materials and the distributions of temperature and thermal stresses. Obata and Noda (1994) considered the steady thermal stresses in a hollow circular cylinder and a hollow sphere made of an FGM in order to understand the effect of the volumetric ratio of constituents and porosity on thermal stresses. They discussed the design of an optimum functionally gradient material by minimizing the thermal stresses.

Takeuti and Furukawa (1981) considered a plate under a state of thermal shock. They analyzed the quasi-static coupled thermoelastic problem and the uncoupled dynamical thermoelastic problem for a plate under the same initial and boundary conditions. The results showed that the thermomechanical coupling term plays a more significant role for the temperature and stress distributions than the inertia term. More recently, Takeuti and Tanigawa (1981) solved an axisymmetric coupled thermal-stress problem of an infinite solid cylinder with temperature changes in the radial and axial directions. The results of this paper showed that the thermal coupling has a considerable effect on the temperature and stress distributions. Most recently, Praveen and Reddy (1990) carried out a thermomechanical analysis of axisymmetric cylinders and Mindlin plates.
2. Formulation

Consider a functionally graded circular plate of total thickness $h$ and subjected to axisymmetric transverse load $q$. The $r$-coordinate is taken radially outward from the center of the plate, the $z$-coordinate along the thickness of the plate, and the $\theta$-coordinate is taken along a circumference of the plate. Suppose that the grading of the material, applied loads, and boundary conditions are axisymmetric so that the displacement $u_\theta$ is identically zero and $(u_r, u_z)$ are only functions of $r$ and $z$. At the moment, we assume that $E = E(z)$ and $\nu = \nu(z)$, and their specific variation will be discussed in the sequel.

The classical plate theory (CPT) is based on the displacement field (see Reddy, 1984, 1997, 1999)

$$
\begin{align*}
  u_r(r, z) &= u_0(r) - z \frac{du_0}{dr} \\
  u_z(r, z) &= u_0(r)
\end{align*}
$$

where $u_0$ is the radial displacement and $u_0$ is the transverse deflection of the point $(r, 0)$ of a point on the midplane (i.e., $z = 0$) of the plate. The displacement field (1) is based on the Kirchhoff hypothesis, which amounts to neglecting both transverse shear and transverse normal effects, i.e., deformation is due entirely to bending and inplane stretching.

The first-order shear deformation plate theory (FST) (see e.g. Reddy, 1984, 1999; Reddy and Chin, 1990) is the simplest theory that accounts for nonzero transverse shear strain. It is based on the displacement field

$$
\begin{align*}
  u_r(r, z) &= u_0(r) + z \phi(r) \\
  u_z(r, z) &= u_0(r)
\end{align*}
$$

where $\phi$ denotes rotation of a transverse normal in the plane $\theta = \text{constant}$. The first-order theory includes a constant state of transverse shear strain with respect to the thickness coordinate, and hence requires shear correction factors which depend not only on the material and geometric parameters but also on the loading and boundary conditions.

Both of the theories are governed by the equations

$$
\frac{d}{dr} (r N_{rr}) - N_{\theta \theta} = 0
$$

$$
-\frac{d}{dr} (r Q_r) = rq
$$

$$
 r Q_r = \frac{d}{dr} (r M_{rr}) - M_{\theta \theta}
$$

where $N_{rr}$ and $N_{\theta \theta}$ are the radial and circumferential in-plane forces and $M_{rr}$ and $M_{\theta \theta}$ are the radial and circumferential moments

$$
(N_{rr}, N_{\theta \theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{rr}, \sigma_{\theta \theta}) dz
$$

$$
(M_{rr}, M_{\theta \theta}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\tau_{rr}, \tau_{\theta \theta}) z dz
$$

The plate constitutive equations of the two theories are given below.
- CPT for isotropic plate

\[ N_{rr}^K = A_{11} \frac{du_0^K}{dr} + A_{12} \frac{u_0^K}{r} \]
\[ N_{\theta\theta}^K = A_{12} \frac{du_0^K}{dr} + A_{11} \frac{u_0^K}{r} \]
\[ M_{rr}^K = -D_{11} \frac{d^2 u_0^K}{dr^2} - D_{12} \frac{1}{r} \frac{du_0^K}{dr} \]
\[ M_{\theta\theta}^K = -D_{12} \frac{d^2 u_0^K}{dr^2} - D_{11} \frac{1}{r} \frac{du_0^K}{dr} \]
\[ Q_r^K = \frac{1}{r} \frac{d}{dr} \left[ r M_{rr}^K - M_{\theta\theta}^K \right] \]

- FST for functionally graded plate

\[ N_{rr}^F = A_{11} \frac{du_0^F}{dr} + A_{12} \frac{u_0^F}{r} + B_{11} \frac{d\phi}{dr} + B_{12} \frac{\phi}{r} \]
\[ N_{\theta\theta}^F = A_{12} \frac{du_0^F}{dr} + A_{11} \frac{u_0^F}{r} + B_{12} \frac{d\phi}{dr} + B_{11} \frac{\phi}{r} \]
\[ M_{rr}^F = B_{11} \frac{du_0^F}{dr} + B_{12} \frac{u_0^F}{r} + D_{11} \frac{d\phi}{dr} + D_{12} \frac{\phi}{r} \]
\[ M_{\theta\theta}^F = B_{12} \frac{du_0^F}{dr} + B_{11} \frac{u_0^F}{r} + D_{12} \frac{d\phi}{dr} + D_{11} \frac{\phi}{r} \]
\[ Q_r^F = A_{55} \left( \phi + \frac{d u_0^F}{dr} \right) \]

where superscript \( K \) denotes quantities in CPT and \( F \) denotes quantities in FST. Of course, \( \phi \) appears only in FST. The plate stiffnesses \( A_{ij}, B_{ij}, \) and \( D_{ij} \) are defined by (see Reddy, 1997)

\( (A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{b}{2}}^{\frac{b}{2}} Q_{ij}(1, z, z^2) dz \quad (i, j = 1, 2) \) (18-a)

\[ A_{55} = K \int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{E}{2(1+\nu)} dz \] (18-b)

\[ Q_{11} = Q_{22} = \frac{E}{(1-\nu^2)}, \quad Q_{12} = \nu Q_{11} \] (18-c)

where \( E \) is the modulus of elasticity, \( \nu \) the Poisson ratio, and \( K \) the shear correction factor.

The strategy is to develop relations for the deflections, forces, and moments of functionally graded plates based on the first-order shear deformation theory in terms of the associated quantities of isotropic plates based on the classical plate theory. Then the relations developed are specialized for plates with various boundary conditions.
3. Relationships between CPT and FST solutions

From Eqs (3), (13) and (14), we obtain

\[
0 = \frac{d}{dr} (rN_{rr} - N_{\theta\theta}) = A_{11} \left[ \frac{d}{dr} \left( r \frac{d}{dr} u^{F}_{0} \right) - u^{F}_{0} \right] + B_{11} \left[ \frac{d}{dr} \left( r \frac{d}{dr} \phi \right) - \frac{\phi}{r} \right] \\
= A_{11} \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (ru^{F}_{0}) \right) \right] + B_{11} \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right]
\]  

(19-a)

or

\[
\left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (ru^{F}_{0}) \right) \right] = -\frac{B_{11}}{A_{11}} \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right]
\]  

(19-b)

Upon integration, we obtain

\[
\frac{d}{dr} (ru^{F}_{0}) = -\frac{B_{11}}{A_{11}} \frac{d}{dr} (r\phi) + k_{1} r
\]  

(19-c)

and

\[
u^{F}_{0} = -\frac{B_{11}}{A_{11}} \phi + k_{1} \frac{r}{2} + \frac{k_{2}}{r}
\]  

(19-d)

from which we can compute

\[
\frac{d}{dr} u^{F}_{0} = -\frac{B_{11}}{A_{11}} \frac{d}{dr} \phi + k_{1} \frac{1}{2} - \frac{k_{2}}{r^{2}}
\]  

(20)

where \(k_{1}\) and \(k_{2}\) are integration constants. Using Eqs (19c, d) and (20), the forces and moments of Eqs (13)-(16) can be expressed in terms of \(\phi\) as

\[
M_{rr}^{F} = \Omega_{1} \frac{d\phi}{dr} + \Omega_{2} \frac{\phi}{r} + \frac{1}{2} \Omega_{3} k_{1} + \frac{1}{r^{2}} \Omega_{4} k_{2}
\]  

(21)

\[
M_{\theta\theta}^{F} = \Omega_{2} \frac{d\phi}{dr} + \Omega_{1} \frac{\phi}{r} + \frac{1}{2} \Omega_{3} k_{1} - \frac{1}{r^{2}} \Omega_{4} k_{2}
\]  

(22)

\[
N_{rr} = \Omega_{5} \frac{\phi}{r} + \frac{1}{2} \Omega_{6} k_{1} + \frac{1}{r^{2}} \Omega_{7} k_{2}
\]  

(23)

\[
N_{\theta\theta}^{F} = \Omega_{5} \frac{d\phi}{dr} + \frac{1}{2} \Omega_{6} k_{1} - \frac{1}{r^{2}} \Omega_{7} k_{2}
\]  

(24)

where

\[
\Omega_{1} = D_{11} - \frac{B_{11}^{2}}{A_{11}}, \quad \Omega_{2} = D_{12} - \frac{B_{11} B_{12}}{A_{11}}
\]  

(25-a)

\[
\Omega_{3} = B_{11} + B_{12}, \quad \Omega_{4} = B_{12} - B_{11}, \quad \Omega_{5} = B_{12} - \frac{A_{12} B_{11}}{A_{11}}
\]  

(25-b)

\[
\Omega_{6} = A_{11} + A_{12}, \quad \Omega_{7} = A_{12} - A_{11}
\]  

(25-c)

Based on load equivalence, we have

\[
\frac{d}{dr} (rQ_{r}^{F}) = \frac{d}{dr} (rQ_{r}^{K})
\]  

(26)
which after integration yields

\[ rQ_r^F = rQ_r^K + c_1 \]  \hspace{1cm} (27)

where \( c_1 \) is a constant on integration. But from Eqs (5), (21) and (22), we have

\[ rQ_r^F = \frac{d}{dr} \left( rM_r^F \right) - M_{\theta\theta}^F = \Omega_1 \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right] \]  \hspace{1cm} (28)

Similarly

\[ rQ_r^C = \frac{d}{dr} \left( rM_r^K \right) - M_{\theta\theta}^K = -D \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right] \]  \hspace{1cm} (29)

Using relations (28) and (29) in (27), we obtain

\[ \Omega_1 \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right] = -D \left[ r \frac{d}{dr} \left( \frac{1}{r} \frac{d}{dr} (r\phi) \right) \right] + c_1 \]  \hspace{1cm} (30)

Upon integrations, we have

\[ \Omega_1 \frac{d}{dr} (r\phi) = -D \frac{d}{dr} \left( r \frac{dw_0^K}{dr} \right) + c_1 r \log r + c_2 r \]  \hspace{1cm} (31-a)

\[ \Omega_1 \phi = -D \frac{dw_0^K}{dr} + \frac{1}{4} c_1 r (2 \log r - 1) + \frac{1}{2} c_2 r + \frac{1}{r} c_3 \]  \hspace{1cm} (31-b)

Next, from Eqs (17) and (28), we have

\[ A_{55} \left( \phi + \frac{dw_0^K}{dr} \right) = \frac{dM^K}{dr} + \frac{1}{r} c_1 \]  \hspace{1cm} (32)

where \( M^K \) is the moment sum

\[ M^K = \frac{M_{\phi\phi}^K + M_{\theta\theta}^K}{(1 + \nu)} \]  \hspace{1cm} (33)

Substituting for \( \phi \) from Eq. (31-b) into Eq. (32), we obtain

\[ A_{55} \left[ -\frac{D}{\Omega_1} \frac{dw_0^K}{dr} + \frac{c_1}{4\Omega_1} r (2 \log r - 1) + \frac{c_2}{2\Omega_1} r + \frac{c_3}{\Omega_1} r + \frac{dw_0^K}{dr} \right] = \frac{dM^K}{dr} + \frac{1}{r} c_1 \]  \hspace{1cm} (34)

Integrating the above expression, we obtain

\[ w_0^F = \frac{D_1}{\Omega_1} w_0^K + \frac{c_1}{\Omega_1} \left[ r^2 \frac{1}{4} (1 - \log r) + \frac{\Omega_1}{A_{55}} \log r \right] - \frac{c_2}{4\Omega_1} r^2 - \frac{c_3}{\Omega_1} \log r - c_4 + \frac{M^K}{A_{55}} \]  \hspace{1cm} (35)

From Eq. (31-b), we have

\[ \Omega_1 \frac{d\phi}{dr} = -D \frac{d^2 w_0^K}{dr^2} + \frac{c_1}{2} \left[ \frac{1}{2} (2 \log r - 1) + 1 \right] + \frac{c_2}{r} - \frac{c_3}{r^2} \]  \hspace{1cm} (36)
Substituting Eqs (31-b) and (36) into Eq. (21), we obtain

\[
M_{rr}^F = -D \frac{d^2 u_0^K}{dr^2} + \frac{c_1}{2} \left[ \frac{1}{2} (2 \log r - 1) + \frac{1}{r^2} \right] + \frac{c_2}{r} \frac{\Omega_2}{r^2} + \frac{c_3}{r} + \frac{1}{r^2}
+ \frac{\Omega_2}{r} \left[ \frac{1}{2} \frac{d u_0^K}{dr} + \hat{c}_1 \frac{r}{2} (2 \log r - 1) + \hat{c}_2 \frac{r}{2} \frac{1}{r^2} \right] + \frac{\Omega_3}{r} \frac{k_1}{2} + \frac{\Omega_4}{r^2} \frac{k_2}{r^2}
\]  

(37-a)

or

\[
M_{rr}^F = M_{rr}^K + D \frac{d u_0^K}{dr} \left( \nu - \hat{\Omega}_2 \right) + \frac{c_1}{2} \left[ \frac{1}{2} \left( 1 - \hat{\Omega}_2 \right) + \left( 1 + \hat{\Omega}_2 \right) \log r \right]
+ \frac{c_2}{2} \left( 1 + \hat{\Omega}_2 \right) - \frac{c_3}{r^2} \left( 1 - \hat{\Omega}_2 \right) + \frac{\Omega_3}{r} \frac{k_1}{2} + \frac{\Omega_4}{r^2} \frac{k_2}{r^2}
\]

(37-b)

where \( \hat{D} = \frac{D}{\Omega_1}, \hat{c}_i = \frac{c_i}{\Omega_1} \) and so on. Similarly, we can write

\[
M_{\theta\theta}^F = M_{\theta\theta}^K + D \frac{d^2 u_0^K}{dr^2} \left( \nu - \hat{\Omega}_2 \right) + \frac{c_1}{2} \left[ \frac{1}{2} \left( 1 - \hat{\Omega}_2 \right) + \left( 1 + \hat{\Omega}_2 \right) \log r \right]
+ \frac{c_2}{2} \left( 1 + \hat{\Omega}_2 \right) + \frac{c_3}{r^2} \left( 1 - \hat{\Omega}_2 \right) + \frac{\Omega_3}{r} \frac{k_1}{2} - \frac{\Omega_4}{r^2} \frac{k_2}{r^2}
\]

(38)

Next, we substitute (31-b) into (23) and obtain

\[
N_{rr}^F = \Omega_5 \left[ -\hat{D} \frac{1}{r} \frac{d u_0^K}{dr} + \hat{c}_1 \frac{1}{4} (2 \log r - 1) + \frac{1}{2} \hat{c}_2 + \frac{1}{r^2} \hat{c}_3 \right] + \Omega_6 \frac{k_1}{2} + \frac{\Omega_7}{r^2} \frac{k_2}{r^2}
\]

(39)

Substituting (36) into (24), we obtain

\[
N_{\theta\theta}^F = \Omega_5 \left[ -\hat{D} \frac{d^2 u_0^K}{dr^2} + \hat{c}_1 \frac{1}{4} (2 \log r) + 1 + \frac{1}{2} \hat{c}_2 - \frac{1}{r^2} \hat{c}_3 \right] + \Omega_6 \frac{k_1}{2} - \Omega_7 \frac{k_2}{r^2}
\]

(40)

Define

\[
M^F \equiv \frac{M_{rr}^F + M_{\theta\theta}^F}{(1 + \Omega_2)}, \quad \hat{\Omega}_2 = \frac{\Omega_2}{\Omega_1}
\]

(41)

\[
N^F \equiv \left( N_{rr}^F + N_{\theta\theta}^F \right) \Omega_1
\]

(42)

Then we have

\[
M^F = M^K + c_1 \log r + c_2 + \left( \frac{\Omega_3}{1 + \Omega_2} \right) k_1
\]

(43)

and

\[
N^F = \Omega_5 \left( M^K + c_1 \log r + c_2 \right) + \Omega_1 \Omega_6 k_1
\]

(44)

This completes the development of equations for the deflections, forces, and moments of functionally graded plates based on the first-order shear deformation theory in terms of the associated quantities of isotropic plates.
based on the classical plate theory. Thus, it remains that we develop particular relationships for axisymmetric bending of plates with various boundary conditions (see Wang and Lee, 1996).

4. Bending relationships for various boundary conditions

4.1. Roller-supported circular plate

Consider a solid circular plate with a roller support at \( r = a \), \( a \) being the radius of the plate. The boundary conditions are

\[
\begin{align*}
  \text{At } & r = 0: \quad u = 0, \quad \phi = 0, \quad \frac{dw_0^K}{dr} = 0, \quad Q_r = 0 \quad \text{(45-a)} \\
  \text{At } & r = a: \quad w = 0, \quad N_{rr} = 0, \quad M_{rr} = 0 \quad \text{(45-b)}
\end{align*}
\]

The above boundary conditions give

\[
\begin{align*}
  k_1 &= -\frac{2\Omega_5}{a^3\Omega_6} \left( \hat{\varphi} \frac{dw_0^K(a)}{dr} + \frac{\hat{c}_2}{2} a \right), \quad k_2 = 0, \quad c_1 = c_3 = 0 \quad \text{(46-a)} \\
  c_2 &= -\frac{2D}{a} \left( \frac{\nu - \hat{\Omega}_2 + \hat{\Omega}_5}{1 + \hat{\Omega}_2 - \hat{\Omega}_5} \right) \frac{dw_0^K(a)}{dr}, \quad c_4 = \frac{M^K(a)}{A_{55}} - \frac{\hat{c}_2 a^2}{4} \quad \text{(46-b)}
\end{align*}
\]

where \( \Omega_5 = \hat{\Omega}_3\Omega_5/\Omega_6 \). Hence, we have the following relations between the deflections, forces, and moments of the two theories

\[
\begin{align*}
  w_0^F(r) &= \hat{\varphi} w_0^K(r) + \frac{\mathcal{M}^K(r) - M^K(a)}{A_{55}} + \frac{1}{4} c_2 (a^2 - r^2) \quad \text{(47)} \\
  Q_r^F(r) &= Q_r^K(r) \quad \text{(48)} \\
  N_{rr}^F(r) &= \Omega_5 \hat{\varphi} \left( \frac{1}{a} \frac{dw_0^K(a)}{dr} - \frac{1}{r} \frac{dw_0^K}{dr} \right) \quad \text{(49)} \\
  N_{\theta\theta}^F(r) &= \Omega_5 \hat{\varphi} \left( \frac{1}{a} \frac{dw_0^K(a)}{dr} - \frac{d^2 w_0^K}{dr^2} \right) \quad \text{(50)} \\
  M_{rr}^F(r) &= M_{rr}^K(r) + D \frac{1}{r} \frac{dw_0^K}{dr} \left( \nu - \hat{\Omega}_2 \right) + \frac{1}{2} c_2 \left( 1 + \hat{\Omega}_2 \right) + \frac{1}{2} k_1 \hat{\Omega}_3 \quad \text{(51)} \\
  M_{\theta\theta}^F(r) &= M_{\theta\theta}^K(r) + D \frac{d^2 w_0^K}{dr^2} \left( \nu - \hat{\Omega}_2 \right) + \frac{1}{2} c_2 \left( 1 + \hat{\Omega}_2 \right) + \frac{1}{2} k_1 \hat{\Omega}_3 \quad \text{(52)}
\end{align*}
\]

4.2. Hinged circular plate

Consider a solid circular plate with a hinged support at \( r = a \). The boundary conditions are

\[
\begin{align*}
  \text{At } & r = 0: \quad u = 0, \quad \phi = 0, \quad \frac{dw_0^K}{dr} = 0, \quad Q_r = 0 \quad \text{(53-a)} \\
  \text{At } & r = a: \quad u = 0, \quad w = 0, \quad M_{rr} = 0 \quad \text{(53-b)}
\end{align*}
\]
The boundary conditions give

\[
\begin{align*}
    k_1 &= \frac{2B_{11}}{aA_{11}} \left( -\hat{D} \frac{dw_0^K}{dr} + \frac{\hat{c}_2 a}{2} \right), \quad k_2 = 0, \quad c_1 = c_3 = 0 \\
    c_2 &= -\frac{2D}{a} \left( \frac{\nu - \hat{\Omega}_2 - \hat{\Omega}_2}{1 + \hat{\Omega}_2 + \hat{\Omega}_2} \right) \frac{dw_0^K}{dr}, \quad c_4 = \frac{M^K_0}{A_{55}} - \frac{\hat{c}_2 a^2}{4}
\end{align*}
\]

(54-a)

(54-b)

where \( \Omega_2 = \hat{\Omega}_3 B_{11}/A_{11} \). The relations follow the same form as those given in (47) to (52) but \( k_1, c_2, \) and \( c_4 \) take the expressions given in (54-a) and (54-b).

4.3. Clamped circular plate

Consider a solid circular plate with a clamped support at \( r = a \). The boundary conditions are

\[
\begin{align*}
    \text{At } r = 0: & \quad u = 0, \quad \phi = 0, \quad \frac{du_0^K}{dr} = 0, \quad Q_r = 0 \\
    \text{At } r = a: & \quad u = 0, \quad w = 0, \quad \phi = 0, \quad \frac{dw_0^K}{dr} = 0
\end{align*}
\]

(55-a)

(55-b)

The boundary conditions give

\[
\begin{align*}
    k_1 = k_2 = 0, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = \frac{M^K(a)}{A_{55}}
\end{align*}
\]

(56)

Hence, we have the following relations between the deflections, forces, and moments of the two theories

\[
\begin{align*}
    w_0^F(r) &= \hat{D} w_0^K(r) + \frac{M^K(r) - M^K(a)}{A_{55}} \\
    Q_r^F(r) &= Q_r^K(r) \\
    N_{rr}(r) &= -\Omega_2 \hat{D} \frac{1}{r} \frac{dw_0^K}{dr} \\
    N_{\theta\theta}(r) &= -\Omega_2 \hat{D} \frac{d^2 w_0^K}{dr^2} \\
    M_{rr}^F(r) &= M_{rr}^K(r) + D \frac{1}{r} \frac{dw_0^K}{dr} (\nu - \hat{\Omega}_2) \\
    M_{\theta\theta}^F(r) &= M_{\theta\theta}^K(r) + D \frac{d^2 w_0^K}{dr^2} (\nu - \hat{\Omega}_2)
\end{align*}
\]

(57)

(58)

(59)

(60)

(61)

(62)

4.4. Clamped-free annular plate

Consider an annular plate with a clamped support at the inner edge \( r = b \) and free at the outer edge \( r = a \). The boundary conditions are

\[
\begin{align*}
    \text{At } r = b: & \quad u = 0, \quad w = 0, \quad \phi = 0, \quad \frac{dw_0^K}{dr} = 0 \\
    \text{At } r = a: & \quad N_{rr} = 0, \quad M_{rr} = 0, \quad Q_r = 0
\end{align*}
\]

(63-a)

(63-b)
The boundary conditions give

\[ k_1 = \Omega_5 \left( \frac{-2a \dot{D} \frac{dw_0^K}{dr}}{b^2 \Omega_7 - a^2 \Omega_6} \right), \quad k_2 = \frac{-b^2 k_1}{2}, \quad c_1 = 0 \]  \hfill (64-a)

\[ c_2 = -2 \alpha_1 D \frac{dw_0^K}{dr}, \quad c_3 = \frac{b^2 c_2}{2}, \quad c_4 = \frac{\mathcal{M}^K(b)}{A_{55}} - \frac{\dot{\Omega}_2 \Omega_5 d_2}{4} (1 - 2 \log b) \]  \hfill (64-b)

\[ d_1 = \frac{\nu - \Omega_2 - \dot{\Omega}_5 d_2}{a^2 (1 + \Omega_2) + b^2 (1 - \Omega_2 + \dot{\Omega}_5 d_2 (a^2 - b^2))}, \quad d_2 = \frac{a^2 \Omega_3 - b^2 \Omega_4}{b^2 \Omega_7 - a^2 \Omega_6} \]  \hfill (64-c)

4.5. Clamped-clamped annular plate

Consider an annular plate with clamped inner edge \( r = b \) and outer edge \( r = a \). The boundary conditions are

At \( r = b \): \( u = 0, \ w = 0, \ \phi = 0, \ \frac{dw_0^K}{dr} = 0 \) \hfill (65-a)

At \( r = a \): \( u = 0, \ w = 0, \ \phi = 0, \ \frac{dw_0^K}{dr} = 0 \) \hfill (65-b)

The boundary conditions give

\[ k_1 = k_2 = 0, \quad c_1 = \frac{\Omega_1}{A_{55} c_5} \left( \mathcal{M}^K(b) - \mathcal{M}^K(a) \right) \]  \hfill (66-a)

\[ c_2 = \left( \frac{b^2 \log b - a^2 \log a}{a^2 - b^2} + \frac{1}{2} \right) c_1, \quad c_3 = \left( \frac{a^2 b^2}{2(a^2 - b^2)} \log \frac{a}{b} \right) c_1 \]  \hfill (66-b)

\[ c_4 = \left[ \frac{a^2 + b^2}{16} + \frac{\Omega_1}{2 A_{55}} \log ab + \frac{a^2 b^2 (\log \frac{b}{a} - (\log ab)^2)}{4(b^2 - a^2)} \right] c_1 + \frac{\mathcal{M}^K(a) + \mathcal{M}^K(b)}{2 A_{55}} \]  \hfill (66-c)

\[ c_5 = \frac{\frac{a^2 - b^2}{8} + \frac{\Omega_1}{A_{55}} \log \frac{a}{b}}{2(a^2 - b^2)} \left( \log \frac{a}{b} \right)^2 \]  \hfill (66-d)

4.6. Clamped-roller supported annular plate

Lastly, consider an annular plate with clamped inner edge \( r = b \) and roller supported at the outer edge \( r = a \). The boundary conditions are

At \( r = b \): \( u = 0, \ w = 0, \ \phi = 0, \ \frac{dw_0^K}{dr} = 0 \) \hfill (67-a)

At \( r = a \): \( w = 0, \ N_{rr} = 0, \ M_{rr} = 0 \) \hfill (67-b)

The boundary conditions yield

\[ k_1 = \Omega_5 \left( \frac{-2a \dot{D} \frac{dw_0^K}{dr}}{b^2 \Omega_7 - a^2 \Omega_6} + \frac{\dot{\Omega}_2 a^2}{2} (2 \ln a - 1) + \dot{\Omega}_2 a^2 + 2 \dot{\Omega}_2 \right), \quad k_2 = \frac{-b^2 k_1}{2} \]  \hfill (68-a)
Axisymmetric bending of functionally graded circular and annular plates

\[
c_1 = \frac{f_1e_4 - f_2e_2}{e_1e_4 - e_2e_3}, \quad c_2 = \frac{f_1 - c_1e_1}{e_2}, \quad c_3 = -\frac{c_1b^2}{4}(2\ln b - 1) - \frac{c_2b^2}{2}
\]

\[
c_4 = \frac{M^K(b) + M^K(a)}{2A_{55}} + \frac{\hat{c}_1}{2} \left[ \frac{b^2}{4}(1 - \ln b) + \frac{a^2}{4}(1 - \ln a) + \frac{\Omega_1}{A_{55}} \ln ab \right] - \frac{\hat{c}_2}{8}(a^2 + b^2) - \frac{1}{2} \hat{c}_3 \ln ab
\]

\[
e_1 = \frac{b^2}{4}(1 - \ln b) - \frac{a^2}{4}(1 - \ln a) + \frac{b^2}{4}(2\ln b - 1) \ln \frac{b}{a} + \frac{\Omega_1}{A_{55}} \ln \frac{b}{a}
\]

\[
e_2 = \frac{b^2}{2} \ln \frac{b}{a} - \frac{(b^2 - a^2)}{4}
\]

\[
e_3 = \frac{1}{2} \left( 1 + \hat{\Omega}_2 \right) \ln a + \frac{1}{4} \left( 1 - \hat{\Omega}_2 \right) + \frac{b^2}{4a^2} \left( 1 - \hat{\Omega}_2 \right)
\]

\[
e_4 = \frac{1}{2} \left[ \left( 1 + \hat{\Omega}_2 \right) + \frac{b^2}{a^2} \left( 1 - \hat{\Omega}_2 \right) \right]
\]

\[
f_1 = \frac{M^K(a) - M^K(b)}{A_{55}} \Omega_1, \quad f_2 = \frac{D}{a} \left( \hat{\Omega}_2 - \nu \right) \frac{dw_0^K(a)}{dr}
\]

5. Illustrative examples

For exemplification of the relationships derived herein, we provide some examples. Consider the case of circular plates under uniformly distributed transverse load of intensity \( q_0 \). The classical plate solutions are given by (see Timoshenko and Woinowsky-Krieger, 1970; Reddy, 1999)

\[
w_0^K(r) = \begin{cases} \frac{qo_0^4}{64D} \left[ \left( \frac{r}{a} \right)^4 - 2 \left( \frac{3 + \nu}{4 + \nu} \right) \left( \frac{r}{a} \right)^2 + \frac{1 + \nu}{4 + \nu} \right], & \text{for simple support} \\ \frac{qo_0^4}{64D} \left[ 1 - \left( \frac{r}{a} \right)^2 \right], & \text{for clamped support} \end{cases}
\]

Now under a functionally graded plate whose modulus is assumed to be of the form

\[
E^F(z) = E_m \left( \frac{h - 2z}{2h} \right)^n + E_c \left[ 1 - \left( \frac{h - 2z}{2h} \right)^n \right]
\]

and \( \nu^F = \nu \) (i.e., independent of \( z \)). Here \( E_c \) and \( E_m \) denote the moduli of two different constituents, namely ceramic and metal. Let \( h \) denote the total plate thickness and \( K \) the shear correction factor.

We have

\[
A_{ij} = (Q_{ij}^c - Q_{ij}^m)h \left( \frac{1}{1 + n} \right) + Q_{ij}^m h
\]

\[
B_{ij} = (Q_{ij}^c - Q_{ij}^m) \frac{h^2}{2} \left( \frac{n}{(1 + n)(2 + n)} \right)
\]

\[
D_{ij} = (Q_{ij}^c - Q_{ij}^m) \frac{h^3}{12} \left( \frac{3(2 + n + n^2)}{(1 + n)(2 + n)(3 + n)} \right) + Q_{ij}^m \frac{h^3}{24}
\]

or

\[
A_{11} = A_{22} = \frac{h(E_m + nE_c)}{(1 + n)(1 - \nu^2)}, \quad A_{12} = \nu A_{11}
\]
\[ A_{55} = A_{44} = \frac{hK^2(E_m + nE_r)}{2(1 + n)(1 + \nu)} \quad (72-b) \]
\[ B_{11} = B_{22} = \frac{nh^2(E_r - E_m)}{2(1 + n)(2 + n)(1 - \nu^2)}, \quad B_{12} = \nu B_{11} \quad (72-c) \]
\[ D_{11} = D_{22} = \frac{h^3}{12(1 + n)(2 + n)(3 + n)(1 - \nu^2)} \left[ n(n^2 + 3n + 8)E_r + 3(n^2 + n + 2)E_m \right], \quad D_{12} = \nu D_{11} \quad (72-d) \]

If we define
\[ D_0 = \frac{E_r h^3}{12(1 - \nu^2)}, \quad G = \frac{E_r}{2(1 + \nu)}, \quad E_r = \frac{E_m}{E_c} \quad (73) \]
then the expressions in (72-a) to (72-d) can be written as
\[ A_{11} = \frac{12(n + E_r)D}{h^2(1 + n)}, \quad A_{55} = \frac{K^2 Gh(n + E_r)}{(1 + n)}, \quad B_{11} = \frac{6n(1 - E_r)D}{h(1 + n)(2 + n)} \quad (74-a) \]
\[ D_{11} = \frac{[n(n^2 + 3n + 8) + 3(n^2 + n + 2)E_r]D}{(1 + n)(2 + n)(3 + n)} \quad (74-b) \]

The constants \( \Omega_i \) have the following values: \( \Omega_5 = \Omega_8 = 0 \) and
\[ \Omega_1 = D_{11} - \frac{B_{11}^2}{A_{11}} \quad (75-a) \]
\[ = \frac{D[(n^4 + 4n^3 + 7n^2) + n(n^3 + 4n^2 + 7n + 3E_r)]}{(n + E_r)(3 + n)(2 + n)^2} \]
\[ \Omega_2 = D_{12} - \frac{B_{12} B_{11}}{A_{11}} = \nu \Omega_1 \quad (75-b) \]
\[ \Omega_3 = B_{11} + B_{12} = \frac{6n(1 - E_r)(1 + \nu)D}{h(1 + n)(2 + n)} \quad (75-c) \]
\[ \Omega_4 = B_{12} - B_{11} = -\frac{6n(1 - E_r)(1 - \nu)D}{h(1 + n)(2 + n)} \quad (75-d) \]
\[ \Omega_5 = 0 \quad (75-e) \]
\[ \Omega_6 = \frac{12(n + E_r)(1 + \nu)D}{h^2(1 + n)} \quad (75-f) \]
\[ \Omega_7 = -\frac{12(n + E_r)(1 - \nu)D}{h^2(1 + n)} \quad (75-g) \]
\[ \Omega_8 = 0 \quad (75-h) \]
\[ \Omega_9 = \frac{3(1 - E_r)^2 n^2(3 + n)(1 + \nu)}{(1 + n)[7n^2 + 4n^3 + n^4 + 4E_r(3E_r + 7n + 4n^2 + n^3)]} \quad (75-i) \]

By substituting the CPT solution given by Eq. (69) into (47) and (57), one obtains the deflection of the FGM plate as
\[ \bar{w}_0^F = \frac{64w_0^F D_r}{q_0 a^4} = \frac{D_r}{\Omega_1} \left[ \left( \frac{r}{a} \right)^4 - \frac{3 + \nu}{1 + \nu} \left( \frac{r}{a} \right)^2 + \frac{5 + \nu}{1 + \nu} \right] \]
\[ + \frac{8}{3K_r^2(1 - \nu_r)} \left[ 1 - \left( \frac{r}{a} \right)^2 \right] \left[ \frac{1 + n}{E_r + n} \right] - \frac{4\eta D_r}{(1 + \nu_r)\Omega_1} \left( \frac{\Omega_0}{1 + \nu + \Omega_9} \right) \left[ 1 - \left( \frac{r}{a} \right)^2 \right] \quad (76) \]
Axisymmetric bending of functionally graded circular and annular plates

\[ \eta = \begin{cases} 
0 & \text{for roller supported edge} \\
1 & \text{for simply supported edge} 
\end{cases} \]  

(77)

For clamped edge, we obtain

\[ w_0^F = \frac{64 E_c}{q_0 a^4} D_c \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^2 + \frac{8}{3K^2(1 - \nu_c)} \left( \frac{h}{a} \right)^2 \left[ 1 - \left( \frac{r}{a} \right)^2 \right] \left( \frac{1 + n}{E_r + n} \right) \]  

(78)

Table I. Non-dimensionalized maximum deflection \( w_0^F \) of functionally graded roller-supported circular plates (\( \nu_c = 0.288, E_r = 0.396, K_r = 5/6 \)).

<table>
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<tr>
<th>( n )</th>
<th>Thickness radius ratio, ( h/a )</th>
<th>0.0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
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<tbody>
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<td>10.396</td>
<td>10.481</td>
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<td>10.822</td>
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</tr>
<tr>
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</tr>
<tr>
<td>10^4</td>
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<td>4.158</td>
<td>4.214</td>
<td>4.293</td>
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</tr>
<tr>
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<td>4.151</td>
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<td>4.286</td>
<td></td>
</tr>
</tbody>
</table>

Table II. Non-dimensionalized maximum deflection \( w_0^F \) of functionally graded simply supported circular plates (\( \nu_c = 0.288, E_r = 0.396, K_r = 5/6 \)).

<table>
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<tr>
<th>( n )</th>
<th>Thickness radius ratio, ( h/a )</th>
<th>0.0</th>
<th>0.05</th>
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<th>0.15</th>
<th>0.2</th>
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<tr>
<td>0</td>
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<td>10.623</td>
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<td>4.158</td>
<td>4.214</td>
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<td></td>
</tr>
<tr>
<td>10^5</td>
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<td>4.208</td>
<td>4.286</td>
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</tr>
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</table>
Table III. Non-dimensionalized maximum deflection $\tilde{\omega}^F_n$ of functionally graded clamped circular plates ($\nu = 0.288$, $E_r = 0.396$, $K_s = 5/6$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h/a$</th>
<th>0.0</th>
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<th>0.15</th>
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</tr>
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</table>

In the above equations $K_s$ denotes the shear correction factor.

Considering a titanium-Zirconia FGM plate, i.e., $\nu = 0.288$, $E_r = 0.396$ msi, and taking $K_s^2 = 5/6$, the maximum deflection parameters at the plate center are tabulated in tables I–III for various values of $n$ and $h/a$ ratio.

From Eqs (48)–(52) and (59)–(62), the stress resultants for the FGM plates are (since $\Omega_5 = \Omega_6 = c_2 = k_1 = 0$)

$$N_{rr}^F = 0, \quad N_{\theta\theta}^F = 0, \quad M_{rr}^F = M_{rr}^K, \quad M_{\theta\theta}^F = M_{\theta\theta}^K$$

(79)

6. Conclusions

In this paper exact relationships between the bending solutions of the classical plate theory (CPT) and the first-order plate theory (FST) are developed for functionally graded circular plates. Then, exact solutions of functionally graded plates using the first-order theory are presented in terms of the solutions of the classical plate theory for a number of boundary conditions. Finally, numerical solutions of functionally graded plates are presented as a function of the modulus ratio and ratios of volume fractions.

Acknowledgement

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References