Kinetostatics and analysis methods for the impact problem

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Abstract – General kinetostatics and analytical methods are presented in this paper for solving the impact problem. The kinetostatics approach is obtained by using the mechanism of D’Alembert’s principle in the impact process. This approach uses the likely statical equilibrium equations to solve the dynamical impact problem. On the basis of kinetostatics, using the analytical dynamics method, the general dynamical equation and analytical equations for the impact problem are deduced. Examples of the impact problem from existing mechanics literature are solved to illustrate the merits of the methods presented in this paper. It is shown that the methods examined in this paper in solving the impact problem are simple and convenient, especially for complex systems with multi-freedom. © Elsevier, Paris

impact / kinetostatics method / analytic dynamics equation

1. Introduction

Impact within mechanical systems is a common phenomenon in applied engineering. To solve this problem, the general approach using Newton’s laws is certainly well known, but it is often difficult to handle. In classical mechanics, with Newton’s laws, the method to solve the impact problems is using the theorems of momentum and those of moment of momentum, which are also called theorems of impulse and theorems of moment of impulse (see Alfred and Chenoweth, 1983; Beer and Russel Johnston, 1984; Anand and Cunniff, 1984). This approach is convenient for solving the impact of a simple system, but it is inconvenient for complex impact systems which are often encountered in real industry.

The procedure followed in this paper is first, using D’Alembert’s principle (see Beer and Russel Johnston, 1984; Anand and Cunniff, 1985) the kinetostatics method which can be used to solve the impact problem of bodies is presented; also simplifying of the system of inertial impulse is carried out. Following this, with the combination of the kinetostatics method and the concept of virtual velocity using the dynamics method, the general dynamical equation and analytical equation for impact of bodies are presented. Finally, several examples are provided to illustrate the efficiency of the method developed in this paper.

2. Kinetostatics in the particle impact process

Assume that a particle has mass $m$, the impact impulse acting on this particle is $S$, the reaction impulse is $S_N$, the initial velocity and final velocity is $v$ and $u$ respectively. From the theorem of impact, the following equation is held

$$mu - mv = S + S_N$$

(1)

if we define

$$S_g = -(mu - mv)$$

(2)
then we have the following equation

\[ S + S_N + S_g = 0 \]  \hspace{1cm} (3)

where \( S_g \) is the reaction impact impulse because of the particle inertia, so it is called the inertial impulse. It acts on the body of imposing impulse. For example, a small rivet is clinched by hammering and the inertial impulse acts on the hammer. Following this, the formal equilibrium impulse system is combined of the impulse, reaction impulse and inertial impulse.

3. Kinetostatics in the particle system impact process

Assume a particle system has \( n \) particles, the mass for the \( i \)-th particle is \( m_i \), the external impact impulse acting on \( m_i \) is \( S_i^e \), the internal impulse acting on \( m_i \) is \( S_i^i \), the reaction impulse is \( S_{Ni} \), the inertial impulse is

\[ S_{gi} = -(m_i \mathbf{u}_i - m_i \mathbf{v}_i) \]  \hspace{1cm} (4)

then the equilibrium equation for \( m_i \) is

\[ S_i^e + S_i^i + S_{Ni} + S_{gi} = 0 \]  \hspace{1cm} (5)

Regarding the particle system, the kinetostatics can be written as the following equations

\[ \begin{align*}
\sum_{i=1}^{n} S_i^e + \sum_{i=1}^{n} S_{Ni} + \sum_{i=1}^{n} S_{gi} &= 0 \\
\sum_{i=1}^{n} m_i (S_i^i) + \sum_{i=1}^{n} m_i (S_{Ni}) + \sum_{i=1}^{n} m_i (S_{gi}) &= 0
\end{align*} \]  \hspace{1cm} (6)

In three-dimensional Cartesian coordinates, if we express the \( x, y, z \) axis with \( x_j (j = 1, 2, 3) \), the vector equations above can be written as follows

\[ \begin{align*}
\sum S_{ij}^e + \sum S_{Ni,j} + \sum S_{gi,j} &= 0 \\
\sum m_{ij} (S^i) + \sum m_{ij} (S^N) + \sum m_{ij} (S^g) &= 0
\end{align*} \]  \hspace{1cm} (7)

4. Reduction of the inertial impulse system of rigid bodies

4.1. General motion of a rigid body in the impact process

When a rigid body takes general motion after impact, the inertial impulse imposes on every particle and a general impulse system is formed which can be simplified by a principal impulse vector and a principal impulse moment vector. The principal impulse vector is free from the simplifying center, and the principal moment is related to the simplifying center. When the simplifying center is the mass center \( C \), the relative motion principal moment is equal to the absolute motion principal moment, and the impact process can be taken as the combination of translation of mass center \( C \) and rotation relative to mass center \( C \), as is shown in figure 1; in this case the equation of translation can be expressed as

\[ S_g = -(M \mathbf{u}_r - M \mathbf{v}_r) \]  \hspace{1cm} (8)
where

\[ u_c = u_i, \quad v_c = v_i \quad \text{and} \quad M = \sum_{i=1}^{n} m_i \]

The equation of relative rotation can be obtained

\[ L_g = \sum_{i=1}^{n} m_i (S_{gi}) \]  \hspace{1cm} (9)

If Eq. (9) is written in component form in three-dimensional Cartesian coordinates, the impulse moments to the \(x, y, z\) axis are

\[
\begin{align*}
L_{gx} &= \sum_{i=1}^{n} m_{x'i'}(S_{gi}) = I_{x'z'}\omega_2 - I_{x'z'}\omega_1 \\
L_{gy} &= \sum_{i=1}^{n} m_{y'i'}(S_{gi}) = I_{y'z'}\omega_2 - I_{y'z'}\omega_1 \\
L_{gz} &= \sum_{i=1}^{n} m_{z'i'}(S_{gi}) = -I_{z'}\omega_2 + I_{z'}\omega_1 \\
L_g &= \sqrt{L_{gx}^2 + L_{gy}^2 + L_{gz}^2} \hspace{1cm} (10)
\end{align*}
\]

where

\[
I_{x'z'} = \sum_{i=1}^{n} m_i x_i'z_i', \quad I_{y'z'} = \sum_{i=1}^{n} m_i y_i'z_i', \quad I_{z'} = \sum_{i=1}^{n} m_i r_i'^2 \hspace{1cm} (12)
\]

\(\omega_1\) and \(\omega_2\) are the angular velocities of the rigid body before and after impact.
4.2. General plane motion of the rigid body in the impact process

The plane motion is the special case of the general motion of a rigid body. When the rigid body takes the general plane motion, the following equation is held $I_{x'z'} = I_{y'z'} = 0$, $L_{yx} = L_{yy} = 0$. At this time, the simplifying of the inertial impulse to the mass center is an impulse acting on the mass center $C$, and a couple impulse, which can be described by the following equation

$$
\begin{align*}
S_y &= -Mu_e + Mv_e \\
L_y &= -I_e \omega_2 + I_e \omega_1
\end{align*}
$$

(13)

The direction of vectors is shown in figure 2.

![Figure 2. General plane motion of a rigid body in the impact process.](image)

The pure translation and the pure rotation of rigid bodies are also the special cases of general motion. So, the translation can be described by Eq. (8) and rotation can be described by Eqs (8) and (10).

5. Virtual velocity and generalized impulse

From the virtual displacement theory, regarding the particle system with idealized constraints, one has the following equation

$$
\sum F_i \cdot \delta r_i = 0
$$

(14)

The above equation can also be written as

$$
\sum S_i \cdot \delta \dot{r}_i = 0
$$

(15)

where $S$ is the external impact impulse acting on the particle system, and $\delta \dot{r}_i$ is the virtual velocity.

For the particle system with $n$ particles, if there are $m$ idealized and completed constraints, then this system has $3n - m$ freedom, that is to say, this system has $3n - m$ generalized coordinates, if $u_i$ is the velocity of a particle at any time of the impact process, and $v_i$ is the velocity of impact start, then the following equation is held

$$
\dot{r}^* = u_i - v_i \quad i = 1, 2, 3, ..., n
$$

(16)

It is easy to see that the following equation is also held

$$
\sum S_i \cdot \delta \dot{r}_i^* = 0
$$

(17)
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If \( \dot{q}_1, \dot{q}_2, \dot{q}_3, ..., \dot{q}_{3n-m} \) are the generalized velocity coordinates of this system regarding the impact problem, the equation can be written as a function of virtual velocity, that is

\[
\dot{r}_i^* = \dot{r}_i^*(\dot{q}_1, \dot{q}_2, \dot{q}_3, ..., \dot{q}_{3n-m})
\]  

(18)

If variation is taken on the above equation, then

\[
\delta \dot{r}_i^* = \sum_{k=1}^{3n-m} \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} \delta \dot{q}_k
\]

(19)

Substitute Eq. (19) for Eq. (17), then

\[
\sum_{i=1}^{n} S_i \cdot \left( \sum_{k=1}^{3n-m} \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} \delta \dot{q}_k \right) = 0
\]

(20)

The above equation can also be written as

\[
\sum_{k=1}^{3n-m} \left( \sum_{i=1}^{n} S_i \cdot \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} \right) \delta \dot{q}_k = 0
\]

(21)

If we define the generalized impact impulse as

\[
Q_k^* = \sum_{i=1}^{n} S_i \cdot \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k}, \quad k = 1, 2, 3, ..., 3n - m
\]

(22)

then

\[
\sum_{i=1}^{n} S_i \cdot \delta \dot{r}_i^* = \sum_{k=1}^{3n-m} Q_k^* \delta \dot{q}_k
\]

(23)

The generalized impact impulse can be written in three-dimensional Cartesian coordinates as the following

\[
Q_k^* = \sum_{i=1}^{n} \left( S_{ix} \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} + S_{iy} \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} + S_{iz} \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} \right)
\]

(24)

where

\[
x_i^* = u_{ix} - v_{ix}, \quad y_i^* = u_{iy} - v_{iy}, \quad z_i^* = u_{iz} - v_{iz}
\]

(25)

There is another method to obtain the \( Q_k^* \). If the generalized velocity \( \delta \dot{q}_k \neq 0 \), and

\[
\delta \dot{q}_i = 0, \quad i = 1, 2, 3, ..., 3n - m - 1, \quad i \neq k
\]

(26)

from

\[
\sum \delta W'_s = Q_k^* \delta \dot{q}_k
\]

(27)

the following equation is obtained

\[
Q_k^* = \frac{\sum \delta W'_s}{\delta \dot{q}_k}
\]

(28)

where \( \sum \delta W'_s \) is the multiplication of active impulse and virtual velocity.
6. The equation of analysis method for the impact problem

If the particle system consists of \( n \) particles, the mass of a particle is \( m_i \), the active impulse is \( S_i \), and the inertial impulse is \( S_{gi} \), the velocities of the particle before impact and in the impact process are \( v_i \) and \( u_i \) respectively. From Eqs (4) and (16) we have

\[
S_{gi} = -(m_i u_i - m_i v_i) = -m_i \dot{r}_i^*
\]  

(29)

As for the particle system with idealized constraints, if the kinetostatics and virtual velocity concept are combined the generalized impact dynamics equation is

\[
\sum_{i=1}^{n} (S_i + S_{gi}) \cdot \delta \dot{r}_i^* = 0
\]

(30)

In three-dimensional Cartesian coordinates, the above equation can be written as

\[
\sum_{i=1}^{n} [(S_{ix} + S_{gix}) \delta \dot{x}_i^* + (S_{iy} + S_{giy}) \delta \dot{y}_i^* + (S_{iz} + S_{giz}) \delta \dot{z}_i^*] = 0
\]

(31)

If Eqs (19), (23) and (29) are substituted for Eq. (30), then

\[
\sum_{k=1}^{3n-m} \left[ Q_k - \sum_{i=1}^{n} m_i \dot{r}_i^* \cdot \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} \right] \delta \dot{q}_k = 0
\]

(32)

According to the concept of displacement of difference, \( \delta \dot{q}_k \neq 0 \), is derived the following equation

\[
Q_k - \sum_{i=1}^{n} m_i \dot{r}_i^* \cdot \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} = 0
\]

(33)

Take Eq. (16) is changed into Eq. (33), after some manipulation then the following equation can be obtained

\[
\sum_{i=1}^{n} m_i \dot{r}_i^* \cdot \frac{\partial \dot{r}_i^*}{\partial \dot{q}_k} = \sum_{i=1}^{n} m_i (u_i - v_i) \cdot \frac{\partial (u_i - v_i)}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k} \left[ \sum_{i=1}^{n} \frac{1}{2} m_i (u_i - v_i)^2 \right]
\]

(34)

If we define

\[
T^* = \sum_{i=1}^{n} \frac{1}{2} m_i (u_i - v_i)^2
\]

(35)

then Eq. (33) can be written as

\[
\frac{\partial T^*}{\partial \dot{q}_k} = Q_k^*, \quad k = 1, 2, ..., 3n - m
\]

(36)

The above equation is the equation of the analysis method for the impact problem, and the equation numbers are equal to the freedom numbers of particle system. \( T^* \) is the changing of kinetic energy in the impact process, and \( \partial T^*/\partial \dot{q}_k \) is the changing value of the generalized momentum.

From Eq. (36), it can be known that the generalized momentum is conservational under the condition of the generalized impulse \( Q_k^* = 0 \).

7. Examples

In order to demonstrate the above theory, two examples of rigid body impact are presented, and each example is solved by two methods provided above.
Example 1. As figure 3 shows, each of bars OA and AB is of length \( l \), and mass \( m \), and determine the angular velocity of each bar and velocity of point C immediately after the impulse \( S \) is applied at C.

Solution 1. This is obtained using the kinetostatics method.

The impact problem appearing in this example is the problem of the impact system. Assume, at the start and end of impacting, the bar OA has the angular velocity \( \omega_0 \) and \( \omega_1 \) respectively; regarding bar AB, the angular velocity is \( \omega_2 \) and \( \omega_2 \) respectively, the mass center C has the velocity \( v_c \). Before impacting, there is no velocity, so \( \omega_0 = \omega_2 = 0 \). After the start of impact, OA rod is rotated around impact point O, the motion of AB bar is plane motion, the moment of impulse is simplifying to point C, as shown in figure 3.

\[
\begin{align*}
I_c &= I_1 = \frac{1}{12} ml^2, \quad I_0 = \frac{1}{3} ml^2 \\
\mathbf{u}_c &= l\mathbf{\omega}_1 + \frac{l}{2}\mathbf{\omega}_2
\end{align*}
\]
Take the system as the study object

\[ \sum m_0(S) = 0 \quad \frac{3}{2}Sl - \frac{3}{2}mu_l - I_1\omega_2 - I_0\omega_1 = 0 \]  

(39)

thus, we have the following equation

\[ 11ml\omega_1 + 5ml\omega_2 = 9S \]  

(40)

With the bar \( AB \) taken as the study object

\[ \sum m_A(S) = 0 \quad \frac{1}{2}Sl - \frac{1}{2}mu_l - I_1\omega_2 = 0 \]  

(41)

then we have the following equation

\[ 3ml\omega_1 + 2ml\omega_2 = 3S \]  

(42)

Thus the solution is

\[ \omega_1 = \frac{3S}{7ml}, \quad \omega_2 = \frac{6S}{7ml}, \quad u_c = \frac{6S}{7m} \]  

(43)

Solution II. This is obtained using the method of analysis.

The system has two freedoms, and the generalized velocities are taken as \( \omega_1 \) and \( \omega_2 \).

So we have the following equations

\[ T^* = \frac{1}{2}I_0\omega_1^2 + \frac{1}{2}mu_l^2 + \frac{1}{2}I_1\omega_2^2 \]  

(44)

\[ Q_1^* = lS, \quad Q_2^* = \frac{l}{2}S \]

From Eq. (36) we have the following equations

\[ 8ml\omega_1 + 3ml\omega_2 = 6S \]  

(45)

\[ 3ml\omega_1 + 2ml\omega_2 = 3S \]

then the solution is

\[ \omega_1 = \frac{3S}{7ml}, \quad \omega_2 = \frac{6S}{7ml}, \quad u_c = \frac{6S}{7m} \]  

(46)

If the example is solved by using classical mechanics method, the procedure will be much more complex than by using the method in the paper.

Example II. A slider block with mass \( m \) slides on a horizontal, frictionless track MM, which is connected with rod \( AB \) at point \( A \) of rod \( AB \). There a uniform mass disk is connected at point \( B \), the mass of rod \( AB \) is negligible, as shown in figure 4. The disk is rotated along the horizontal plane. At a moment, points \( A, B, C, D \) are at the same line; at this time, the disk is impacting on a step at point \( D \). At start of impacting, the disk has angular velocity \( \omega_0 \); the slider block has velocity \( v_A \); rod \( AB \) has the angular \( \alpha \) with horizontal line, the impact is plastic. Find the angular velocity \( \Omega \) of disk, \( u_A, u_B \), and the impact impulse \( S_A \) of track MM to the slider block.

Solution I. This is obtained using the kinestatics method.
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Figure 4. Impact Example II using the kinestatics method.

Assume before and after impact, the velocities of points $A$, $B$, $C$ are $v_A$, $v_B$, $v_C$ and $u_A$, $u_B$, $u_C$ respectively. The simplifying of the impact impulse system is shown in figure 4. From the theory of velocity projection, at start of impact we have

$$v_A = v_c = r\omega_0$$

At end of impact we have

$$u_A = 0$$

With the disk taken as the study object

$$\sum m_D(S) = 0, \quad Mu_Cr + I\Omega - Mv_Cr \sin \alpha - I\omega_0 = 0$$

where

$$u_C = r\Omega, \quad v_C = r\omega_0, \quad I = \frac{1}{2}Mr^2$$

Thus we have

$$\Omega = \frac{(1 + 2\sin \alpha)\omega_0}{3}, \quad u_B = 2r\Omega$$

Take the slider block as the study object

$$\sum S_t = 0, \quad -S_A \cos \alpha + m v_A \sin \alpha = 0,$$

$$S_A = m v_A \tan \alpha$$

so the impact impulse of slider block to the track is

$$S'_A = -m v_A \tan \alpha$$
Solution II. This is obtained using the method of analysis.

Because in equation (36) there is no constraint impulse, so the constraint of point A is removed, and the impact impulse is applied. Assume $\Omega$ and $u_A y$ are generalized velocities. In figure 5 as shown from the motion of system we have. Regarding the disk

\[
\begin{align*}
    u_{\text{en}} - v_{\text{en}} &= r\omega_0 \cos \alpha \\
    u_{e\tau} - v_{e\tau} &= r\Omega - r\omega_0 \sin \alpha
\end{align*}
\]  

(54)

\[\text{Figure 5. Impact Example II using the method of analysis.}\]

Regarding bar $AB$

\[
    u_A x = -u_A y \tan \alpha, \quad v_A = v_C = r\omega_0
\]  

(55)

so, we have the following equation

\[
    T^* = \frac{1}{2} M(r\omega_0 \cos \alpha)^2 + \frac{1}{2} M(r\Omega - r\omega_0 \sin \alpha)^2 + \frac{1}{2} I(\Omega - \omega_0)^2 \\
    + \frac{1}{2} m(-u_A y \tan \alpha - v_A)^2 + \frac{1}{2} m u_A y^2
\]  

(56)

Differentiate $T^*$ with $\Omega$ and substitute into Eq. (36), we have following equation

\[
    M(r^2 \Omega - r^2 \omega_0 \sin \alpha) + I(\Omega - \omega_0) = 0
\]  

(57)

thus we have

\[
    \Omega = \frac{(1 + 2 \sin \alpha)\omega_0}{3}, \quad u_B = 2r\Omega
\]  

(58)

Take variation to generalized velocity $u_A y$ and then taken into the equation above, we have

\[
    m(-u_A y \tan \alpha - v_A)(-\tan \alpha) + m u_A y = S_A
\]  

(59)

From the constraint condition $u_A y = 0$, we have

\[
    S_A = mv_A \tan \alpha, \quad u_A = u_A x = 0
\]  

(60)
From the above examples, it is shown that using the methods provided in this paper the solution process is very simple, especially regarding the multi-freedom problem.

8. Conclusion

This paper provides kinetostatics and analysis methods for the impact problem. These methods forming an independent system, are the extension of classical mechanics, and new methods. Using these methods, the impact problem can be solved easily and conveniently, especially regarding the multi-freedom system.

From the examples provided in this paper, regarding the solution of impact impulse of the impact problem, the kinetostatics method is convenient, and as regards the solution of velocity of the impact problem, the analysis method is convenient.

As the classical method, the methods provided in this paper, combined with the coefficient of restitution form the complete method for solving the generalized impact problem.

References