A study of chaotic motion in elastic cylindrical shells

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Abstract – The chaotic motion of an elastic cylindrical shell has been studied in this paper; its dynamic equation contains square and cubic nonlinear items. By means of the Galerkin approach and the Melnikov method, the critical condition for chaotic motion has been obtained. Two demonstrative examples have been discussed through Poincaré mapping, phase portrait and time history. © Elsevier, Paris

chaotic motion / elastic cylindrical shell / single mode model

1. Introduction

In recent years, chaos in nonlinear dynamic systems has aroused more and more interest. Chaotic motion is regarded as a natural extension of the study object in nonlinear vibration. In solid mechanics, the study of chaotic motion is only about buckled beams, and this phenomenon has been more fully understood; but the chaotic motion of plates or shells has not been studied.

This paper discusses the chaotic motion of an elastic cylindrical shell; its dynamic equation contains square and cubic nonlinear items. The Duffing equation is a special case within this system. It indicates many governing equations of nonlinear vibration. In the paper, the critical condition that the system enters chaotic states is given by the Melnikov method. By Poincaré mapping, phase portrait and a time-displacement history diagram, whether the chaos occurs is determined, and some beneficial enquiry into this system has been made.

2. The fundamental equations

As shown in figure 1, an elastic cylindrical shell of diameter 2R, thickness h and length L is subjected to the distributed radial load q. One end of the shell is pinned, the axial displacement of the other end is a constant under the axial load; then this end is also pinned.

\[ q = F \cos \omega t \] (1)
According to Reissner's variational principle, the dynamic equation of the shell is the following:

\[
\begin{aligned}
\frac{\partial N_x}{\partial x} &= \rho h \left[ \frac{\partial^2 u}{\partial t^2} + \frac{h^2}{12R} \frac{\partial^2 \varphi}{\partial t^2} \right] \\
\frac{\partial Q_x}{\partial x} - \frac{N_\theta}{R} &= \rho h \ddot{W} + \delta \ddot{W} - F \cos \omega t \\
\frac{\partial M_x}{\partial x} - Q_x + \frac{\partial}{\partial x} (N_x W) &= \frac{\rho h^3}{12} \left( \frac{\partial^2 \varphi}{\partial t^2} + \frac{r^2}{R} \frac{\partial^2 u}{\partial t^2} \right)
\end{aligned}
\]  

(2)

The forces and the bending moment in unit length of the shell are given in the following equations:

\[
\begin{aligned}
N_x &= \frac{Eh}{1-\mu^2} \left[ \varepsilon_x + \frac{\mu}{R} W + \frac{h^2}{12R} \frac{\partial \varphi}{\partial x} \right] \\
N_\theta &= \frac{Eh}{1-\mu^2} \left[ \mu \varepsilon_x + \frac{W}{R} \left( 1 + \frac{h^2}{12R^2} \right) \right] \\
Q_x &= KGh \left( \varphi + \frac{\partial W}{\partial x} \right) \\
M_x &= D \frac{\partial \varphi}{\partial x} \\
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial W}{\partial x} \right)^2
\end{aligned}
\]  

(3)

Where \(u(x, t)\) is the axial displacement, \(W(x, t)\) the radial displacement, \(\varphi(x, t)\) the rotational angle of normal line, \(E\) the elastic modulus, \(G\) the shear modulus, \(\mu\) the Poisson ratio, \(K\) the shear deforming coefficient, \(\delta\) the damping coefficient, and \(D = Eh^3/12(1-\mu^2)\).

Firstly we assume that the following hypotheses hold true: 1) all of the variables are functions only with \(x\) and \(t\), the axial symmetric deformation is only taken into consideration; 2) the axial inertia and the rotational inertia are not taken into account; 3) the Kirchhoff hypothesis is true \(\varphi = -\frac{\partial W}{\partial x}\); 4) \(h \ll R\).

Using Eqs (2) and (3), we get the dynamic equation in the following form:

\[
D \frac{\partial^4 W}{\partial x^4} - N_x \frac{\partial^2 W}{\partial x^2} + \rho h \frac{\partial^2 W}{\partial t^2} + \frac{Eh}{R^2} W + \frac{\mu}{R} N_x = F \cos \omega t - \delta \ddot{W}
\]  

(4)

The first formula in Eq. (3) can be changed into Eq. (5):

\[
N_x = \frac{Eh}{1-\mu^2} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial W}{\partial x} \right)^2 + \frac{\mu}{R} W - \frac{h^2}{12R} \frac{\partial^2 W}{\partial x^2} \right]
\]  

(5)

Equation (5) is integrated in the range \((0, L)\):

\[
N_x = \frac{Eh}{(1-\mu^2) L} \left[ u(L, t) - u(0, t) + \frac{1}{2} \int_0^L \left( \frac{\partial W}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W}{\partial x^2} dx \right]
\]  

(6)

For the pinned shell the axial boundary conditions are:

\[u(L, t) = 0, \quad u(0, t) = C > 0 \quad (C = \text{const})\]

(7)
and

\[ N_r = \frac{Eh}{(1 - \mu^2) L} \left[ -C + \frac{1}{2} \int_0^L \left( \frac{\partial W}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W}{\partial x^2} dx \right] \]  

(8)

When Eq. (8) is substituted into the dynamic equation, we can get the following nonlinear equation

\[
D \frac{\partial^4 W}{\partial x^4} - \frac{Eh}{(1 - \mu^2) L} \left[ -C + \frac{1}{2} \int_0^L \left( \frac{\partial W}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W}{\partial x^2} dx \right] \frac{\partial^2 W}{\partial x^2} + \rho \frac{\partial^2 W}{\partial t^2} \\
+ \frac{Eh}{R^2} W + \frac{\mu}{R} \frac{Eh}{(1 - \mu^2) L} \left[ -C + \frac{1}{2} \int_0^L \left( \frac{\partial W}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W}{\partial x^2} dx \right] \\
= F \cos \omega t - \delta \dot{W}
\]  

(9)

Let

\[ P = \frac{EhC}{L}, \quad W = W^* + \frac{\mu RP}{Eh} \]

(10)

Equation (9) can be changed into the following form

\[
D \frac{\partial^4 W^*}{\partial x^4} + P \frac{\partial^2 W^*}{\partial x^2} - \frac{Eh}{(1 - \mu^2) L} \left[ \frac{1}{2} \int_0^L \left( \frac{\partial W^*}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W^* dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W^*}{\partial x^2} dx \right] \\
\times \frac{\partial^2 W^*}{\partial x^2} + \rho \frac{\partial^2 W^*}{\partial t^2} + \frac{Eh}{R^2} W^* + \frac{\mu}{R} \frac{Eh}{(1 - \mu^2) L} \\
\times \left[ \frac{1}{2} \int_0^L \left( \frac{\partial W^*}{\partial x} \right)^2 dx + \frac{\mu}{R} \int_0^L W^* dx - \frac{h^2}{12R} \int_0^L \frac{\partial^2 W^*}{\partial x^2} dx \right] = F \cos \omega t - \delta \dot{W}^*
\]  

(11)

Its nondimensional equation is as follows

\[
\bar{W}''' + \bar{W}'' + \bar{W} + \alpha^2 \bar{W} = - \left[ \beta_1 \int_0^L \bar{W}'^2 d\bar{x} + \beta_2 \int_0^L \bar{W} d\bar{x} - \beta_3 \int_0^L \bar{W}'' d\bar{x} \right] \bar{W}'' + \beta_4 \int_0^L \bar{W}'^2 d\bar{x} \\
+ \beta_5 \int_0^L \bar{W} d\bar{x} - \beta_6 \int_0^L \bar{W}'' d\bar{x} = f \cos \bar{\omega} \cdot \bar{t} - \gamma \dot{\bar{W}}
\]  

(12)

where

\[
\begin{align*}
\alpha^2 &= \frac{EhD}{P^2 R^2}, \quad \beta_1 = \frac{EhR^2}{2(1 - \mu^2) LP} \sqrt{\frac{P}{D}}, \quad \beta_2 = \frac{Eh\mu}{(1 - \mu^2) LP} \sqrt{\frac{D}{P}}, \quad \beta_3 = \frac{1}{L} \sqrt{\frac{D}{P}}, \quad \beta_4 = \frac{1}{2} \beta_2 \\
\beta_5 &= \frac{Eh\mu}{(1 - \mu^2) R^2 LP} \left( \frac{D}{P} \right)^{3/2}, \quad \beta_6 = \frac{\mu}{R^2 L} \left( \frac{D}{P} \right)^{3/2}, \quad f = \frac{FD}{P^2 R}, \quad \bar{\omega} = \omega \sqrt{\frac{\rho hD}{P}}, \quad \frac{\bar{W}}{R} = \frac{W^*}{R}, \\
\gamma &= \frac{D\delta}{P \sqrt{\rho hD}}, \quad \bar{t} = \frac{\sqrt{\rho hD}}{P} x, \quad t = \frac{\sqrt{\rho hD}}{P}
\end{align*}
\]  

(13)
According to the Galerkin method one obtains

\[
\int_0^L \left\{ \dddot{W} + \ddot{W} + \dot{W} + \alpha^2 \dddot{W} - \left[ \beta_1 \int_0^L \dot{W}^2 \, d\bar{x} + \beta_2 \int_0^L W \, d\bar{x} - \beta_3 \int_0^L W'' \, d\bar{x} \right] \right. \\
\times \dddot{W} + \beta_4 \int_0^L \dddot{W}^2 \, d\bar{x} + \beta_5 \int_0^L \dot{W} \, d\bar{x} - \beta_6 \int_0^L W'' \, d\bar{x} - \int_0^L W'' \cos \dot{\bar{x}} - \gamma \ddot{W} \left. \right\} \delta \ddot{W} \, d\bar{x} = 0 \quad (14)
\]

The boundary conditions are

\[
\dddot{W}(0, \bar{t}) = \dddot{W}(L, \bar{t}) = \dddot{W}(0, \bar{t}) = \dddot{W}(L, \bar{t}) = 0 \quad (15)
\]

So, let

\[
\bar{W}(\bar{x}, \bar{t}) = T(\bar{t}) \sin \frac{\pi \bar{x}}{L} \quad (16)
\]

Substituting into Eq. (14) gives the nonlinear dynamic system

\[
\dddot{T} - \lambda_1 T + \lambda_2 T^2 + \lambda_3 T^3 = \varepsilon (g \cos \dot{\bar{x}} - \varepsilon' \dot{T}) \quad (17)
\]

where

\[
\begin{align*}
\lambda_1 &= \left( \frac{\pi}{L} \right)^2 - \left( \frac{\pi}{L} \right)^4 - \alpha^2 - \frac{8}{L} \beta_5 \left( \frac{L}{\pi} \right)^2 - \frac{8}{L} \beta_6 \\
\lambda_2 &= 2 \left[ \beta_2 + \beta_3 \left( \frac{\pi}{L} \right)^2 + \beta_4 \right] \left( \frac{\pi}{L} \right) > 0 \\
\lambda_3 &= \frac{L}{2} \left( \frac{\pi}{L} \right) \beta_1 > 0 \\
g &= \frac{4f}{\pi \varepsilon} > 0, \quad \varepsilon' = \gamma / \varepsilon > 0
\end{align*} \quad (18)
\]

3. The chaotic motion for \( \lambda_1 > 0 \)

When \( \lambda_1 > 0 \), the following condition is true

\[
\frac{D L}{E h y^*} > \frac{C}{D L} \quad (19)
\]

where

\[
\begin{align*}
y^* &= \left( \frac{1}{2} \left( \frac{d}{a} \right) + \sqrt{\frac{1}{4} \left( \frac{d}{a} \right)^2 + \frac{1}{27} \left( \frac{b}{a} \right)^3} \right)^{1/3} + \left( \frac{1}{2} \left( \frac{d}{a} \right) - \sqrt{\frac{1}{4} \left( \frac{d}{a} \right)^2 + \frac{1}{27} \left( \frac{b}{a} \right)^3} \right)^{1/3} \\
\left( \frac{d}{a} \right) &= \left( 1 - \mu^2 \right) \frac{R^2 L^2 D}{8E h \mu^2} \\
\left( \frac{b}{a} \right) &= \left[ \frac{8 \mu}{R^2 L^2} + \frac{E h}{R^2 D} + \left( \frac{\pi}{L} \right)^4 \right] \left( \frac{L}{\pi} \right)^2 \frac{1 - \mu^2}{8E h \mu^2} \frac{R^2 L^2 D}{8E h \mu^2}
\end{align*} \quad (20)
\]
Let us consider the equation \( \ddot{\theta} - \lambda_1 T + \lambda_2 T^2 + \lambda_3 T^3 = \varepsilon (g \cos \omega l - \varepsilon' \dot{\theta}) (\lambda_1, \lambda_2, \lambda_3 > 0) \), where \( \varepsilon \) is a small parameter. It is similar to the Duffing equation, but contains a square item. The dynamic system corresponding to Eq. (17) is

\[
\begin{align*}
\dot{y} &= \dot{\theta} \\
\dot{y} &= \lambda_1 T - \lambda_2 T^2 - \lambda_3 T^3 + \varepsilon (g \cos \omega l - \varepsilon' \dot{\theta})
\end{align*}
\] (21)

Its unperturbed system is

\[
\ddot{\theta} - \lambda_1 T + \lambda_2 T^2 + \lambda_3 T^3 = 0
\] (22)

Equation (22) has the first integration

\[
\frac{1}{2} \dot{\theta}^2 - \frac{1}{2} \lambda_1 T^2 + \frac{1}{3} \lambda_2 T^3 + \frac{1}{4} \lambda_3 T^4 = H
\] (23)

For the different values of \( H \), they indicate the different curves in the phase portrait; its value is determined by the initial conditions.

Equation (22) can be changed into the following form

\[
\begin{align*}
\dot{y} &= \dot{\theta} \\
\dot{y} &= \lambda_1 T - \lambda_2 T^2 - \lambda_3 T^3
\end{align*}
\] (24)

Let \( y = \dot{y} = 0 \); the three fixed points are \( O, A \) and \( B \)

\[
O (0, 0), \quad A \left( \frac{-\lambda_2 - \sqrt{\lambda_2^2 + 4\lambda_1 \lambda_3}}{2\lambda_3}, 0 \right), \quad B \left( \frac{-\lambda_2 + \sqrt{\lambda_2^2 + 4\lambda_1 \lambda_3}}{2\lambda_3}, 0 \right)
\]

Where \( O \) is a hyperbolic-type fixed point and the others are stable fixed points that lie at the two sides of point \( O \).

Let

\[
\xi_1 = \frac{-\lambda_2 - \sqrt{\lambda_2^2 + 4\lambda_1 \lambda_3}}{2\lambda_3} < 0, \quad \xi_2 = \frac{-\lambda_2 + \sqrt{\lambda_2^2 + 4\lambda_1 \lambda_3}}{2\lambda_3} > 0
\] (25)

The conditions that the three points exist are

\[
\begin{align*}
H &\geq -\frac{1}{2} \lambda_1 \xi_1^2 + \frac{1}{3} \lambda_2 \xi_1^3 + \frac{1}{4} \lambda_3 \xi_1^4 \\
H &\geq -\frac{1}{2} \lambda_1 \xi_2^2 + \frac{1}{3} \lambda_2 \xi_2^3 + \frac{1}{4} \lambda_3 \xi_2^4
\end{align*}
\] (26)

Now let us find the homoclinic motion; with \( H = 0 \), one has

\[
\dot{T}^* = \pm \sqrt{\lambda_1 T^{*2} - \frac{2}{3} \lambda_2 T^{*3} - \frac{1}{2} \lambda_3 T^{*4}}
\] (27)

or

\[
\int \frac{dT^*}{T^* \sqrt{\lambda_1 - \frac{2}{3} \lambda_2 T^* - \frac{1}{3} \lambda_3 T^{*2}}} = \pm \dot{T} + C_1
\] (28)

where \( C_1 \) is a constant. So the homoclinic orbit is obtained

\[
\left| \sqrt{\lambda_1 - \frac{2}{3} \lambda_2 T^* - \frac{1}{3} \lambda_3 T^{*2} + \sqrt{\lambda_1}} - \frac{\lambda_2}{3 \sqrt{\lambda_1}} \right| = C_2 e^{\pm \sqrt{\lambda_1} \dot{T}}
\] (29)

and figure 2 is its phase portrait.
When $H < 0$, two closed orbits around $A$ and $B$ can be obtained; which indicate the vibration in the neighborhood of the stable equilibrium positions. When $H$ increases, the period of vibration also increases. When $H > 0$, it indicates another kind of vibration. The Melnikov function is

$$M(\tau) = \int_{-\infty}^{+\infty} \dot{T}^* \cdot \psi [T^*, \dot{T}^*, \bar{\omega}(\bar{t} + \tau)] \, d\bar{t}$$

(30)

where

$$\psi = \psi(T, \dot{T}, \bar{\omega}) = g \cos \bar{\omega} \bar{t} - \varepsilon \dot{T}$$

(31)

so

$$M(\tau) = -\varepsilon' \int_{-\infty}^{+\infty} \dot{T}^* \, d\bar{t} + g \int_{-\infty}^{+\infty} \dot{T}^* \cos (\bar{t} + \tau) \, d\bar{t}$$

(32)

In Eq. (32), the first integration is the area surrounded by the homoclinic orbit. Obviously, the homoclinic orbit is symmetric about the $T$ axis. Let $H = \dot{T} = 0$, $C$ and $D$ points are determined (shown in figure 2 $C(a, 0)$ and $D(b, 0)$)

$$\left\{ \begin{array}{l}
  a = \frac{-\frac{3}{2} \lambda_2 - \sqrt{\frac{3}{2} \lambda_2^2 + 2 \lambda_1 \lambda_3}}{\lambda_3} < 0 \\
  b = \frac{-\frac{3}{2} \lambda_2 + \sqrt{\frac{3}{2} \lambda_2^2 + 2 \lambda_1 \lambda_3}}{\lambda_3} > 0 
\end{array} \right.$$ 

(33)

Note that the homoclinic orbit can be rewritten

$$T^* = \frac{2C_2 \sqrt{\lambda_1} e^{\pm \sqrt{\lambda_1} \bar{t}}}{(C_2 e^{\pm \sqrt{\lambda_1} \bar{t}} + \frac{\lambda_2}{3 \sqrt{\lambda_1}})^2 + \frac{1}{3} \lambda_3}$$

(34)

Where $C_2$ is a constant, if the initial conditions are: $\dot{S}(0) = 0$, $T^*(0) = b > 0$ we can get

$$C_2 = \frac{\sqrt{\lambda_1}}{b} - \frac{\lambda_2}{3 \sqrt{\lambda_1}} > 0$$

(35)

According to the residual law we get the Melnikov function

$$M(\tau) = -\frac{4 \varepsilon'}{3 \lambda_3} \left[ \frac{3}{2} \lambda_1 + \frac{\lambda_2 \sqrt{\lambda_1}}{3 \lambda_3} + \lambda_2 \cdot \frac{2 \lambda_2^2 + 9 \lambda_1 \lambda_3}{9 \lambda_3 \sqrt{2 \lambda_3}} \arcsin \left\{ \sqrt{2} \cdot \frac{\lambda_2}{2 \lambda_2^2 + 9 \lambda_1 \lambda_3} \right\} \right]$$

$$+ \frac{4 \pi g \overline{\omega} \sin (\xi + \tau) (ch \overline{\omega} \eta_1 - ch \overline{\omega} \eta_2)}{\sqrt{2 \lambda_3} \left( ch \frac{2 \pi \overline{\omega}}{\sqrt{\lambda_1}} - 1 \right)}$$

(36)
where

\[
\begin{aligned}
\xi &= \frac{1}{\sqrt{\lambda_1}} \ln \left[ \frac{1}{3C_2} \sqrt{\frac{2\lambda_2^2 + 9\lambda_1\lambda_3}{2\lambda_1}} \right] \\
\eta_1 &= \frac{1}{\sqrt{\lambda_1}} \left[ \pi + \arctg \left( \frac{3}{\lambda_2} \sqrt{\frac{\lambda_1\lambda_3}{2}} \right) \right] \\
\eta_2 &= \frac{1}{\sqrt{\lambda_1}} \left[ \pi - \arctg \left( \frac{3}{\lambda_2} \sqrt{\frac{\lambda_1\lambda_3}{2}} \right) \right]
\end{aligned}
\] (37)

If the Melnikov function has simple zero points, the stable and unstable manifolds intersect, the Poincaré map has a horse-shoe, so there exists the strange constant set, it is possible for the dissipative system to enter chaos.

According to \( M(\tau) = 0 \) we have

\[
\sin \bar{\omega} (\xi + \tau) = \frac{\gamma \left( \frac{2\pi \bar{\omega}}{\sqrt{\lambda_1}} - 1 \right)}{6\sqrt{2} f \bar{\omega} \sqrt{3} (\chi \bar{\omega} \eta_1 - \chi \bar{\omega} \eta_2)} \left[ \lambda_1^\frac{3}{2} + \frac{\lambda_2^2 \sqrt{\lambda_1}}{3\lambda_3} + \lambda_2 \cdot \frac{2\lambda_2^2 + 9\lambda_1\lambda_3}{9\lambda_3 \sqrt{2\lambda_3}} \arcsin \frac{\sqrt{2} \cdot \lambda_2}{\sqrt{2\lambda_2^2 + 9\lambda_1\lambda_3}} \right]
\] (38)

Because \( \sin \bar{\omega} (\xi + \tau) \leq 1 \), the critical condition that the system enters chaotic states is as follows

\[
\frac{f}{\gamma} \geq \frac{ch \left( \frac{2\pi \bar{\omega}}{\sqrt{\lambda_1}} - 1 \right)}{6\sqrt{2} \bar{\omega} \sqrt{3} (\chi \bar{\omega} \eta_1 - \chi \bar{\omega} \eta_2)} \left[ \lambda_1^\frac{3}{2} + \frac{\lambda_2^2 \sqrt{\lambda_1}}{3\lambda_3} + \lambda_2 \cdot \frac{2\lambda_2^2 + 9\lambda_1\lambda_3}{9\lambda_3 \sqrt{2\lambda_3}} \arcsin \frac{\sqrt{2} \cdot \lambda_2}{\sqrt{2\lambda_2^2 + 9\lambda_1\lambda_3}} \right] = R(\bar{\omega})
\] (39)

4. Numerical computations and conclusions

There are 3000 computation points for each curve shown in Figures 3–6. The initial conditions are \( T(0) = \dot{T}(0) = 0.01 \). Figures 3 and 5 indicate steady motions. In figure 4 the time-displacement history curve does not have a regular pattern, its phase portrait is like as 8 placed upside down, the Poincaré map is like a fried dough twist; at the same time, the group of parameters is at the chaotic range given by the Melnikov function, so we say that this is a chaotic motion. Figure 6 is similar to figure 4; it corresponds to a chaotic motion.

According to the analysis discussed above, the dynamic system \( \ddot{T} - \lambda_1 T + \lambda_2 T^2 + \lambda_3 T^3 = \varepsilon (g \cos \bar{\omega} t - \varepsilon \dot{T}) \) represents the governing equation of many nonlinear vibration problems. With the aid of a further study of this system, we shall obtain a better understanding of the chaotic mechanism.

1) For a given frequency \( \bar{\omega} (\lambda_1 > 0) \), chaos occurs when the ratio of the force \( f \) to the damping coefficient \( \gamma \) exceeds some critical value. For three known values \( \gamma, f \) and \( \bar{\omega} \), the system does not enter chaotic states if \( f/\gamma \) is smaller than \( R(\bar{\omega}) \).

2) If the condition (39) is satisfied, it is only possible that chaos should occur because the strange attractors and the other attractors may exist simultaneously.

3) The Melnikov function and corresponding critical condition are obtained in this paper for the nonlinear dynamic system \( \ddot{T} - \lambda_1 T + \lambda_2 T^2 + \lambda_3 T^3 = \varepsilon (g \cos \bar{\omega} t - \varepsilon \dot{T}) \) (\( \lambda_1 > 0 \)). It is not difficult to find that the Duffing equation and the corresponding conclusions about chaotic motion constitute a special case in our paper. For example, if \( \lambda_2 = 0 \), the nonlinear dynamic system discussed above can be degenerated into the Duffing equation.

4) The dynamic system represents a kind of nonlinear governing equations of forced vibration. For example, if \( \lambda_1, \lambda_2, \lambda_3 > 0 \), it is the post-buckling dynamic equation of elastic cylindrical shells subjected to axial load; if
\(\lambda_1 < 0, \lambda_2, \lambda_3 > 0\), this corresponds to the pre-buckling situation or elastic arches; when \(\lambda_2 = 0\) it is degenerated into the Duffing equation. When \(\lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0\), this is a new dynamic system which has a central singularity and a hyperbolic saddle point, and chaos may also occur. So the study of the chaotic motion of this dynamic system will help us to better understand the principal chaotic characters of many structural elements.

**Figure 3.** (a) Phase portrait; (b) time-displacement history diagram; (c) Poincaré map; \(\lambda_1 = 3, \lambda_2 = 10, \lambda_3 = 1, \omega = 3, g = 100, \epsilon' = 1\).

**Figure 4.** (a) Phase portrait; (b) time-displacement history diagram; (c) Poincaré map; \(\lambda_1 = 3, \lambda_2 = 10, \lambda_3 = 1, \omega = 3, g = 500, \epsilon' = 1\).
Figure 5. (a) Phase portrait; (b) time-displacement history diagram; (c) Poincaré map; \( \lambda_1 = -40, \lambda_2 = 90, \lambda_3 = 16, \bar{\omega} = \pi, g = 220, \varepsilon = 1 \).

Figure 6. (a) Phase portrait; (b) time-displacement history diagram; (c) Poincaré map; \( \lambda_1 = -40, \lambda_2 = 90, \lambda_3 = 16, \bar{\omega} = \pi, g = 360, \varepsilon = 1 \).
Finally, let us consider the effect of higher-order modes on chaotic vibration. First, the partial differential equation governing the transverse vibration of the shell is derived in this paper. Then, by means of the Galerkin approach, the partial differential equation can be simplified into a single ordinary differential equation or a set of two ordinary differential equations according to the number of mode taken into account. For brevity, the differential equations corresponding to these two cases are called 'single mode model', and 'double mode model', respectively. For the single mode model \( W_1 = f(t) W_1(x) \) and for the double mode model \( W_2 = f_1(t) W_1(x) + f_2(t) W_2(x) \), when \( |f(t) - f_1(t)| \ll 1 \) and \( |f_2 W_2| \ll |f_1 W_1| \), \( W_2 \) can be a better approximate solution than \( W_1 \). If \( f(t) \) is chaotic, \( f_1(t) \) is also chaotic, and vice versa. This is the reason why the single mode model is used to analyze the chaotic motion of the shell.

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