Statics of curved rods on account of torsion and flexion

Jacqueline Sanchez-Hubert a, b, Evarisre Sanchez Palencia b

a Laboratoire de Mécanique, Université de Caen, bd. Marechal Juin, 14032 Caen, France
b Laboratoire de Modélisation en Mécanique, Université Paris VI, 4 place Jussieu, 75252 Paris, France

(Received 11 May 1998; revised and accepted 22 March 1999)

Abstract – We adress the problem of a thin curved rod in linear elasticity for small displacements. We use an asymptotic two-scale method based on the small parameter \( \varepsilon \), ratio of the thickness to the overall lengths. The two leading order terms have the Bernoulli’s structure and the corresponding displacement is inextensional. The constitutive law involve torsion effects at the same order as flexion effects, so that the description of the kinematics involve an angle \( \theta \) which is the rotation of the sections. The Lagrange multiplier, associated with the constraint of inextensibility, is discussed. The variational formulation of the problem in the subspace of the inextensional displacements is given, as well as the equations involving the Lagrange multiplier. © Elsevier, Paris

1. Introduction

In this paper, we adress the problem of the statics of a thin curved rod. The general framework is that of linear elasticity with small displacements. The main tool is the use of asymptotic analysis of the three dimensional elasticity system linked with a small parameter \( \varepsilon \), which accounts for the ratio of the thickness to any overall length of the rod. Basically, we use a two-scale procedure which is classical in homogenization problems for heterogeneous media, as well as plate theory and straight rod theory (see Sanchez-Hubert and Sanchez Palencia, 1992, Ch. V, VI and VII respectively) where the procedure is worked out in cartesian coordinates. This method is easily adapted to curvilinear coordinates to study shell theory (Sanchez-Hubert and Sanchez Palencia, 1997) and curved rod theory (Jamal and Sanchez Palencia, 1996; Jamal, 1998). In the last case, the large and small scale are associated with coordinates along the middle curve of the rod and in the normal sections respectively. The main features of these works are:

(a) the classical Bernoulli’s structure of the solutions (normal sections remain plane and normal to the deformed middle line) is deduced for the two leading asymptotic terms (but not for the others).

(b) Flexion and torsion effects are of the same order, so that the description of the problem cannot be done with unknowns describing only the deformation of the middle curve: it appears a new variable \( \theta \), describing the rotation angle around the middle curve. Of course, \( \theta \) disappears when the middle curve is plane and submited to in-plane forces.

(c) As a consequence of the traction rigidity which is very high with respect to the flexion and torsion rigidities, a phenomenon of inextensibility appears at the leading order.

According to (c), rods are analogous to “non-inhibited shells”, i.e. shells with a middle surface which is not geometrically rigid (Sanchez-Hubert and Sanchez Palencia, 1997, Ch. VIII). This feature has important consequences in numerical computation. Indeed, the leading order terms “live” in the subspace of inextensional displacements and the finite elements computations exhibit the locking phenomenon, unless very special
conditions are satisfied (see, for instance (Chenais and Paumier, 1994; Chapelle, 1997) as well as (Brezzi and Fortin, 1991) for generalities on this question). Let us remark that Arunakirinathar and Reddy (1993) give a mixed finite element approximation which converges uniformly, i.e. without locking.

The above mentioned works on rod theory (Jamal and Sanchez Palencia, 1996; Jamal, 1998) involve several terms of the asymptotic expansion and lead to the variational formulation of the limit problem in the subspace of the inextensional displacements. The main purpose of the present paper is to show the link with classical equations of statics of curves and exhibit the Lagrange multiplier associated with the constraint of inextensibility.

Moreover, in this paper, we shall use a duality method for obtaining the kinematic properties involved in the constitutive law. In order to explain this, let us recall a little the case of a straight rod (Sanchez-Hubert and Sanchez Palencia, 1991 and 1992, Ch. VII). The kinematic quantities involved in the constitutive law are

\[
\begin{align*}
E_1 &= \ddot{u}_1 = \text{extension at order } \varepsilon, \\
E_2 &= -v'_2 = \text{kinematic flexion in the plane } (y_1, y_2), \\
E_3 &= -v'_3 = \text{kinematic flexion in the plane } (y_1, y_3), \\
E_4 &= \theta' = \text{kinematic torsion}.
\end{align*}
\]

The notations are self-evident and, in any case, will be given later. We only point out that at order \(\varepsilon^0\) the rod behaves as an inextensible one so that \(E_2\) and \(E_3\) involve flexion at order \(\varepsilon^0\), whereas \(E_1\) involves extension at order \(\varepsilon\). Obviously, \(E_4\) accounts for the rotation. The constitutive law, which follows from equations at the “microscopic level” in the two-scale procedure, gives the traction, flexions and torsion moments as functions of \(E_1, \ldots, E_4\).

When passing to the case of curved rods, it appears that the local equations are independent of the curvature (and thus the same as for the straight rods) but the expressions of \(E_1, \ldots, E_4\) are modified. They take the form (5.12) hereafter. These new expressions (rather expressions equivalent to (5.12) on account of the inextensibility) were obtained in Jamal and Sanchez (1996) and Jamal (1998) using higher order terms in the asymptotic expansion. In this paper (Section 5), we obtain these expressions in a much easier way which only involves lower order of the asymptotic expansion and a duality procedure using the equations of statics of curves.

We shall insist on the fact that the constitutive law only depends on the local geometry of the section and on the elasticity coefficients. These coefficients may be anisotropic and depending on the point of the section. A local problem, analogous to the local problems in homogenization theory, furnishes the constitutive law (see Sanchez-Hubert and Sanchez Palencia, 1991, or 1992, Ch. VII). Obviously, the geometry of the section and the coefficients may depend of the longitudinal “macroscopic” variable and the constitutive law then depends on this variable.

Our problem is worked out in the case when the rod is clamped at its extremities, but slight modifications allow to consider other cases; in fact an example with a free extremity is given in Section 6. Moreover, the applied forces and moments by unit length are of order \(\varepsilon^3\), so that the leading term of the displacement is of order \(\varepsilon^0\), as the flexion and torsion rigidities are of order \(\varepsilon^4\). But, according to the linearity of the problem, we may consider given forces of any order as well.

Other works on asymptotic theory of curved rods should be mentioned. They are based on the method of dilatation of the normal section and reduction to a constant reference domain. The problems involved are less general than ours, as the material is isotropic and homogeneous and only special geometries are considered. Let us mention, for instance, Alvarez-Dios and Viaño (1998) for shallow arches and Madani (1998) for plane
curves. Let us also quote here the above mentioned work (Arunakirinathar and Reddy, 1993) which deals with general non plane middle curve. The starting point of this work is the Timoshenko phenomenological model so that it is essentially different from ours, which starts from three dimensional elasticity.

The paper is organized as follows. Classical statics of curves is recalled in Section 2. The asymptotic two-scale procedure is worked out in Section 3 for the elasticity problem in curvilinear coordinates, leading to the Bernoulli’s structure and the inextensibility property at order \( \varepsilon^0 \). In Section 4, we consider the somewhat general case when the constitutive law is such that the traction effects are uncoupled from flexion and torsion ones; the role of the Lagrange multiplier is explained. Section 5 is devoted to the above duality method giving the expressions of the \( E_2, E_3, E_4 \). In Section 6, we give a short account of the theory for practical utilization as well as an example. The case of a general constitutive law is addressed in Section 7. Finally, Section 8 is devoted to a special problem with very particular forces of order \( \varepsilon^3 \) by unit length.

The required differential geometry reduces to a little space curve theory and equations in curvilinear coordinates, which may be found in any treatise of classical differential geometry (let us quote (Lichnerowicz, 1960) for instance).

### 2. Equations of the statics of a curve

In this section we consider a curve in the mathematical sense, that is to say without thickness; we shall write the classical equilibrium equations when considering it as a material system. We shall see later that they are, at some asymptotic orders, the limit equations for thin rods.

The curve is parametrized by its arc \( s \) and the running point will be denoted by \( \textbf{OP} = \textbf{r}(s) \). The orthonormal Frenet frame at a point \( P \) is denoted by \( \textbf{a}_1 \equiv \textbf{t}, \textbf{a}_2 \equiv \textbf{n}, \textbf{a}_3 \equiv \textbf{b} \) where \( \textbf{t}, \textbf{n} \) and \( \textbf{b} \) denote respectively the unit tangent, principal normal and binormal vectors. For the sake of completeness, we recall the Frenet’s formulae

\[
\begin{align*}
\frac{d\textbf{t}}{ds} &= k(s)\textbf{n}, \\
\frac{d\textbf{n}}{ds} &= -k(s)\textbf{t} + \tau(s)\textbf{b}, \\
\frac{d\textbf{b}}{ds} &= -\tau(s)\textbf{n},
\end{align*}
\]

where \( k(s) \) and \( \tau(s) \) denote respectively the curvature and the torsion at the point \( s \).

The equilibrium equations are derived according to the classical procedure: Let us denote respectively by \( \textbf{f}(s_0) \) and \( \textbf{m}(s_0) \) the linear densities and moments of the applied forces and by \( \textbf{T} \) and \( \textbf{M} \) the force and moment describing the mechanical actions of the part \( s > s_0 \) upon the part \( s < s_0 \) of the curve, then by writing the equilibrium equations of a part \( s_1 < s < s_2 \) and passing to the limit \( s_1, s_2 \to s_0 \), we obtain

\[
\begin{align*}
\frac{dT}{ds} + \textbf{f} &= 0, \\
\frac{dM}{ds} + \textbf{a}_1 \wedge \textbf{T} + \textbf{m}(s) &= 0,
\end{align*}
\]

(2.2)
or, writing these equations in the Frenet frame,

\[
\begin{aligned}
\frac{dT_1}{ds} - kT_2^2 &= -f^1, \\
\frac{dT_2}{ds} + kT_1^1 - \tau T_3 &= -f^2, \\
\frac{dT_3}{ds} + \tau T_2^2 &= -f^3,
\end{aligned}
\]

\[
\frac{dM_1}{ds} - kM_2^2 = -m_1(s),
\]

\[
\frac{dM_2}{ds} + kM_1^1 - \tau M_3 = T^3 - m_2(s),
\]

\[
\frac{dM_3}{ds} + \tau M_2^2 = -T_2^2 - m_3(s)
\]

which will be called system of statics of curves.

Clearly, the roles of the two components \(T_2^2\) and \(T_3^3\) and of the component \(T_1^1\) are very different. In the sequel, it will prove useful to eliminate \(T_2^2\) and \(T_2^3\). Using the last two equations, we obtain:

\[
\begin{aligned}
k \left( \frac{dM_3}{ds} + \tau M_2^2 \right) + \frac{dT_1}{ds} &= -f^1, \\
- \frac{d}{ds} \left( \frac{dM_3}{ds} + \tau M_2^2 \right) - \tau \left( \frac{dM_2}{ds} + kM_1^1 - \tau M_3 \right) + kT_1^1 &= -f^2 + \frac{dm_3}{ds} - \tau m_2, \\
\frac{d}{ds} \left( \frac{dM_2}{ds} + kM_1^1 - \tau M_3 \right) - \tau \left( \frac{dM_2}{ds} + \tau M_2^2 \right) &= -f^3 - \frac{dm_2}{ds} - \tau m_3,
\end{aligned}
\]

which we will called reduced system of statics of curves.

Of course, the equilibrium equations may be written in others frames; this may be useful in cases where the curve is not easily rectifiable. If the position of the points in a neighbourhood of the curve may be expressed in curvilinear coordinates \((y^1, y^2, y^3)\), with \(y^2 = y^3 = 0\) on the curve, then we shall have

\[
ds = |a_1| \ dy^1, \quad a_1 = \frac{dr}{dy^1}
\]

and the equations of equilibrium of the forces are given by

\[
D_i T^i = f^i,
\]

where \(D_i\) denotes the covariant derivative in curvilinear coordinates which are expressed by

\[
D_i T^j \equiv \partial_i T^j - \Gamma^j_{ik} T^k,
\]
where $\Gamma_{j,k}^i$ are the Christoffel symbols. Analogously, for the moments we have the equilibrium equations:

\[
\begin{cases}
D_1M^1 = 0, \\
D_1M^2 - |a_1|T^3 = 0, \\
D_1M^3 + |a_1|T^2 = 0,
\end{cases}
\]

where $T^2$ and $T^3$ may be eliminated as before. In the sequel, we shall use the Frenet frame which gives easier formulas and provides a better insight of the geometric properties.

3. Modelling from the three-dimensional elasticity

3.1. Preliminary computations

In this section, the rod will be considered as an elastic body filling a “slender” domain around the curve $C$ (this will be precised in the sequel), see figure 1.

The curve will be parametrized by its arc $s$, $s \in \mathcal{I} = (0, l)$ where $l$ is the total length of the curve. The Cartesian coordinates of the running point in the rod are

\[
x_1 = r_1(s), \quad x_2 = r_2(s), \quad x_3 = r_3(s)
\]

the vector function $r$ will be sufficiently smooth. We shall consider the unit tangent, the principal normal and the binormal forming the Frenet frame at the point $P(s)$:

\[
\begin{cases}
a_1 = r'(s), \\
a_2 = \frac{1}{k}a_1', \\
a_3 = a_1 \wedge a_2,
\end{cases}
\]

where

\[
k(s) = |a_1'|, \quad |\tau(s)| = |a_3|.
\]
In the case when $k(s)$ vanishes anywhere, we shall admit the existence of an extension by continuity of $a_2$.

In a neighbourhood of $C$, the geometric points will be defined by curvilinear coordinates $y^1 \equiv s$, $y^2$, $y^3$ where $y^2$, $y^3$ denote the distances along $a_2$ and $a_3$ so that the position of the point of curvilinear coordinates $y^1$, $y^2$, $y^3$ is

$$\rho(y^1, y^2, y^3) = r(y^1) + y^2 a_2 + y^3 a_3$$

(3.4)

which is well-defined for sufficiently small $y^2$, $y^3$.

Now, let $\Sigma(s)$ be a connected domain of the $(y^2, y^3)$-plane depending, in a smooth manner on the parameter $s$ and let $\varepsilon$ be a small parameter (describing the smallness of the cross section with respect to the length of the curve). The rod is the set $P^\varepsilon$ of the points whose the curvilinear coordinates belong to the domain

$$\Omega_\varepsilon = \{(y^1, y^2, y^3), \ y^1 \in I, \ (y^2, y^3) \in \varepsilon \Sigma(y^1)\}.$$

At each point of the space (in the considered neighbourhood) we define the covariant basis $g_i = \partial_i \rho$ as well as the contravariant basis $g^i$ ($g_i \cdot g^j = \delta^j_i$, where $\delta^j_i$ are the Kronecker symbols). We note that we have $g_i(y^1, 0, 0) = a_i = g^i(y^1, 0, 0)$. We easily obtain

$$\begin{cases}
  g_1 = (1 - ky^2) a_1 - \tau y^3 a_2 + \tau y^2 a_3, \\
  g_2 = a_2, \\
  g_3 = a_3,
\end{cases} \quad \begin{cases}
  g^1 = \frac{1}{1 - ky^2} a_1, \\
  g^2 = \frac{\tau y^3}{1 - ky^2} a_1 + a_2, \\
  g^3 = -\frac{\tau y^2}{1 - ky^2} a_1 + a_3.
\end{cases}$$

(3.5)

The determinant of the metric tensor of components $g_{ij} = g_i \cdot g_j$ is

$$g = (1 - k^2 y^2)$$

(3.6)

and the volume element in curvilinear coordinates is given by

$$dx \cdot dx_2 \cdot dx_3 = \sqrt{g} dy^1 dy^2 dy^3 = \sqrt{g} dy.$$  

(3.7)
The Christoffel symbols are easily computed from (3.5) on account of the Frenet formulas, they are

\[
\begin{align*}
\Gamma^1_{11} &= \frac{-k'(s)y^2 + k(s)\tau(s)y^3}{1 - ky^2}, \\
\Gamma^2_{11} &= k - (k^2 + \tau^2)y^2 - \tau'y^3 + \tau y^2 - \frac{k' y^2 + k\tau y^3}{1 - ky^2}, \\
\Gamma^3_{11} &= -\tau^2 y^3 + \tau' y^2 - \tau y^2 - \frac{k'y^2 + k\tau y^3}{1 - ky^2}, \\
\Gamma^1_{12} &= \Gamma^1_{21} = \frac{-k}{1 - ky^2}, \\
\Gamma^2_{12} &= \Gamma^2_{21} = \frac{-k\tau y^3}{1 - ky^2}, \\
\Gamma^3_{12} &= \Gamma^3_{21} = \tau + \frac{k\tau y^2}{1 - ky^2}, \\
\Gamma^2_{13} &= \Gamma^2_{31} = -\tau,
\end{align*}
\]

(3.8)

the other symbols vanish.

Let us recall that the derivatives of a vector field \( u = u_k g^k \) are given by

\[ \partial_h u = D_h u_k g^k, \]

where \( D_h \) denotes the covariant differentiation:

\[ D_h u_k = \partial_h u_k - \Gamma^m_{hk} u_m. \]

(3.9)

### 3.2. Lengthening of the curve \( C \)

We must deal with both the curve before and after deformation under the action of the applied forces. According to the small displacement theory, we shall consider the two curves as close to each other and we shall linearize with respect to the small displacement. For this purpose, the general framework consists of immersing the two curves in a family of curves depending on a parameter \( t \) and to express the difference between the two curves by the differentiation with respect to \( t \). Let us denote by \( C_t \) the curves of the family, \( y^1 \) being the parameter describing the curve, so that

\[ r_t : y^1 \mapsto r_t(y^1) \in \mathbb{E}^3. \]

The position of a point \( P \) on the “non-deformed” curve \( C_t \) is thus given by \( r_t(y^1) \) whereas the position of the same point “after deformation” is

\[ \mathbf{OP}' = r_{t+\delta t}(y^1) = r_t(y^1) + \delta r(y^1), \]

where the symbol “variation \( \delta \)” is defined by

\[ \delta = dt \frac{\partial}{\partial t}. \]

Consequently, \( \delta r \) is the infinitesimal displacement of \( P \), higher order terms being neglected. Our theory being linearized with respect to the small displacements, we shall call \( \delta r \) the displacement that we shall denote \( u \).
According to Section 2, we emphasize that $y^1$ is the arc of the “non-deformed curve $C_t$” but, in a general deformation, $y^1$ is no more the arc of $C_{t+bt}$.

The displacement vector may be referred to the Frenet frame, so that

$$u(s) = u^1(s)a_1(s) + u^2(s)a_2(s) + u^3(s)a_3(s)$$

and, as the variations of $r$ are independent of the differentiation with respect to $y^1$, we have

$$\frac{du}{dy^1} = \delta a_1,$$

so that the lengthening $\delta a_1 \cdot a_1$ of the tangent vector is given by

$$\delta a_1 \cdot a_1 = \frac{du^1}{ds} - ku^2. \tag{3.11}$$

### 3.3. The strain tensor in curvilinear coordinates

Coming back to the three-dimensional body $P^c$, a point $P \in P^c$ is defined by $\rho(y)$ and the local frame by (3.5), so that in this case the analogue of formula (3.10) is

$$\partial_k u = \delta g_k. \tag{3.12}$$

By developing the displacement vector in the contravariant basis we obtain

$$\partial_k u = \partial_k u_i^j g^i + u_i \partial_k g^j = (\partial_k u_i - u^h_{ik} \Gamma^i_{hk}) g^j = D_k u_i g^j$$

and so

$$D_k u_i g^j \cdot g_h = D_k u_h = \delta g_k \cdot g_h.$$  

The variations of the components of the metric tensor are given by

$$\delta g_{hk} = g_h \cdot \delta g_k + g_k \cdot \delta g_h$$

so that the covariant components of the strain tensor may be defined by

$$\gamma_{ij}(u) = \frac{1}{2} (D_i u_j + D_j u_i) = \epsilon_{ij}(u) - \Gamma^k_{ij} u_k, \tag{3.13}$$

where $\epsilon_{ij}$ are the expressions

$$\epsilon_{ij}(u) = \frac{1}{2} (u_{i,j} + u_{j,i})$$

which coincide with the classical Cartesian expressions of the strain components.

### 3.4. Formulation of the elasticity problem in the three-dimensional domain $P^c$

Let $\epsilon^{ijkl}$ be the elasticity coefficients of the material. They are expressed in the contravariant components associated with the frame (3.5). In general, they are functions of $y^1$, $y^2$, $y^3$ and they satisfy the classical properties of symmetry and positivity.
The underformed position of the rod is the domain $P^\varepsilon$. We shall consider it fixed by its extremities ($y^1 = 0$ and $y^1 = l$) and submitted to outer forces with linear density $\mathbf{f}^\varepsilon$. We emphasize that this is the total force by unit length of the rod, not the three-dimensional density of the forces. The dependence of $\mathbf{f}^\varepsilon$ with respect to $\varepsilon$ will be specified later.

The space of the kinematically admissible displacements is

$$V^\varepsilon = \{ \mathbf{v}; \mathbf{v} \in H^1(P^\varepsilon), \mathbf{v}(0) = \mathbf{v}(l) = 0 \},$$

(3.14)

where $H^1(P^\varepsilon)$ is the space of the vectors such that each component belongs to the classical Sobolev space $H^1$ of the domain $P^\varepsilon$.

As the volume element is given by (3.7), the variational formulation of the problem is:

Find $\mathbf{u}^\varepsilon \in V^\varepsilon$ such that

$$\int_{P^\varepsilon} a^{jkh} \gamma_{kh}(\mathbf{u}^\varepsilon) \gamma_{ij}(\mathbf{v}) \sqrt{\varepsilon} \, dy = \int_0^l \mathbf{f}^\varepsilon \cdot \mathbf{v} \, ds \quad \forall \mathbf{v} \in V^\varepsilon.$$  

(3.15)

Since $\mathbf{v}$ is defined in $P^\varepsilon$, we note that the expression in the right hand side of (3.15) is a little ambiguous; this point will be clarified later.

3.5. Asymptotic procedure. Bernoulli structure

In this section we shall develop a procedure in order to obtain the asymptotic structure of the solution $\mathbf{u}^\varepsilon$ as $\varepsilon \rightarrow 0$. To this end, we use a two-scale expansion as $y^2, y^3$ are of order $\varepsilon$ in the domain $P^\varepsilon$ whereas $y^1$ is of order of the unity. We shall write

$$z^2 = \frac{y^2}{\varepsilon}, \quad z^3 = \frac{y^3}{\varepsilon},$$

(3.16)

so that $z^2$ and $z^3$ are of order unity. We shall search for an asymptotic expansion of the form

$$\mathbf{u}^\varepsilon = \mathbf{u}^0(s) + \varepsilon \mathbf{u}^1(s, z^2, z^3) + \varepsilon^2 \mathbf{u}^2(s, z^2, z^3) + \cdots,$$

(3.17)

where, of course, we may write $s$ as well as $y^1$.

Accordingly, the expansion of the components $\gamma_{ij}(\mathbf{u}^\varepsilon)$ write

$$\gamma_{ij}(\mathbf{u}^\varepsilon) = \gamma_{ij}^0 + \varepsilon \gamma_{ij}^1 + \varepsilon^2 \gamma_{ij}^2 + \cdots,$$

(3.18)

where

$$\gamma_{ij}(\mathbf{u}^\varepsilon) = \epsilon_{ij}(\mathbf{u}^\varepsilon) - \Gamma_{ij}^k u^\varepsilon_k,$$

(3.19)

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}).$$

(3.20)

It will prove useful to look at (3.17) formally as a two-scale expansion where $\mathbf{u}^\varepsilon$ depend on both the macroscopic variable $y = (y^1, y^2, y^3)$ and on the microscopic ones $z = (z^2, z^3)$ taking account of

$$\frac{\partial}{\partial y^2} = \frac{\partial}{\partial y^3} = \frac{\partial}{\partial z^3} = 0.$$  

(3.21)
Moreover the Christoffel symbols are functions of \((y^1, y^2, y^3)\) so that a Taylor expansion allows us to write them in terms of \(y^1, z^2, z^3\):

\[
\Gamma^k_{ij}(y) = \Gamma^k_{ij}(s, 0, 0) + \varepsilon z^2 \Gamma^k_{ij,2}(s, 0, 0) + \varepsilon z^3 \Gamma^k_{ij,3}(s, 0, 0) + \cdots .
\]

(3.22)

From the previous considerations, it follows that the two first terms of the expansion (3.18) are

\[
\gamma^0_{ij} = e_{ijy}(u^0) - \Gamma^k_{ij}(s, 0) u^0_k + e_{ijz}(u^1) = \gamma_{ijy}(u^0) + e_{ijz}(u^1),
\]

(3.23)

\[
\gamma^1_{ij} = e_{ijy}(u^1) - \Gamma^k_{ij}(s, 0) u^1_k - z^2 \Gamma^k_{ij,2} - z^3 \Gamma^k_{ij,3}(s, 0) + e_{ijz}(u^2),
\]

(3.24)

where the indices \(y\) or \(z\) mean that the differentiation is only taken with respect to \(y\) or \(z\) respectively (but keeping always in mind (3.21)).

Taking \(v = u^e\) in (3.15) and on account of the previous asymptotic expansions, we have

\[
\varepsilon^2 \int_0^l \int_\Sigma a^{ijkh} \gamma_{kh}^0(u^e) \gamma_{ij}^0(v) \, dy^1 \, dz^2 \, dz^3 + \cdots = \int_0^l f^e \cdot u^0 \, ds + \cdots
\]

(3.25)

because of \(\sqrt{g} = \varepsilon^2 (1 + \cdots) \, dy^1 \, dz^2 \, dz^3\) (where (3.6) was used).

We are now analyzing the hypotheses leading to the Bernoulli structure. Let us recall that, classically, this structure amounts to:

(a) The middle line is inextensible.
(b) The displacement of the cross-sections are displacements of rigid solid.
(c) After deformation, the cross-sections remain normal to the middle line.

The condition (a) cannot be obtained at the leading order of the expansion unless

\[
\varepsilon^{-2} f^e \xrightarrow[\varepsilon \downarrow 0]{} 0.
\]

(3.26)

Indeed, under this hypothesis, (3.25) gives

\[
\int_0^l \int_\Sigma a^{ijkh} \gamma_{kh}^0(u^e) \gamma_{ij}^0(v) \, dy^1 \, dz^2 \, dz^3 = 0
\]

which, on account of the positivity property of the coefficients, is equivalent to

\[
\gamma_{kh}^0(u^e) = 0 \quad \forall i, j
\]

(3.27)
or, equivalently,

\[
\begin{align*}
\gamma_{11}^0 &= \frac{du_1^0}{ds} - ku_2^0 = 0, \\
\gamma_{12}^0 &= \gamma_{21}^0 = \frac{1}{2} \frac{du_2^0}{ds} + ku_1^0 - \tau u_3^0 + \frac{1}{2} \frac{d(u_1^0)^2}{dz^2} = 0, \\
\gamma_{13}^0 &= \gamma_{31}^0 = \frac{1}{2} \frac{du_3^0}{ds} + \frac{1}{2} \frac{d(u_1^0)^2}{dz^3} + \tau u_2^0 = 0, \\
\gamma_{22}^0 &= \frac{d^2u_1^0}{ds} = 0, \\
\gamma_{23}^0 &= \gamma_{32}^0 = \frac{1}{2} \left( \frac{d^2u_3^0}{ds} + \frac{d^2u_2^0}{dz^2} \right) = 0, \\
\gamma_{33}^0 &= \frac{d^2u_3^0}{dz^3} = 0.
\end{align*}
\]

(3.28)

The first Eq. (3.28) amounts to the inextensibility of the middle line (i.e. the curve \(C\)) at the leading order of the expansion (compare with (3.11)).

The other Eqs (3.28) are the local equations which allows to compute \(u^1\) when \(u^0\) is considered as known. Indeed, as \(u^0\) depends only on \(s\), the equations in (3.28) can be integrated with respect to \(z^2, z^3\) to obtain \(u^1\) what gives:

\[
\begin{align*}
\begin{pmatrix}
u_1^0(s) + \varepsilon \hat{u}_1^1(s) \\
\nu_2^0(s) + \varepsilon \hat{u}_2^1(s) \\
\nu_3^0(s) + \varepsilon \hat{u}_3^1(s)
\end{pmatrix} + \begin{pmatrix}
\theta(s) \\
-\frac{du_1^0}{ds} - 2\tau u_3^0(s) \\
\frac{du_2^0}{ds} + 2k u_1^0(s) - 2\tau u_3^0(s)
\end{pmatrix} \wedge \begin{pmatrix}
0 \\
y^2 \\
y^3
\end{pmatrix}
\end{align*}
\]

(3.30)

which involve new unknown functions \(\theta(s)\) and \(\hat{u}^1(s)\) of the macroscopic variable \(s\). They will be determined later, as well as \(u^0(s)\) when solving the global (macroscopic) problem.

Presently, taking into account the two leading terms of the expansion of \(u^e\), in other words the vector \(u^0(s, \varepsilon u^1(s, z))\) of which the components are

\[
\begin{align*}
\begin{pmatrix}
u_1^0(s) + \varepsilon \hat{u}_1^1(s) \\
\nu_2^0(s) + \varepsilon \hat{u}_2^1(s) \\
\nu_3^0(s) + \varepsilon \hat{u}_3^1(s)
\end{pmatrix}
\end{align*}
\]

we recognize a rigid displacement for each given \(s\).

The fixation condition at \(s = 0\) and \(s = l\) give, for \(u^0\):

\[
\begin{align*}
\begin{pmatrix}
u_0^0(0) = 0, \\
\frac{du_0^0}{ds}(0) = \frac{du_0^0}{ds}(0) = 0.
\end{pmatrix}
\end{align*}
\]

(3.31)
which amounts to a clamping condition (at the leading order). As for the new arbitrary functions, we have

\[
\begin{align*}
\mathbf{u}'(0) &= 0, \\
\theta(0) &= 0.
\end{align*}
\] (3.32)

The results of this section may be summed up in the next proposition:

**Proposition 1:** The introduction of the asymptotic expansion (3.17) into the variational formulation (3.15) shows that:

1. \( \mathbf{u}^0(s) \) is an inextensional displacement of the curve \( C \), satisfying the clamped conditions (3.30).
2. \( \mathbf{u}^0(s) + \varepsilon \mathbf{u}'(s) \) has a Bernoulli’s structure, the new unknown functions \( \theta(s) \) and \( \mathbf{u}'(s) \) (coming from the local integration) satisfy the boundary conditions (3.32).
3. The previous results are consistent with \( f^T = o(\varepsilon^2) \), i.e. the total applied force by unit length tends to zero as \( \varepsilon \to 0 \) faster than \( \varepsilon^2 \).

4. The rod problem in the case when the traction is not coupled with the moments

4.1. Local constitutive relation

In this section, we shall follow a procedure analogous to that of Koiter in shell theory. In other words, we shall admit that the local problem, which furnishes the strain-stress relation is analogous to that of the theory of straight rods (see Sanchez-Hubert and Sanchez Palencia, 1992, Ch. VII and 1991); let us recall that properties.

We saw in Section 3.5 that the leading terms \( \gamma^0_{ij} \) of the expansion of the strain tensor \( \gamma^\varepsilon_{ij} \) vanish, so that the corresponding expansion is

\[ \gamma^\varepsilon_{ij} = \varepsilon \gamma^1_{ij} + \cdots \]

and, accordingly, we have for the stress tensor

\[ \sigma^\varepsilon_{ij} = \varepsilon \sigma^1_{ij} + \cdots. \]

Then (see Sanchez-Hubert and Sanchez Palencia, 1992), the components \( \sigma^1_{ij} \) are determined as functions of of four functions \( E_i, i = 1, \ldots, 4 \), which depend on \( \mathbf{u}^0(s), \mathbf{u}'(s) \) and \( \theta(s) \). Roughly speaking, \( E_1 \) is the lengthening (at order \( \varepsilon \)), \( E_2, E_3 \) the kinematic flexions and \( E_4 \) the kinematic torsion. Of course, as we shall see later, here the corresponding expressions in terms of \( \mathbf{u}^0(s), \mathbf{u}'(s) \) and \( \theta(s) \) are different since they involve the curvature and the torsion of \( C \). Moreover, when \( \sigma^1 \) is determined, we may compute the components of the moment \( \mathbf{M} \) on the section (which are of order \( \varepsilon^3 \)) and the traction on the section (of order \( \varepsilon^3 \)), the other components of the resultant force on the section vanishing at this leading order. In the sequel, it will prove
useful to denote these four components by $T^2$, $T^3$, $T^4$, $T^1$ respectively, indeed

\[ \varepsilon^3 T^1 = \varepsilon^3 \int a^{11 kh} \gamma^1_{kh}(\mathbf{u}^e) \, dz^2 \, dz^3, \]

traction

\[ \varepsilon^4 T^2 = -\varepsilon^4 M^3 = \varepsilon^4 \left( \sum a^{11 kh} \gamma^1_{kh}(\mathbf{u}^e) \right) z^2 \, dz^2 \, dz^3, \]

flexion moments

\[ \varepsilon^4 T^3 \equiv \varepsilon^4 M^2 = \varepsilon^4 \left( \sum a^{11 kh} \gamma^1_{kh}(\mathbf{u}^e) \right) z^3 \, dz^2 \, dz^3, \]

\[ \varepsilon^4 T^4 \equiv \varepsilon^4 M^1 = \varepsilon^4 \left( \sum a^{13 kh} \gamma^1_{kh}(\mathbf{u}^e) - a^{12 kh} \gamma^1_{kh}(\mathbf{u}^e) \right) z^3 \, dz^2 \, dz^3 \]

torsion moment

Components $T^i$ are thus defined in terms of the functions $E_i$: the corresponding relation is the local constitutive relation which only depends on the local elasticity coefficients and on the section $\Sigma$.

In this section, we only consider the case when the local relation is such that $T^1$ only depends on $E_1$ and $M^1$, $M^2$, $M^3$ on $E_2$, ..., $E_4$. This is classically satisfied in the case of constant coefficients under symmetry properties.

Let us now consider the rod fixed by its extremities and submitted to given forces $f^e$ by unit of length. We saw (Proposition 1) that $f^e$ must be small with respect to $\varepsilon^2$. Moreover, by analogy with the case of straight rods (see Sanchez-Hubert and Sanchez Palencia, 1991 and 1992) we shall suppose

\[ f^e = \varepsilon^4 f(s) \] 

by unit of length on $C$.

For the sake of completeness and for a better insight on the phenomena, we shall prescribe also a given moment

\[ m^e = \varepsilon^4 m^1 a_1. \] 

As the given forces are of order $\varepsilon^4$, the equilibrium equations (1.1) show that $T^e$ is of the same order and we shall write

\[ T^e = \varepsilon^4 T + \cdots. \]

As for the resultant moment $M^e$, it follows from (2.3) that

\[ M^e = \varepsilon^4 M + \cdots. \]

the equilibrium equations write

\[
\begin{align*}
\frac{dT^1}{ds} - kT^2 &= -f^1, \\
\frac{dT^2}{ds} + kT^1 - \tau T^3 &= -f^2, \\
\frac{dT^3}{ds} + \tau T^2 &= -f^3,
\end{align*}
\]
where the factor $\varepsilon^4$ desappears. Now by eliminating $T^2$ and $T^3$ as in Section 2, we obtain the reduced system of statics for the curve $C$: \[
abla \begin{align*}
abla \frac{dM^1}{ds} - kM^2 &= -m^1, \\
abla \frac{dM^2}{ds} + kM^1 - \tau M^3 &= T^3, \\
abla \frac{dM^3}{ds} + \tau M^2 &= -r^2,
abla \end{align*}\] (4.5)

In order to continue our study, we must use some properties of the Lagrange multipliers.

4.2. The constraint of inextensibility and the Lagrange multiplier

At the present state, in order to identify some of the terms in the previous equations, it will prove useful to introduce some elements of the general theory of constrained variational problems. We refer to Brezzi and Fortin (1991) for the general theory but we shall rather follow the presentation of Sanchez-Hubert and Sanchez Palencia (1997, Sect. VIII.1) which is closer to our present context.

As the inextensibility constraint \[
abla \frac{dM^1}{ds} - kM^2 = 0 \] (4.7)
only involves the two variables $v_1(s)$ and $v_2(s)$, we develop here the theory for

$$v = (v^1, v^2) \in V = H^1_0(0, l) \times H^2_0(0, l).$$

(4.8)

Indeed, we shall see later that those spaces are the appropriate ones for the energy bilinear form and the kinematic boundary conditions. Moreover, the results of the present section will be used later in a slightly more complex framework when other unknowns, in addition to $u_1$ and $u_2$, are involved; but this point is irrelevant.

We start with a “preliminary problem” the variational formulation of which is

Find $u \in V$ such that \[
abla a(u, v) = (F, v) \quad \forall v \in V,
abla \] (4.9)

where $a$ is the energy bilinear form, supposed to be positive and coercive on $V$, $F \in V'$ is the given forcing term and where the brackets denote the duality product between $V'$ and $V$. Here $V'$ is the dual of $V$, usually defined in association with another “pivot” space $H$ which is identified to its dual, prescribing that $V$ is dense in $H$. We then have $V \subset H \subset V'$ with dense and continuous embeddings. The pivot space is generally chosen
in order to express the equations and boundary conditions with the help of integrations by parts in a suitable
context, usually it is $L^2$ (here $(L^2)^2$ as we have two components). Then, problem (4.9) is equivalent to

$$ Au = F, $$ \hspace{0.5cm} (4.10)

where $A$ is the operator (continuous from $V$ in $V'$) defined by

$$ a(u, v) = (Au, v). \hspace{0.5cm} (4.11) $$

Now let us introduce a constraint, more precisely, we replace $V$ by its closed subspace

$$ G = \{ v \in V; Bv = 0 \}, \hspace{0.5cm} (4.12) $$

where $B$ is some continuous operator from $V$ to another auxiliary pivot space $H_m$ ($Bv = 0$ is the expression of
the constraint). The constrained problem is then:

Find $u \in G$ such that

$$ a(u, v) = (F, v) \quad \forall v \in G. $$ \hspace{0.5cm} (4.13)

As the test function only runs in $G$ which is only a subspace of $V$, this problem cannot be written under the
form (4.10). To write the “modified equation”, let us introduce the space $M'$ which is the image of $V$ by $B$.
It is a subspace of $H_m$, supposed to be dense in $H_m$ ($H_m$ is chosen to fulfil this condition). Moreover, $M'$ is a
Hilbert space for the norm transported by $B$ from $G^\perp$ (orthogonal of $G$ in $V$). Identifying $H_m$ with its dual, we
define the dual $M$ of $M'$ and we have

$$ M' \subset H_m \subset M. $$ \hspace{0.5cm} (4.14)

We note that $B$ is also a continuous operator from $V$ into $M'$. Let $B^*$ be its adjoint, it is a continuous operator
from $M$ into $V'$ then the equations equivalent to (4.13) are expressed with the aid of a Lagrange multiplier

$$ \lambda \in M: $$

$$ \begin{cases}
Au + B^*\lambda = f, \\
Bu = 0.
\end{cases} \hspace{0.5cm} (4.15) $$

In our problem, (4.7), (4.8), we shall choose as pivot space for the Lagrange multiplier $H_m = L^2(0, l)$ and
we shall prove that the image coincides with $L^2(0, l)$. It means that for a given $\varphi \in L^2(0, l)$ we may find
$(v_1, v_2) \in V$ (defined in (4.8)) such that

$$ \varphi = \frac{dv_1}{ds} - kv_2. $$ \hspace{0.5cm} (4.16)

Indeed, let us consider $v_2$ as known, then

$$ v_1(s) = \int_0^s \left( \varphi(\sigma) + kv_2(\sigma) \right) \, d\sigma $$ \hspace{0.5cm} (4.17)

belongs to $H^1(0, l)$ and satisfies to the prescribed condition $v_1(0) = 0$. It must also satisfy

$$ 0 = v_1(l) = \int_0^l \left( \varphi(\sigma) + kv_2(\sigma) \right) \, d\sigma $$ \hspace{0.5cm} (4.18)
which implies
\[ \int_0^l k v_2(\sigma) \, d\sigma = - \int_0^l \varphi(\sigma) \, d\sigma. \] (4.19)

Consequently, \( v_2 \) must be chosen as an element of \( H_0^2(0, l) \) satisfying (4.19). This is obviously possible: we may find \( v_2 \) with compact support in an interval where \( k \) keeps the same sign and we continue \( v_2 \) with vanishing values out of that interval. Then, the general hypotheses of the theory are satisfied with
\[ V' = H^{-1}(0, l) \times H^{-2}(0, l) \] (4.20)

and
\[ M' = H_m = M = L^2(0, l). \] (4.21)

The adjoint of \( B \) is defined by
\[ \langle B^* p, v \rangle_{V'} = \langle p, Bv \rangle \quad \forall (p, v) \in L^2 \times V \]
that is to say
\[ \langle (B^* p)_1, v_1 \rangle_{H^{-1}H_0^1} + \langle (B^* p)_2, v_2 \rangle_{H^{-2}H_0^2} = \int_0^l p \left( \frac{dv_1}{ds} - kv_2 \right) \, ds \]

then, taking \( v_1 \) and \( v_2 \) in the space \( \mathcal{D}(O, l) \) of test functions, we have
\[ \begin{cases} 
(B^* p)_1 = -\frac{dp}{ds}, \\
(B^* p)_2 = -kp.
\end{cases} \] (4.22)

Moreover, the dualities \( H^{-1}, H_0^1 \) and \( H^{-2}, H_0^2 \) are those of distributions (more exactly, continuations or restrictions, as \( \mathcal{D} \) is dense in \( H_0^1 \) and \( H_0^2 \)), so that
\[ B^* p = \left( -\frac{dp}{ds}, -kp \right). \] (4.23)

It should be noted, for ulterior utilization, that the image of \( B^* \) coincides with the subspace \( G^0 \) of \( V' \) which is the polar set of \( G \), i.e. the subspace formed by the elements of \( V' \) such that
\[ \langle f, v \rangle_{V'V} = 0 \quad \forall v \in G. \] (4.24)

Physically, \( V' \) is the “space of forces” and \( G^0 \) the subspace formed by the forces with vanishing work in the inextensional displacements.

Coming back to the system (4.6), we see that \( T^1 \) appears exactly under the form \(-dT^1/ds\) in Eq. (4.6)\(_1\) and \(-kT^1\) in Eq. (4.6)\(_2\) which are precisely the components of \( B^* p \) in (4.22). As a consequence, the unknown \( T^1 \) appears as the Lagrange multiplier when we take the product of the Eqs (4.6) by respectively \( v^1, v^2, v^3 \) and \( \varphi \) which are the components of a test function kinematically admissible, i.e. satisfying the constraint of inextensibility (4.7). and the boundary conditions (3.31) and (3.32).
It is then convenient to write the system (4.6) in terms of the components $T^i$ defined in (4.1) and to denote by $\lambda$ the Lagrange multiplier $T^1$, i.e. to write the system under the form:

\[
\begin{align*}
&k \left[ \frac{dT^2}{ds} - \tau T^3 \right] - \frac{d\lambda}{dx} = f^1, \\
&\tau \left[ \frac{dT^3}{ds} + kT^4 + \tau T^2 \right] + \frac{d}{ds} \left[ - \frac{dT^2}{ds} + \tau T^3 \right] - k\lambda = f^2, \\
&\tau \left[ - \frac{dT^2}{ds} + \tau T^3 \right] - \frac{d}{ds} \left[ \frac{dT^3}{ds} + kT^4 + \tau T^2 \right] = f^3, \\
&\frac{dT^4}{ds} + kT^3 = m^1. 
\end{align*}
\] (4.25)

The unknowns are thus $u^1, u^2, u^3, \varphi \in H^1_0 \times H^2_0 \times H^3_0$ and the Lagrange multiplier $\lambda \in L^2(0, l)$; the equations are (4.25) and the constraint

\[
\frac{dv_1}{ds} - kv_2 = 0. \tag{4.26}
\]

The system (4.25), (4.26) is the specific form of the system (4.15).

Moreover, the bilinear form of energy is

\[
\int_0^l \left( T^2 E_2 + T^3 E_3 + T^4 E_4 \right) dv^1 \tag{4.27}
\]

which is the analogous of the form $a(u, v)$ of the general theory ((4.11) or (4.13)). The space of the $(u^1, u^2, u^3, \theta) \in V$ is here $H^1_0 \times H^2_0 \times H^3_0$ and the subspace $G$ is

\[
G = \left\{ (u_1, u_2, u_3, u_4 = \theta) \in V; \frac{du_1}{ds} - ku_2 = 0 \right\}. \tag{4.28}
\]

5. Expression of the functions $E_i, \ i = 2, 3, 4$

In usual problems, the expression of the bilinear form of energy $a(u, v)$ is explicitly known, in other words, the expressions of the strain and stress is known in terms of the displacement. It is then possible to deduce the equations, or equivalently, to establish the equivalence of (4.9) and (4.10). But, if the strains are not known in term of $u$, as it is the case at the present state of our problem, it is possible to obtain their expressions from the knowledge of the equations. As this process is not usual, we are handling two elementary “examples” before to carry out our study of the rods. The first example is concerned with an unconstrained problem of type (4.9), (4.10), the second one has a constraint and lies in the context of (4.13) ou (4.15).

5.1. Exercices

Exercise 5.1: In two-dimensional elasticity the equilibrium equations are

\[
\begin{align*}
-\partial_1 \sigma_{11} - \partial_2 \sigma_{12} &= f_1, \\
-\partial_1 \sigma_{12} - \partial_2 \sigma_{22} &= f_2 
\end{align*} \tag{5.1}
\]
and the bilinear form of energy is
\[
\int_\Omega \sigma_{ij}(u) e_{ij}(v) \, dx, \tag{5.2}
\]
where \( e_{ij} \) and \( \sigma_{ij} \) are respectively the strain and stress tensors. In order to deduce the expressions of the components \( e_{ij} \), we multiply the two Eqs (5.1) by respectively \( v_1 \) and \( v_2 \) belonging to \( D(\Omega) \). Integrating by parts, we then obtain:
\[
\int_\Omega (\sigma_{11} \partial_1 v_1 + \sigma_{12} \partial_2 v_1 + \sigma_{21} \partial_1 v_2 + \sigma_{22} \partial_2 v_2) \, dx = \int_\Omega (f_1 v_1 + f_2 v_2) \, dx. \tag{5.3}
\]
and, by indentifying the left hand side with (5.2), the classical relations follow:
\[
\begin{aligned}
e_{11} &= \partial_1 v_1, \\
e_{12} &= \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1), \\
e_{22} &= \partial_2 v_2.
\end{aligned} \tag{5.4}
\]

**Exercise 5.2:** Let us consider now the two-dimensional incompressible elasticity, i.e. we prescribe in the previous problem the constraint \( \text{div} \, v = 0 \). The equilibrium equations are (5.1), but classically, the stress tensor is decomposed into a part \( \sigma_{ij}^{\text{def}} \) associated with the deformation and a spherical tensor \( -p \delta_{ij} \), associated with the pressure. Then, (5.1) becomes
\[
\begin{aligned}
-\partial_1 \sigma_{11}^{\text{def}} - \partial_2 \sigma_{12}^{\text{def}} - \partial_1 p &= f_1, \\
-\partial_1 \sigma_{21}^{\text{def}} - \partial_2 \sigma_{22}^{\text{def}} - \partial_2 p &= f_2, \tag{5.5}
\end{aligned}
\]
where the terms \( \partial_1 p \) account for \( B^* p \). The bilinear form is always (5.2). Proceeding as in the previous example, we obtain in (5.3) the extra term
\[
\int_\Omega p \, \text{div} \, v \, dx
\]
which vanishes on account of the constraint. Then, we obtain again the expressions (5.4) for the strain.

5.2. Computing the functions \( E_i \)

Let us consider again the equations of rods (4.25) and the bilinear form (4.27). In order to handle the Eqs (4.25) in a more suitable form, we shall write them under the form
\[
\sum_{J=2}^4 A_{IJ} T^J + (B^* p)_I = f_I, \quad I = 1, 2, 3, 4, \tag{5.6}
\]
where \( \mathbf{f} = (f_1, f_2, f_3, f_4 = m^1) \) and the expressions of the \( A_{IJ} \) are those of the table

\[
\begin{array}{ccc|ccc}
A_{IJ} & J = 2 & J = 3 & J = 4 \\
\hline
I = 1 & k \frac{d}{ds} & -k \tau & 0 \\
I = 2 & \tau^2 - \frac{d^2}{ds^2} & 2 \tau \frac{d}{ds} + \tau' & k \tau \\
I = 3 & -2 \tau \frac{d}{ds} - \tau' & \tau^2 - \frac{d^2}{ds^2} & -k \frac{d}{ds} - k' \\
I = 4 & 0 & k & -\frac{d}{ds}
\end{array}
\]

(5.7)

Let us take the duality product of (5.6) with \( \mathbf{v} = (v^1, v^2, v^3, v^4 = \theta) \), \( \mathbf{v} \in G \), space of the inextensional displacements. As we have pointed out at the end of Section 3.2, \( B^* p \in G^0 \) so that \( \langle B^* p, \mathbf{v} \rangle = 0 \) and we have

\[
\int_0^l \sum_{J=2}^{4} A_{IJ} T^J, v^J \bigg|_{H^{-1} \Omega} \bigg) ds.
\]

(5.8)

Denoting by \( A_{IJ}^* \) the adjoint of the operator \( A_{IJ} \) (or, in other terms, integrating by parts in the sense of distributions) the left hand side of (5.6) becomes

\[
\int_0^l \sum_{I=1}^{4} \sum_{J=2}^{4} T^I (A_{IJ}^* v_J) \ ds.
\]

(5.9)

Then, identifying with the bilinear form of energy (4.27), we obtain

\[
E_J(\mathbf{v}) = \sum_{I=1}^{4} A_{IJ}^* v_J,
\]

(5.10)

where the table of the \( A_{IJ}^* \) is

\[
\begin{array}{ccc|ccc}
A_{IJ}^* & I = 1 & I = 2 & I = 3 & I = 4 \\
\hline
J = 2 & -k' - k \frac{d}{ds} & \tau^2 - \frac{d^2}{ds^2} & 2 \tau \frac{d}{ds} + \tau' & 0 \\
J = 3 & -k \tau & -\tau' - 2 \tau \frac{d}{ds} & \tau^2 - \frac{d^2}{ds^2} & k \\
J = 4 & 0 & k \tau & -k \frac{d}{ds} & \frac{d}{ds}
\end{array}
\]

(5.11)
so that we obtain from (5.10) the expressions for $E_J, J = 2, 3, 4$:

$$
\begin{align*}
E_2 &= -k'v_1 - k \frac{dv_1}{ds} + \tau^2 v_2 - \frac{d^2v_2}{ds^2} + \tau' v_3 + 2\tau \frac{dv_3}{ds}, \\
E_3 &= -k\tau v_1 - \tau' v_2 - 2\tau \frac{dv_2}{ds} + \tau^2 v_3 - \frac{d^2v_3}{ds^2} + k\theta, \\
E_4 &= k\tau v_2 + k \frac{dv_3}{ds} + \frac{d\theta}{ds}.
\end{align*}
$$

(5.12)

6. Synthesis of the results and complements. Helicoïdal rod with a free extremity

6.1. Summary of the previous results

Let us recall that, in the present case, in the local constitutive law the moments $T^2 \equiv -M^3$, $T^3 \equiv M^2$, $T^4 \equiv M^1$ are expressed as functions of the $E_2, E_3, E_4$ under the form

$$
T^J = K^{IJ} E_J,
$$

(6.1)

where $K$ is a matrix definite positive and symmetric.

The unknowns $u_1^0, u_2^0, u_3^0, u_4^0 \equiv \theta$ are solutions of the variational problem

Find $U \equiv (u_1^0, u_2^0, u_3^0, u_4^0 \equiv \theta) \in G$ such that

$$
\int_0^l \left( T^2(U) E_2(V) + T^3(U) E_3(V) + T^4(U) E_4(V) \right) dy^1 = \int_0^l f_i v_i dy^1 \quad \forall V \in G,
$$

(6.2)

where $G$ is the constrained space defined in (4.28) and, of course, $T^J(U) = K^{IJ} E_J(U)$ and formulas (5.12) are used.

Equivalently these unknowns are solutions of the equations with Lagrange multiplier $\lambda$: Find $U(u_1^0, u_2^0, u_3^0, u_4^0 \equiv \theta, \lambda) \in V \times L^2(0, l)$ satisfying the Eqs (4.25) and the constraint (4.26) with the boundary conditions (3.31) and (3.32). Of course,

$$
V = H_0^1 \times H_0^2 \times H_0^2 \times H_0^1.
$$

(6.3)

The existence and uniqueness of the solution to (6.2) follows classically from the classical Lax–Milgram theorem and was proved in (Jamal, 1998).

6.2. Example: helicoïdal rod with a free extremity

As mentioned in the introduction, slight modifications of the previous theory allows us to consider other situations, mainly concerning the boundary conditions. We adress here a case where one of the extremities is clamped and the other is free.

We consider the helix defined by

$$
\begin{align*}
x_1 &= \cos \theta, \\
x_2 &= \sin \theta, \\
x_3 &= h\theta,
\end{align*}
$$

where $0 < \theta < \pi$. 
The applied forces reduce to a point force $\varepsilon^4 F e_3$ applied at the free extremity, where $(O; e_1, e_2, e_3)$ is the frame associated with the cartesian coordinates $x_1, x_2, x_3$. Moreover, we consider an isotropic homogeneous material, so that the constitutive law is (6.1) with

$$
(K^{I,I}) = \begin{pmatrix}
ES & 0 & 0 & 0 \\
0 & EI_3 & 0 & 0 \\
0 & 0 & EI_2 & 0 \\
0 & 0 & 0 & \frac{E}{2(1+v)}I_1
\end{pmatrix},
$$

where we recognize the classical coefficients of rigidity for traction, flexions and torsion respectively for the section dilated to the variables $\varepsilon$ (and consequently independent of $\varepsilon$). This choice allows us to compute the resultant $T$ and the moment $M$ at each point and to work out the problem almost explicitly.

It is well known that the curvature $k$ and the torsion $\tau$ are constant and respectively equal to

$$
k = \frac{1}{1+h^2}, \quad \tau = \frac{h}{1+h^2}.
$$

Then, from the equilibrium equations (2.1) it follows

$$
\varepsilon^3 \frac{d}{ds}(T^1 a_1) + \varepsilon^4 \frac{dT}{ds} + \cdots = 0 \Rightarrow T^1 a_1 = Cte.
$$
As the given force at $B$ is $F = \varepsilon^4 F \mathbf{e}_3$, we immediately see that $T^1 \equiv 0$ and we have
\[
\frac{dT}{ds} = 0 \Rightarrow T = \text{Const} = F \mathbf{e}_3 = F \left( \frac{h}{\sqrt{1+h^2}} \mathbf{a}_1 + \frac{1}{\sqrt{1+h^2}} \mathbf{a}_1 \right).
\] (6.6)

The components, in the Frenet frame, of the moment satisfy
\[
\begin{align*}
\frac{dM^1}{ds} - kM^2 &= 0, \\
\frac{dM^2}{ds} + kM^1 - \tau M^3 &= \frac{F}{\sqrt{1+h^2}}, \\
\frac{dM^3}{ds} + \tau M^2 &= 0
\end{align*}
\] (6.7)
from which it follows
\[
\frac{d^2 M^2}{ds^2} + (k^2 + \tau^2) M^2 = 0.
\]
The component $M^2$ is of the form:
\[
M^2(s) = \alpha \cos \frac{s}{\sqrt{1+h^2}} + \beta \sin \frac{s}{\sqrt{1+h^2}}.
\]
The condition at the free extremity gives
\[
M^2(2\pi \sqrt{1+h^2}) = 0 \Rightarrow \alpha = 0
\]
from which
\[
M^2(s) = \beta \sin \frac{s}{\sqrt{1+h^2}}.
\] (6.8)
It immediately follows
\[
\begin{align*}
M^1(s) &= -\frac{\beta}{\sqrt{1+h^2}} \cos \frac{s}{\sqrt{1+h^2}} + C_1, \\
M^3(s) &= \frac{\beta h}{\sqrt{1+h^2}} \cos \frac{s}{\sqrt{1+h^2}} + C_3
\end{align*}
\] (6.9)
where $\beta$, $C_1$ and $C_2$ are arbitrary constants. As $B$ is free of applied moments
\[
\begin{align*}
M^1(2\pi \sqrt{1+h^2}) &= -\frac{\beta}{\sqrt{1+h^2}} + C_1 = 0 \Rightarrow C_1 = \frac{\beta}{\sqrt{1+h^2}}, \\
M^3(2\pi \sqrt{1+h^2}) &= \frac{\beta h}{\sqrt{1+h^2}} + C_3 = 0 \Rightarrow C_3 = -\frac{\beta h}{\sqrt{1+h^2}}
\end{align*}
\] (6.10)
Now, we compute the constant $\beta$: the second Eq. (6.7) gives, on account of the relations (6.10)
\[
\frac{dM^2}{ds}(2\pi \sqrt{1+h^2}) = \frac{F}{\sqrt{1+h^2}} \Rightarrow \beta = F.
\] (6.11)
From the constitutive law (6.4) with \( T^1 = 0, T^2 = -M^3, T^3 = M^2, T^4 = M^1 \) and using the expressions (5.12) of the \( E_I \) which are in our case

\[
\begin{align*}
E_2 &= -\frac{d^2 u^0_2}{ds^2} - 2k \frac{du^0_1}{ds} + 2\tau \frac{du^0_3}{ds} + (k^2 + \tau^2)u^0_2, \\
E_3 &= -\frac{d^2 u^0_3}{ds^2} - 2\tau \frac{du^0_2}{ds} + k\theta - k\tau u^0_1 + \tau^2 u^0_3, \\
E_4 &= \theta' + k\left(\frac{du^0_3}{ds} + \tau u^0_2\right) 
\end{align*}
\]

(6.12)

we obtain the system satisfied by the unknowns \( u^0(s) \) and \( \theta(s) \):

\[
\begin{align*}
\frac{du^0_1}{ds} - ku^0_2 &= 0, \\
E_I_3 \left( -\frac{d^2 u^0_3}{ds^2} - 2k \frac{du^0_1}{ds} + 2\tau \frac{du^0_3}{ds} + (k^2 + \tau^2)u^0_2 \right) &= \frac{hF}{\sqrt{1 + h^2}} \left( 1 - \cos \frac{s}{\sqrt{1 + h^2}} \right), \\
E_I_2 \left( -\frac{d^2 u^0_3}{ds^2} - 2\tau \frac{du^0_2}{ds} + k\theta - k\tau u^0_1 + \tau^2 u^0_3 \right) &= F \sin \frac{s}{\sqrt{1 + h^2}}, \\
\mu I_1 \left( \theta' + k\frac{du^0_3}{ds} + k\tau u^0_2 \right) &= \frac{hF}{\sqrt{1 + h^2}}
\end{align*}
\]

(6.13)

which may be written under the matrix form:

\[
\frac{d}{ds} \begin{pmatrix} u^0_1 \\ u^0_2 \\ u^0_3 \\ u^0_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -k & 2\tau & 0 \\ 0 & 0 & 1 \\ -2\tau^3 & 0 & 3\tau^2 + k^2 \end{pmatrix} \begin{pmatrix} u^0_1 \\ u^0_2 \\ u^0_3 \\ u^0_4 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2 \\ 0 \\ f_4 \end{pmatrix},
\]

(6.14)

where

\[
f_2(s) = -\frac{h}\sqrt{k} \frac{E_I_3}{E_I_3} \left( s - \frac{1}{\sqrt{k}} \sin \sqrt{k} \right)
\]

(6.15)

and

\[
f_4(s) = \frac{2\tau\sqrt{k}h}{E_I_3} \left( s - \frac{1}{\sqrt{k}} \sin \sqrt{k} \right) + \frac{hk\sqrt{k}}{\mu I_1} F \sin \sqrt{k} - \frac{F}{E_I_2} \sin \sqrt{k}.
\]

(6.16)

Using Maple, D. Choï gave the eigenvalues of the matrix, which are

\[
\begin{align*}
\lambda_1 &= \sqrt{-2\tau^2 + \sqrt{5\tau^4 + k^4 + 2\tau^2 k^2}}, \\
\lambda_2 &= -\sqrt{-2\tau^2 + \sqrt{5\tau^4 + k^4 + 2\tau^2 k^2}}, \\
\lambda_3 &= \sqrt{-2\tau^2 - \sqrt{5\tau^4 + k^4 + 2\tau^2 k^2}}, \\
\lambda_4 &= -\sqrt{-2\tau^2 - \sqrt{5\tau^4 + k^4 + 2\tau^2 k^2}},
\end{align*}
\]
the solution is thus of the form
\[ \tilde{u}^0(s) = \sum_{i=1}^{4} C_i \, e^{\lambda_is} \, v_i + u_f(s), \]
where the functions \( v_i \) and \( u_f \) are easily computed by formal calculus.

Clearly, (6.14) shows that the torsion angle \( u_0^4(s) \equiv \theta(s) \) is coupled with the others so that the torsion phenomenon is of the same order as the flexions.

7. General coupled case with given forces of order \( \varepsilon^4 \)

We now consider the same problem as in Section 5 in the general case where the local constitutive law involves the traction \( T^1 \). We shall also assume that the curvature \( k \) does not vanish identically. According to the general properties (see Section 4.1) the traction is of order \( \varepsilon^3 \) and the equilibrium equations at that order give
\[ \begin{align*}
\frac{dT^1}{ds} = 0 \Rightarrow T^1 &= \text{Const} \Rightarrow T^1 = 0 \text{ everywhere.} \\
kT^1 = 0 \text{ where } k \neq 0 \Rightarrow T^1 = 0
\end{align*} \]
It then follows that the resultant is, as in Section 5, \( \mathbf{T}^\varepsilon = \varepsilon^4 \mathbf{T} + \cdots \) and we shall denote as in Section 4.2 by \( \lambda \) the first component of \( \mathbf{T} \).

In the present case, we have 6 unknowns: \( u^0, \theta, E_1, \lambda \) and also 6 equations: the four Eqs (4.25), \( T^1 = 0 \) and the constraint (4.26).

In fact, without using the equilibrium equations, we may obtain \( T^1 = 0 \) from the variational formulation. Indeed, it writes:
\[ \int_0^l \sum_{i=1}^{4} T^1 E_i \, ds = \int_0^l (f^1 v_i + m^1 \theta) \, ds \] (7.2)
then, by taking as test element \( (0, 0, 0, 0, E_1^\lambda) \) with arbitrary \( E_1^\lambda \in L^2(0, l) \), we have \( T^1 = 0 \).

As a result, the resultant \( \mathbf{T}^\varepsilon \) is of order \( \varepsilon^4 \) so that the equations and the orders of magnitude are the same as in Section 4, i.e. (4.25). Obviously, the expressions of the functions \( E_i \) are the same as in Subsection 5.2.

8. Case when certain given forces are of order \( \varepsilon^3 \)

Let us recall that, in Subsection 4.1, we introduced a rather artificial term \( \mathbf{m}^\varepsilon = \varepsilon^4 m^1 \mathbf{a}_1 \) (applied moment by unit length). This allowed us to understand better the meaning of the different terms involved in the problem.

Analogously, the problem in Section 7 is not very clear since the unknown \( E_1 \) is involved in the constitutive local law whereas the corresponding force \( T^1 \) vanishes.

It should also be noticed that
\[ E_1 = \frac{d\tilde{u}^1_1}{ds} - k\tilde{u}^1_2 \]
so that \( E_1 \) is the lengthening at order \( \varepsilon \); but it is easier to keep \( E_1 \) as an unknown rather than expressing it in terms of \( \tilde{u}^1 \).
A better understanding of the problem is obtained when applying, in addition to \( F \) and \( \varepsilon^4 m^1 a_1 \), a given abstract “force” \( F \in L^2(0, l) \) associated with \( E_1 \). The variational formulation becomes

\[
\int_0^l \sum_{i=1}^4 T^i E_i^* \, ds = \int_0^l \left( \sum_{i=1}^3 f_i v_i + m_1 v_4 + F E_1^* \right) \, ds \tag{8.1}
\]

then, taking the test function \((0, 0, 0, E_1^*)\) with arbitrary \( E_1^* \in L^2(0, l) \), we see that (8.1) gives \( T^1 = F \) (which agree with Section 7 where \( T^1 = 0 \) for \( F = 0 \)).

Let us search for the equations of statics of the present problem. We saw, in Section 4, that the component in \( C^0 \) of the applied forces give a vanishing work for the inextensional displacements. These forces are of the form

\[
\begin{align*}
\dot{g}^1 &= -\frac{dp}{ds}, \\
\dot{g}^2 &= -kp, \\
\dot{g}^3 &= 0,
\end{align*}
\tag{8.2}
\]

where \( p \in L^2(0, l) \). Let us then consider an auxiliary problem where the applied forces have the form (8.2). The Eqs (2.2) of statics give

\[
T^1 = p \in L^2(0, l).
\]

Consequently, for given applied forces \( F^* = (f^{\varepsilon^1}, f^{\varepsilon^2}, f^{\varepsilon^3}) \), \( m^* = m^{\varepsilon^1} a_1 \) of the form

\[
\begin{align*}
f^{\varepsilon^1} &= \varepsilon^3 \left( -\frac{dF}{ds} \right) + \varepsilon^4 f^1, \\
f^{\varepsilon^2} &= \varepsilon^3 (-kF) + \varepsilon^4 f^2, \\
f^{\varepsilon^3} &= \varepsilon^4 f^3, \\
m^{\varepsilon^1} &= \varepsilon^4 m^1
\end{align*}
\]

with \( F \in L^2(0, l) \), \( f^1, m^1 \in H^{-1}(0, l) \), \( f^2, f^3 \in H^{-2}(0, l) \) the equations of the problem are

\[
T^1 = F \tag{8.3}
\]

with the Eqs (4.25).

It will be noticed that

\[
T^* \cdot a_1 = \varepsilon^3 F + \varepsilon^4 \lambda + \cdots, \tag{8.4}
\]

where \( \lambda \) is the Lagrange multiplier which appears in (4.25), associated with the inextensibility constraint, and \( F \) is associated with the force \( g \) defined by (8.2) with \( p = F \).

References
