Risk analysis for a stochastic cash management model with two types of customers

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Abstract

A stochastic cash management system is studied in which the cash flow is modeled by the superposition of a Brownian motion with drift and a compound Poisson process with positive and negative jumps for “big” deposits and withdrawals, respectively. We derive explicit formulas for the distributions of the bankruptcy time, the time until bankruptcy or the reaching of a prespecified level, the maximum cash amount in the system, and for the expected discounted revenue generated by the system. ©2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

We study a stochastic model of a cash management system. Two types of customers have to be served: many “little” customers frequently deposit and withdraw small amounts of money; the cash fluctuations generated by them will be modeled by a Brownian motion. The transactions of the few “big” customers cause upward or downward jumps in the amount of cash held by the system. The arrival times of the big customers are assumed to form a Poisson process. The cash flow in and out of the system is not directly controllable; it can only be influenced by the specification of the parameters of the underlying stochastic processes.

We suppose that the total drift of the cash flow is negative so that level zero will eventually be reached or downcrossed. The cash fund process \( X = (X(t)| t \geq 0) \) is assumed to start at some initial capital \( X(0) = x > 0 \), to stop at the bankruptcy time \( T = \inf\{t \geq 0| X(t) \leq 0\} \), and in between to fluctuate as a Lévy process formed by a Brownian and a compound Poisson component.

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The main objective of this paper is to derive several characteristic functionals of the cash fund process described above. We will determine the distribution of the bankruptcy time $T$ by means of its Laplace transform (LT) $E(e^{-\beta T} | X(0) = x)$. A suitable functional to measure the revenue generated by the cash fund is

$$E \left( \int_0^T e^{-\beta t} X(t) \, dt | X(0) = x \right), \quad \beta \geq 0,$$

since $\beta$ can be interpreted as a discount factor. We also obtain as a by-product the functional

$$E \left( \int_0^T \exp(-\beta X(t)) \, dt | X(0) = x \right) / E(T | X(0) = x),$$

which can also be interpreted as “quasi-stationary” LT. This makes it possible to compute the quasi-stationary $n$th moment functional. Another variable of interest is the last record value $K = \sup \{X(t)|0 \leq t \leq T\}$, the maximum amount of cash in the system. To compute the distribution of $K$, we need the distributions of the two-sided stopping times $\tau_y = \inf \{t \geq 0|X(t) \neq (0, x+y)\}$, $y > 0$, and of $\sup_{t \geq 0} X(t)$, which are of independent interest. We will also determine the joint distribution of $K$ and $X(T)$. The explicit expressions that we obtain for all these quantities may lay the ground for optimization purposes.

The cash flow model described above is similar to other stochastic storage systems such as queues, inventories, dams, and insurance risk models. The crucial novel feature is the presence of random jumps upwards and downwards. In the classical models random changes occur only in one direction, while the flow in the other direction is deterministic. Moreover, our approach adds a second new dimension by considering small and large inflows and outflows simultaneously. The cash flow process studied here can also be interpreted as the workload for a $GI/G/1$ queue in heavy traffic with occasional “big” customers causing random upward jumps and work removals by “negative customers” (Boucherie and Boxma, 1996; Gelenbe, 1991; Jain and Sigman, 1996). What is here called “bankruptcy”, can, after some reformulation, be considered as a clearing action in inventory systems (Kim and Seila, 1993; Perry and Stadje, 1998; Serfozo and Stidham, 1978; Stidham, 1977, 1986).

Several papers relate certain storage systems to cash flow management. Harrison and Taksar (1983) consider impulse control policies: when the cash fund gets too large, the controller may choose to convert some of his cash into securities. When the amount of cash decreases below some limit, securities may be reconverted into cash. Harrison et al. (1983) model the cash fund as a Brownian motion reflected at the origin. Both papers focus on the proofs that their special policies are optimal regarding transaction costs. Perry (1997) extended the model of Harrison et al. (1983) by considering drift control for a two-sided reflected Brownian motion, also taking into account holding cost and unsatisfied demand cost functionals. The combination of a Brownian and a compound Poisson component as suggested here can make these models more realistic. Milne and Robertson (1996) study the behavior of a firm whose cash flow is determined by a diffusion process and which faces liquidation if the internal cash balance falls below some threshold value. They look for an optimal trade-off between the desire to pay dividends and the need to retain cash as a barrier against possible liquidation. Browne (1995) considers a firm with an uncontrollable cash flow and the possibility to invest in a risky stock. In Asmussen and Taksar (1997) and Asmussen and Perry (1998) jump diffusion models motivated by finance and general storage applications are studied. Further related papers in which the buffer content can be interpreted as a cash fund are Asmussen and Kella (1996), Radner (1998), Zipkin (1992a,b).

2. Analysis of the model

We now introduce our model in a formal manner. The cash amount in the system at time $t$ is given by

$$X(t) = x + B(t) + Y(t), \quad t \geq 0,$$
where
1. \( x > 0 \) is the initial capital;
2. \( B = (B(t) \mid t \geq 0) \) is a Brownian motion with arbitrary drift \( \mu \in \mathbb{R} \) and diffusion parameter \( \sigma^2 = 1 \).
3. \( Y = (Y(t) \mid t \geq 0) \) is a compound Poisson process with arrival rate \( \lambda + \eta \) and jump size distribution having the LT:

\[
\psi(\alpha) = \frac{\lambda}{\lambda + \eta} \frac{\nu}{v + \alpha} + \frac{\eta}{\lambda + \eta} \frac{\xi}{\xi - \alpha}.
\]

The LT is only defined for \(-\nu < \Re \alpha < \xi\) but can, by analytic continuation, be extended to a function on \( \mathbb{C} \setminus \{ -\nu, \xi \} \). The processes \( B \) and \( Y \) are assumed to be independent. Note that \( Y \) itself is the superposition of two independent compound Poisson processes: one with arrival rate \( \lambda \) and \( \exp(-\nu) \)-distributed positive jumps, and one with arrival rate \( \eta \) and \( \exp(\xi) \)-distributed negative jumps. We will show in Section 4 that the analysis below can be extended to the case that the upward and the downward jumps have phase-type or hyperexponential distributions.

The exponent \( \varphi(\alpha) = \log \mathbb{E}(e^{-\alpha(B(1)+Y(1))}) \) of the Lévy process \( B + Y \) is given by

\[
\varphi(\alpha) = \frac{\alpha^2}{2} - \mu \alpha - \frac{\lambda \alpha}{v + \alpha} + \frac{\eta \alpha}{\xi - \alpha}.
\] (2.1)

For future use we note the following properties of \( \varphi \).

**Lemma 1.** For every \( \beta > 0 \) the equation \( \varphi(\alpha) = \beta \) has exactly four distinct roots \( \alpha_i(\beta), i = 1, 2, 3, 4 \), satisfying

\[
\alpha_1(\beta) > \xi > \alpha_2(\beta) > 0 > \alpha_3(\beta) > -\nu > \alpha_4(\beta).
\]

If \( \beta = 0 \) and \( \mu - (\eta/\xi) + (\lambda/\nu) < 0 \), there are exactly four roots of the equation \( \varphi(\alpha) = 0 \), and they satisfy \( \alpha_1(0) > \xi > \alpha_2(0) = 0 > \alpha_3(0) > -\nu > \alpha_4(0) \).

**Proof.** The equation \( \varphi(\alpha) = \beta \) can be written as

\[
g(\alpha) \equiv \frac{\eta \alpha}{\xi - \alpha} - \frac{\lambda \alpha}{v + \alpha} = \beta + \mu \alpha - \frac{\alpha^2}{2} \equiv h(\alpha).
\]

Note that

\[
\lim_{\alpha \to -\infty} g(\alpha) = \lim_{\alpha \to \infty} g(\alpha) = -\eta - \lambda < 0,
\]

\[
\lim_{\alpha \uparrow -\nu} g(\alpha) = \lim_{\alpha \downarrow \xi} g(\alpha) = -\infty.
\]

Since \( h(0) = \beta \geq 0 \) and \( \lim_{\alpha \to \pm\infty} h(\alpha) = -\infty \), there are roots \( \alpha_1(\beta) > \xi \) and \( \alpha_4(\beta) < -\nu \). Furthermore, \( g(0) = 0 \) and \( \lim_{\alpha \uparrow -\nu} g(\alpha) = \lim_{\alpha \downarrow \xi} g(\alpha) = \infty \). The function \( h \) is continuous in \([-\nu, \xi]\) and \( h(0) = \beta \), so that if \( \beta > 0 \), there are roots \( \alpha_2(\beta) \in (0, \xi) \) and \( \alpha_3(\beta) \in (-\nu, 0) \). If \( \beta = 0 \), then \( \alpha_2(0) = 0 \) is a root and as

\[
g'(0) = (\eta/\xi) - (\lambda/\nu) > \mu = h'(0),
\]

there must be a root \( \alpha_3(0) \in (-\nu, 0) \). Since \((\xi - \alpha)(v + \alpha)(\varphi(\alpha) - \beta)\) is a polynomial of degree 4 in \( \alpha \), the four roots we have found are the only ones. \( \square \)

One of the main tools of our analysis is the process

\[
M(t) = (\varphi(\alpha) - \beta) \int_0^t e^{-\alpha X(s) - \beta s} \, ds + e^{-\alpha x} - e^{-\alpha X(t) - \beta t}, \quad t \geq 0.
\]
It is easy to check that $M = (M(t) | t \geq 0)$ is a martingale for every $\alpha$ satisfying $-\nu < \text{Re} \alpha < \xi$ and every $\beta \geq 0$. (Actually, $M$ is a special case of a martingale introduced by Kella and Whitt (1992).)

First we will use $M$ to derive the distribution of the stopping time

$$\tau_y = \inf \{ t \geq 0 | X(t) \leq 0 \text{ or } X(t) \geq x + y \}, \quad y > 0.$$  

The LT $\phi^y(\beta)$ of $\tau_y$ can be decomposed as

$$\phi^y(\beta) = \phi_1^y(\beta) + \phi_2^y(\beta) + \phi_3^y(\beta) + \phi_4^y(\beta),$$

where

$$\phi_1^y(\beta) = E(e^{-\beta \tau_y} 1_{\{X(\tau_y) = 0\}}), \quad \phi_2^y(\beta) = E(e^{-\beta \tau_y} 1_{\{X(\tau_y) < 0\}}),$$

$$\phi_3^y(\beta) = E(e^{-\beta \tau_y} 1_{\{X(\tau_y) = x + y\}}), \quad \phi_4^y(\beta) = E(e^{-\beta \tau_y} 1_{\{X(\tau_y) > x + y\}},$$

The four components $\phi_i^y(\beta)$ are also of independent interest. They are given in the following theorem.

**Theorem 1.** Fix $x, y > 0$ and define the $(4 \times 4)$-matrix $A(\beta)$ by

$$A(\beta) = \begin{pmatrix}
1, & \frac{\nu}{\nu - \alpha_1(\beta)}, & e^{-\alpha_1(\beta)(x+y)}, & e^{-\alpha_1(\beta)(x+y)} & \frac{\xi}{\xi + \alpha_1(\beta)} \\
1, & \frac{\nu}{\nu - \alpha_2(\beta)}, & e^{-\alpha_2(\beta)(x+y)}, & e^{-\alpha_2(\beta)(x+y)} & \frac{\xi}{\xi + \alpha_2(\beta)} \\
1, & \frac{\nu}{\nu - \alpha_3(\beta)}, & e^{-\alpha_3(\beta)(x+y)}, & e^{-\alpha_3(\beta)(x+y)} & \frac{\xi}{\xi + \alpha_3(\beta)} \\
1, & \frac{\nu}{\nu - \alpha_4(\beta)}, & e^{-\alpha_4(\beta)(x+y)}, & e^{-\alpha_4(\beta)(x+y)} & \frac{\xi}{\xi + \alpha_4(\beta)}
\end{pmatrix}.$$  

Then

$$\begin{pmatrix}
\phi_1^y(\beta) \\
\phi_2^y(\beta) \\
\phi_3^y(\beta) \\
\phi_4^y(\beta)
\end{pmatrix} = A(\beta)^{-1} \begin{pmatrix} e^{-\alpha_1(\beta)x} \\
e^{-\alpha_2(\beta)x} \\
e^{-\alpha_3(\beta)x} \\
e^{-\alpha_4(\beta)x}
\end{pmatrix}. \quad (2.2)$$

The inversion of $A(\beta)$ is cumbersome. Explicit expressions for the $\phi_i^y(\beta)$ are given in the Appendix A.

**Proof.** There are four possibilities for $X$ to exit from the interval $(0, x + y) : X(\tau_y) = 0, X(\tau_y) < 0, X(\tau_y) = x + y, X(\tau_y) > x + y$. In the second and the fourth case, the overshoot is exponentially distributed with parameters $\nu$ and $\xi$, respectively, and conditionally independent of $(t, X(t)), t \leq \tau_y$, given that $X(\tau_y) < 0$ or $X(\tau_y) > x + y$, respectively. Now we apply the optional sampling theorem to the martingale $M$ and the stopping time $\tau_y$. The equation $E(M(\tau_y)) = E(M(0))$ yields
\[(\varphi(\alpha) - \beta)E\left(\int_0^{\tau_Y} e^{-\alpha X(t) - \beta t} dt\right)\]
\[= -e^{-\alpha x} + E(e^{-\alpha x} - \beta \tau_Y)\]
\[= -e^{-\alpha x} + E(e^{-\beta \tau_Y} 1_{(X(\tau_Y) = 0)} + P(X(\tau_Y) < 0)E(e^{-\alpha X(\tau_Y)} - \beta \tau_Y)\]
\[\leq 0 + e^{-\alpha x} E(e^{-\beta \tau_Y} 1_{(X(\tau_Y) = x + y)} + P(X(\tau_Y) > x + y)E(e^{-\alpha X(\tau_Y)} - \beta \tau_Y)\]
\[= -e^{-\alpha x} + \phi_1^y(\beta) + P(X(\tau_Y) < 0)\frac{v}{v - \alpha} E(e^{-\beta \tau_Y} X(\tau_Y) < 0)\]
\[+ e^{-\alpha(x+y)} \phi_2^y(\beta) + P(X(\tau_Y) > x + y)e^{-\alpha(x+y)} \frac{\xi}{\xi + \alpha} E(e^{-\beta \tau_Y} X(\tau_Y) > x + y)\]
\[= -e^{-\alpha x} + \phi_1^y(\beta) + \frac{v}{v - \alpha} \phi_2^y(\beta) + e^{-\alpha(x+y)} \frac{\xi}{\xi + \alpha} \phi_3^y(\beta) + e^{-\alpha(x+y)} \frac{\xi}{\xi + \alpha} \phi_4^y(\beta). \quad (2.3)\]

Eq. (2.3) holds for every \(\alpha \in (-v, \xi)\); but since all terms are analytic functions for \(\alpha \in \mathbb{C}\setminus\{-v, \xi\}\), it follows from the identity theorem for analytic functions that (2.3) holds for all \(\alpha \in \mathbb{C}\setminus\{-v, \xi\}\) and all \(\beta \geq 0\). In particular, we can set \(\alpha = \alpha_i(\beta), \ i = 1, 2, 3, 4\), and obtain the four equations

\[0 = -e^{-\alpha_i(\beta) x} + \phi_1^y(\beta) + \frac{v}{v - \alpha_i(\beta)} \phi_2^y(\beta)\]
\[+ e^{-\alpha_i(\beta)(x+y)} \phi_3^y(\beta) + e^{-\alpha_i(\beta)(x+y)} \frac{\xi}{\xi + \alpha_i(\beta)} \phi_4^y(\beta), \quad i = 1, 2, 3, 4. \quad (2.4)\]

\(A(\beta)\) is the coefficient matrix of this system of linear equations, and the solution is thus given by (2.2). \(\square\)

3. The revenue functionals

In this section we study the behavior of \(X\) until the bankruptcy time \(T = \inf\{t \geq 0|X(t) \leq 0\}\). We assume from now on that \(\mu - (\eta/\xi) + (\lambda/\nu) < 0\), i.e. the process has a downward drift. This ensures \(E(T) < \infty\). We start from the martingale relation \(E(M(T) = E(M(0))\), which can be written more explicitly as

\[(\varphi(\alpha) - \beta)E\left(\int_0^T e^{-\alpha X(s) - \beta s} ds\right) = -e^{-\alpha x} + E(e^{-\alpha X(T)} - \beta T). \quad (3.1)\]

One can obtain the LT of \(T\) by letting \(y\) in \(\phi^y(\beta)\) tend to infinity, but it is easier to use (3.1) and the decomposition

\[E(e^{-\alpha X(T)} - \beta T) = E(e^{-\beta T} 1_{(X(T) = 0)}) + \frac{\xi}{\xi - \alpha} E(e^{-\beta T} 1_{(X(T) > 0)})\]
\[= \rho_1(\beta) + \frac{\xi}{\xi - \alpha} \rho_2(\beta), \quad (3.2)\]

where \(\rho_i(\beta) = \lim_{y \to \infty} \phi_i^y(\beta)\). Setting \(\alpha = \alpha_i(\beta), \ i = 2, 3\), in (3.1) yields

\[0 = -e^{-\alpha_i(\beta) x} + \rho_1(\beta) + \frac{\xi}{\xi - \alpha_i(\beta)} \rho_2(\beta), \quad i = 2, 3. \quad (3.3)\]

The solution of (3.3) is given by

\[\rho_2(\beta) = \frac{(\xi - \alpha(\beta))(\xi - \alpha_3(\beta))}{\xi(\alpha_2(\beta) - \alpha_3(\beta))}(e^{-\alpha_2(\beta)} - e^{-\alpha_3(\beta)}), \quad (3.4)\]
\[\rho_1(\beta) = e^{-\alpha_2(\beta)} - \frac{\alpha_2(\beta)}{\xi - \alpha_2(\beta)} \rho_2(\beta). \quad (3.5)\]
Hence, by (3.1) and (3.2),

\[ E \left( \int_0^T e^{-\alpha X(s) - \beta s} \, ds \right) = (\varphi(\alpha) - \beta)^{-1} \left[ \rho_1(\beta) + \frac{\xi}{\xi - \alpha} \rho_2(\beta) - e^{-\alpha x} \right]. \tag{3.6} \]

Inserting (2.1) in (3.6), taking the derivative in (3.6) with respect to \( E \) gives the desired revenue functional \( E \left( \int_0^T X(s) e^{-\beta s} \, ds \right) \). We omit the tedious algebra and state the result.

**Theorem 2.** The revenue functionals are given by

\[ E(e^{-\beta T}) = \rho_1(\beta) + \rho_2(\beta), \]

\[ E \left( \int_0^T X(s) e^{-\beta s} \, ds \right) = \frac{(\rho_1(\beta) + \rho_2(\beta) - 1)(\mu - (\eta/\xi) + (\lambda/\nu))}{\beta^2} + \frac{x}{\beta} + \frac{\rho_2(\beta)}{\beta^2}. \]

The quasi-stationary distribution of \( X \) can also be derived from (3.6). Its LT is given by

\[ \frac{E \left( \int_0^T e^{-\alpha X(s)} \, ds \right)}{E(T)} = \frac{e^{-\alpha x} - \rho_1(0) + \frac{\xi}{\xi - \alpha} \rho_2(0)}{\varphi(\alpha)(\rho'_1(0) + \rho'_2(0))}. \tag{3.7} \]

The numerator on the left-hand side of (3.7) is computed from (3.6) by setting \( \beta = 0 \).

4. Extension to phase-type and hyperexponential jumps

In order to see how the above martingale approach can be extended to phase-type and hyperexponential jumps, we consider as an example the case that the positive jumps are Erlang \((\nu, 2)\) while the distribution of the negative jumps is a mixture of \( \exp(\xi_1) \) and \( \exp(\xi_2) \) with weights \( p \) and \( 1 - p \), respectively. Thus, the LT of the jump size is

\[ \psi(\alpha) = \frac{\lambda}{\lambda + \eta} \left( \frac{\nu}{\nu + \alpha} \right)^2 + \frac{\eta}{\lambda + \eta} \left( p \frac{\xi_1}{\xi_1 - \alpha} + (1 - p) \frac{\xi_2}{\xi_2 - \alpha} \right), \]

for some \( p \in (0, 1) \) and \( \nu, \xi_1, \xi_2 > 0 \). The exponent of \( X \) is given by

\[ \varphi(\alpha) = \frac{\alpha^2}{2} - \mu \alpha - \lambda \left( 1 - \left( \frac{\nu}{\nu + \alpha} \right)^2 \right) - \eta \left( 1 - \frac{p \xi_1}{\xi_1 - \alpha} - \frac{(1 - p) \xi_2}{\xi_2 - \alpha} \right). \]

**Lemma 2.** There is a \( \beta^* > 0 \) such that for the equation \( \varphi(\alpha) - \beta = 0 \) the following holds:

1. If \( \beta \in (0, \beta^*) \), there are six distinct real roots.
2. If \( \beta \in (\beta^*, \infty) \), there are four distinct real roots and two (conjugate) non-real roots.
3. If \( \beta = \beta^* \), there are five distinct real roots, one of them having the multiplicity 2.

**Proof.** The equation \( \varphi(\alpha) = \beta \) is equivalent to a polynomial equation of degree six so that there are exactly six complex roots, counted with their multiplicities. Every intersection point of the real curves \( h(\alpha) = \beta + \mu \alpha - (\alpha^2/2) \) and \( g(\alpha) = \lambda (1 - [\nu/(\nu + \alpha)]^2) - \eta (1 - (p \xi_1/\xi_1 - \alpha)) - ((1 - p) \xi_2/\xi_2 - \alpha) \) gives a root. The parabola \( h \) satisfies \( h(0) = \beta \leq 0 \), and because of \( h'(0) = \mu > 0 \), has its maximum at some positive value. Considering the behavior of \( g \) at its three vertical asymptotes at \( \alpha = -\nu, \xi_1, \xi_2 \), it is easily seen that there are exactly four intersection points (one in each of the intervals \((-\nu, 0), (0, \xi_1), (\xi_1, \xi_2), (\xi_2, \infty))\), while for sufficiently large \( \beta \) there are two more in \((-\infty, -\nu)\); only for one value \( \beta^* \) of \( \beta \) there is a real double root. In the case of only four real roots (i.e. if \( \beta < \beta^* \)), there are two more non-real roots, which are conjugate complex and thus distinct. \( \square \)
Any downward jump of \( X \) can be thought of as the realization of a two-stage experiment: choose \( \xi_1 \) or \( \xi_2 \) with probability \( p \) or \( 1 - p \), respectively, and then take an \( \exp(\xi_1) \) or on \( \exp(\xi_2) \) random variable. An upward jump is composed of two independent phases, each being \( \exp(\nu) \)-distributed.

Now the process \( X \) can leave the interval \((0, x+y)\) in six different ways: (1) by an \( \exp(\xi_1) \) downward jump, (2) by an \( \exp(\xi_2) \) downward jump, (3) in the first phase of an upward jump, (4) in the second phase of an upward jump, (5) without jump, i.e. by the Brownian component, at 0, (6) without jump at \( x+y \). Define the random variable \( Z \) by setting \( Z = i \) if and only if case (i) occurs, \( 1 \leq i \leq 6 \). The LT \( \psi^y(\beta) = E(e^{-\beta X}) \) of \( X \) can be written as

\[
\psi^y(\beta) = \sum_{i=1}^{6} \psi^y_i(\beta), \text{ where } \psi^y_i(\beta) = E(e^{-\beta \xi_i 1_{Z=i}}).
\]

We now show how to compute \( \psi^y_i(\beta), 1 \leq i \leq 6 \). The basic observation is that \( X(\tau_y) \) and \( \tau_y \) are conditionally independent, given \( Z \); moreover, \(-X(\tau_y)\) is \( \exp(\xi) \)-distributed given \( Z = i, i = 1, 2 \), while \( X(\tau_y) \) is \( \exp(\nu) \)-distributed if \( Z = 3 \), and \( \exp(\nu) \ast \exp(\nu) \)-distributed if \( Z = 4 \). If \( Z = 5 \) or \( Z = 6 \), we have \( X(\tau_y) \equiv 0 \) or \( X(\tau_y) \equiv x+y \), respectively. Therefore, by using again the martingale equation \( E(M(\tau_y)) = E(M(0)) \), we obtain

\[
(\varphi(\alpha) - \beta) E \left( \int_0^{\tau_y} e^{-\alpha X(t) - \beta t} \, dt \right) = -e^{-\alpha x} + E(e^{-\alpha X(\tau_y) - \beta \tau_y})
\]

\[
= -e^{-\alpha x} + \frac{\xi_1}{\xi_1 - \alpha} \psi^y_1(\beta) + \frac{\xi_2}{\xi_2 - \alpha} \psi^y_2(\beta) + e^{-\alpha (x+y)} \frac{\nu}{\nu + \alpha} \psi^y_3(\beta)
\]

\[
+ e^{-\alpha (x+y)} \left( \frac{\nu}{\nu + \alpha} \right)^2 \psi^y_4(\beta) + \psi^y_5(\beta) + e^{-\alpha (x+y)} \psi^y_6(\beta).
\]  

(4.1)

Let \( \beta \neq \beta^* \). Then \( \varphi(\alpha) - \beta = 0 \) has six distinct roots \( \alpha_i(\beta), 1 \leq i \leq 6 \), and inserting them in (4.1) yields a system of six linear equations for the six unknowns \( \psi^y_i(\beta), 1 \leq j \leq 6 \). The closed-form solution of these equations is easily available, e.g. via MATHEMATICA, but not very illuminating.

If \( \beta = \beta^* \), we may of course take limits to obtain \( \psi^y_i(\beta^*) = \lim_{\beta \neq \beta^* \rightarrow \beta^*} \psi^y_i(\beta) \). But we can also proceed as follows. Let \( \alpha_1(\beta^*) \) be the double root of \( \varphi(\alpha) - \beta^* = 0 \). Taking the derivative with respect to \( \alpha \) in the first equation of (4.1) and inserting \( \alpha = \alpha_1(\beta^*) \) we find that the left-hand side is equal to 0, while the right-hand side is given by

\[
x e^{-\alpha_1(\beta^*) x} = E(X(\tau_y) e^{-\alpha_1(\beta^*) X(\tau_y) - \beta \tau_y}).
\]

Again, \( X(\tau_y) \) and \( \tau_y \) are conditionally independent given \( Z = i \), so that

\[
x e^{-\alpha_1(\beta^*) x} = E(X(\tau_y) e^{-\alpha_1(\beta^*) X(\tau_y) - \beta \tau_y}) = \sum_{i=1}^{6} E(X(\tau_y) e^{-\alpha_1(\beta^*) X(\tau_y)} | Z = i) \psi^y_i(\beta^*).
\]  

(4.2)

The conditional distribution of \( X(\tau_y) \), given \( Z = i \), was given above for \( i = 1, \ldots, 6 \); it is either a point mass or exponential or Erlang. Thus, (4.2) provides the “missing” sixth linear equation for the \( \psi^y_i(\beta^*) \).

The above technique can be applied in the case of general phase-type or hyperexponential jump size distributions (upwards and downwards). However, there may be more roots of higher multiplicity. Then one has to take higher derivatives to obtain a sufficiently large number of linear equations for the partial LTs.

5. The maximal value of the cash fund before bankruptcy

Let \( K = \sup \{ X(t) : 0 \leq t \leq T \} \) be the maximal amount of cash before bankruptcy. In this section we determine the joint distribution of \( K \) and \( X(T) \). Note first that the marginal distributions of \( K \) and \( X(T) \) are easily obtained from the results of Section 2.
Lemma 3.

\[ P(X(T) \leq u) = \begin{cases} \rho_2(0)e^{\xi u}, & \text{if } u \leq 0, \\ 1, & \text{if } u \geq 0, \end{cases} \]

\[ P(K \leq x + y) = \phi_1^x(0) + \phi_2^x(0), \quad y > 0. \]

Proof.

1. \(X(T)\) has an atom at 0; the event \(\{X(T) < 0\}\) occurs with probability \(\rho_2(0)\). Furthermore, by the memoryless property, \(X(T)\) and \(T\) are conditionally independent, given \(\{X(T) < 0\}\), and \(-X(T) \sim \exp(\xi)\).

2. The event \(\{K < x + y\}\) occurs if and only if \(\{X(t) \leq 0\}\). The probability of the latter event is \(\phi_1^x(0) + \phi_2^x(0)\), and \(P(K = x + y) = 0\).

To determine the joint distribution of \((K, X(T))\) is a more intricate problem. It turns out that it can be expressed in terms of the distribution of \(K\) and some exponential distributions. Define the Lévy process \(X_0(t) = B(t) + Y(t)\) (i.e. \(X_0 = X - x\)) and consider its supremum

\[ S = \sup_{0 \leq t < \infty} X_0(t), \]

the local time process

\[ L_0(t) = -\inf \{X_0(s) : s < t\}, \]

and the reflected process

\[ W(t) = X_0(t) + L_0(t). \]

Kella and Whitt (1992) have already shown (in a more general setting) that the process

\[ \varphi(\alpha) \int_0^t e^{-\alpha W(s)} ds + 1 - e^{-\alpha W(t)} - \alpha L_0(t), \quad (5.1) \]

is a martingale. Under the negative drift assumption \((\lambda/\nu) + \mu - (\eta/\xi) < 0\), the process \(W\) is regenerative. It is well-known that the distribution of \(S\) is the same as the limiting distribution of \(W(t)\), as \(t \to \infty\) (see e.g. Asmussen, 1987, Chapter III.7). We now use this fact, the limit theorem for regenerative processes and optional sampling for the martingale (5.1) to derive this distribution.

Lemma 4. For \(u > 0\), let \(E_1(u)\), \(E_2(u)\), \(E_3(u)\) be independent \(\exp(u)\)-distributed random variables which are independent of \(S\). Then

\[ S + E_1(\nu) \overset{D}{=} E_2(|\alpha_3(0)|) + E_3(|\alpha_4(0)|). \quad (5.2) \]

Proof. The sample paths of \(X_0\) are right-continuous with left-hand side limits, while those of \(L_0\) are left-continuous with right-hand side limits (and are non-decreasing). Let

\[ T_0 = \inf \{t > 0 : W(t) < 0\}. \]

Note that \(T_0\) is the time of the first negative jump that is also a negative record value of \(X_0(\cdot)\). \(T_0\) is a stopping time with respect to \(W\) and gives a regeneration point for \(W\). Note that

\[ W(T_0-) > 0 = W(T_0+) > W(T_0). \]
and by the memoryless property, \( L_0(T_0^+) - L_0(T_0) = -W(T_0) \), and \(-W(T_0)\) has an exponential distribution with mean \( 1/\xi \). Using the martingale (5.1) for \( T_0 \) we obtain
\[
\varphi(\alpha)E \left( \int_0^{T_0} e^{-\alpha W(s)} \, ds \right) = -1 + E(e^{-\alpha W(T_0)}) + \alpha E(L_0(T_0)). \tag{5.3}
\]
By Lemma 1, the function \( \varphi(\alpha) \) is the ratio of a polynomial of degree 4 and a quadratic function:
\[
\varphi(\alpha) = \frac{\alpha(\alpha - \alpha_1(0))}{(v + \alpha)(\xi - \alpha)} \frac{(v + \alpha)(\xi - \alpha)}{(\alpha - \alpha_1)(\alpha - \alpha_3)(\alpha - \alpha_4)}.
\]
(Recall that \( \alpha_1(0) > 0 = \alpha_2(0) > \alpha_3(0) > \alpha_4(0) \).) Let \( \alpha_i = \alpha_i(0), \ i = 1, 3, 4 \). By the memoryless property of the exponential distribution we have
\[
E(e^{-\alpha W(T_0)}) = \frac{\xi}{\xi - \alpha}.
\]
Setting \( \alpha = \alpha_1 \) in (5.3) we find that
\[
E(L_0(T_0)) = \frac{1}{\alpha_1 - \xi}.
\]
Hence,
\[
E \left( \int_0^{T_0} e^{-\alpha W(s)} \, ds \right) = \frac{\alpha - \alpha_1}{(\alpha_1 - \xi)(\xi - \alpha)} \frac{(v + \alpha)(\xi - \alpha)}{(\alpha - \alpha_1)(\alpha - \alpha_3)(\alpha - \alpha_4)}
\]
\[
= \frac{v + \alpha}{(\alpha_1 - \xi)(\alpha + |\alpha_3|)(\alpha + |\alpha_4|)}, \tag{5.4}
\]
and as \( \alpha \to 0 \) in (5.4) we obtain
\[
E(T_0) = \frac{v}{(\alpha_1 - \xi) |\alpha_3| |\alpha_4|}. \tag{5.5}
\]
Using \( T_0 \) and its replications as cycle lengths makes \( W \) a regenerative process so that \( W(t) \) converges weakly to some random variable as \( t \to \infty \), say \( W(\infty) \), having the stationary distribution of \( W \), so that \( W(\infty) \overset{D}{=} S \). By renewal theory,
\[
E(e^{-\alpha W(\infty)}) = \frac{E \left( \int_0^{T_0} e^{-\alpha W(s)} \, ds \right)}{E(T_0)}. \tag{5.6}
\]
Now insert (5.4) and (5.5) in (5.6), we get the LT of \( S \):
\[
E(e^{-\alpha S}) = E(e^{-\alpha W(\infty)}) = \frac{(v + \alpha) |\alpha_3| |\alpha_4|}{v(\alpha + |\alpha_3|)(\alpha + |\alpha_4|)}. \tag{5.7}
\]
Multiplying (5.7) by \( v/(v + \alpha) \) and inverting the resulting identity for LTs yields (5.2).
\( \Box \)

The main result of this section is the following.

**Theorem 3.** The joint distribution of \( K \) and \( X(T) \) is given by
\[
P(K \leq x + y, X(T) = 0) = \frac{P(S \leq y) - P(K \leq x + y)P(S \leq x + y)}{P(S \leq x + y) - P(S \leq x + y)}, \quad y > 0, \tag{5.8}
\]
\[
P(K \leq x + y, X(T) < -u) = e^{-ku} (P(K \leq x + y) - P(K \leq x + y, X(T) = 0)), \quad u > 0. \tag{5.9}
\]
where the distributions of $K$ and $S$ are given by Lemmas 3 and 4 and $S_1 = S - E_1(\xi)$, so that the LT of $S_1$ is

$$E(e^{-\alpha S_1}) = \frac{\vert \alpha \vert \vert \alpha_4 \vert (v + \alpha)(\xi - \alpha)}{v \xi (\alpha + \vert \alpha_3 \vert ) (\alpha + \vert \alpha_4 \vert )}.$$ 

Proof. We use the law of total probability and the strong Markov property for the Markov process $X = x + X_0$:

$$P(S \leq y) = P \left( \sup_{t \geq 0} X_0(t) \leq y \right) = P \left( \sup_{t \geq 0} X(t) \leq x + y \right)$$

$$= \int_{(-\infty,0]} P \left( \sup_{0 \leq t \leq T} X(t) \leq x + y, \sup_{T < t < \infty} X(t) \leq x + y \big| X(T) = u \right) P(X(T) \in du)$$

$$= \int_{(-\infty,0]} P \left( \sup_{0 \leq t \leq T} X(t) \leq x + y \big| X(T) = u \right) P \left( \sup_{T < t < \infty} X(t) \leq x + y \big| X(T) = u \right)$$

$$\times P(X(T) \in du). \quad (5.10)$$

Given that $X(T) < 0$, the overshoot $-X(T)$ below 0 is exp$(\xi)$-distributed and independent of $(X(t) | t < T)$. Thus, splitting the right-hand side of (5.11) into the integral over $(0, \infty)$ and the contribution at 0, we see that it is equal to

$$P(K \leq x + y | X(T) < 0) \int_0^\infty P \left( \sup_{t \geq 0} X_0(t) \leq x + y + v \big| X(T) < 0 \right) P(X(T) < 0) e^{-\xi v} dv$$

$$+ P(K \leq x + y | X(T) = 0) P \left( \sup_{t \geq 0} X_0(t) \leq x + y \big| X(T) = 0 \right) P(X(T) = 0).$$

It follows that

$$P(S \leq y) = P(K \leq x + y, X(T) < 0) P(S \leq x + y + E_1(\xi))$$

$$+ P(K \leq x + y, X(T) = 0) P(S \leq x + y). \quad (5.11)$$

and we have the obvious second equation

$$P(K \leq x + y) = P(K \leq x + y, X(T) < 0) + P(K \leq x + y, X(T) = 0). \quad (5.12)$$

The distributions of $K$ and of $S$ are given by Lemmas 3 and 4. Hence, from the two linear equations (5.11) and (5.12) we get $P(K \leq x + y, X(T) = 0)$ and $P(K \leq x + y, X(T) < 0)$. A short calculation shows (5.8). Since $K$ and $X(T)$ are conditionally independent, given $X(T) < 0$, and $-X(T)$ is conditionally exp$(\xi)$-distributed, we obtain

$$P(K \leq x + y, X(T) < -u) = e^{-\xi u} P(K \leq x + y, X(T) < 0)$$

$$= e^{-\xi u} \left[ P(K \leq x + y) - P(K \leq x + y, X(T) = 0) \right]$$

for all $u > 0$, yielding (5.9). \qed

Appendix A

A closed-form solution of (2.2) can be found by using MATHEMATICA. Write $A(\beta)$ as
Then $\Delta = \det A(\beta)$ is given by

$$\Delta = b_3 c_2 d_1 - b_4 c_2 d_1 - b_2 c_3 d_1 + b_4 c_3 d_1 + b_4 c_4 d_1 - b_3 c_1 d_2 + b_4 c_1 d_2 + b_1 c_3 d_2$$

$$-b_4 c_3 d_2 - b_1 c_4 d_2 + b_3 c_4 d_2 + b_2 c_1 d_3 - b_4 c_1 d_3 - b_1 c_2 d_3 + b_4 c_2 d_3 + b_1 c_4 d_3 - b_2 c_4 d_3$$

$$-b_2 c_1 d_4 + b_3 c_1 d_4 + b_1 c_2 d_4 - b_3 c_2 d_4 - b_1 c_3 d_4 + b_2 c_3 d_4.$$  

we find that

$$\phi_1^\gamma (\beta) = \Delta^{-1} \left( e^{-a_1(\beta)} [-b_4 c_3 d_2 + b_3 c_4 d_2 + b_4 c_2 d_3 - b_2 c_4 d_3 - b_3 c_2 d_3 + b_2 c_3 d_4]$$

$$+ e^{-a_2(\beta)} [b_4 c_3 d_1 - b_3 c_4 d_1 - b_4 c_1 d_3 + b_3 c_1 d_3 + b_3 c_1 d_4 - b_1 c_3 d_4]$$

$$+ e^{-a_3(\beta)} [-b_4 c_2 d_1 + b_2 c_4 d_1 + b_4 c_1 d_2 - b_1 c_4 d_2 - b_2 c_4 d_4 + b_1 c_2 d_4]$$

$$+ e^{-a_4(\beta)} [b_3 c_2 d_1 - b_2 c_3 d_1 - b_3 c_1 d_2 + b_1 c_2 d_3 + b_2 c_1 d_3 - b_1 c_2 d_3] \right),$$

$$\phi_2^\gamma (\beta) = \Delta^{-1} \left( e^{-a_1(\beta)} [c_3 d_2 - c_4 d_2 - c_2 d_3 + c_4 d_3 + c_2 d_4 - c_3 d_4]$$

$$+ e^{-a_2(\beta)} [-c_3 d_1 + c_4 d_1 + c_1 d_3 - c_4 d_3 - c_1 d_4 + c_3 d_4]$$

$$+ e^{-a_3(\beta)} [c_2 d_1 - c_4 d_1 - c_1 d_2 + c_4 d_2 + c_1 d_4 - c_2 d_4]$$

$$+ e^{-a_4(\beta)} [-c_2 d_1 + c_3 d_1 + c_1 d_2 - c_3 d_2 - c_1 d_3 + c_2 d_3] \right),$$

$$\phi_3^\gamma (\beta) = \Delta^{-1} \left( e^{-a_1(\beta)} [-c_3 d_2 + b_4 d_2 + b_2 d_3 - b_4 d_3 - b_2 d_4 + b_3 d_4]$$

$$+ e^{-a_2(\beta)} [b_3 d_1 - b_4 d_1 - b_1 d_3 + b_4 d_3 + b_1 d_4 - b_3 d_4]$$

$$+ e^{-a_3(\beta)} [-b_2 d_1 + b_4 d_1 + b_1 d_2 - b_4 d_2 - b_1 d_4 + b_2 d_4]$$

$$+ e^{-a_4(\beta)} [b_2 d_1 - b_3 d_1 - b_1 d_2 + b_3 d_2 + b_1 d_3 - b_2 d_3] \right),$$

$$\phi_4^\gamma (\beta) = \Delta^{-1} \left( e^{-a_1(\beta)} [b_3 c_2 - b_4 c_2 - b_2 c_3 + b_4 c_3 + b_2 c_4 - b_3 c_4]$$

$$+ e^{-a_2(\beta)} [-b_3 c_1 + b_4 c_1 + b_1 c_3 - b_4 c_3 - b_1 c_4 + b_3 c_4]$$

$$+ e^{-a_3(\beta)} [b_2 c_1 - b_4 c_1 - b_1 c_2 + b_4 c_2 + b_1 c_4 - b_2 c_4]$$

$$+ e^{-a_4(\beta)} [-b_2 c_1 + b_3 c_1 + b_1 c_2 - b_3 c_2 - b_1 c_3 + b_2 c_3] \right).$$

References


Stidham, S., 1986. Clearing systems and (s, S) inventory systems with nonlinear costs and positive lead times. Operations Research 34, 276–280.
