Ruin probabilities based at claim instants for some non-Poisson claim processes

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Received 2 October 1998; received in revised form 2 March 1999; accepted 3 May 1999

Abstract

The paper presents a recursive method of calculating ruin probabilities for non-Poisson claim processes, by looking at the surplus process embedded at claim instants. The developed method is exact. The processes considered have both claim sizes and the inter-claim revenue following selected phase type distributions. The numerical section contains figures derived from the exact approach, as well as a tabular example using the numerical approach of De Vylder and Goovaerts. The application of the method derived in the paper through numerical examples reveals the sensitivity of the value of probability of ruin to changes in claim number process. © 2000 Published by Elsevier Science B.V. All rights reserved.

JEL classifications: C690; C220

Keywords: Ruin probability; Recursive methods; Phase-type distributions; Non-Poisson risk processes

1. Introduction

This article is an extension of the results presented in Stanford and Stroiński (1994), dealing with the Sparre–Andersen risk process as defined, for instance, by Bowers et al. (1986). The process can be described as follows. The initial surplus is $u$, and premiums are earned at a constant rate. The inter-claim revenues (namely, the incomes received between successive claims) form a sequence of i.i.d. random variables (RVs) with a common distribution $A(x)$. Claim sizes are assumed to be non-negative i.i.d. RVs with a common distribution $B(x)$.

The current extension to the earlier work considers the effect of some non-Poisson claim processes on the probability of ruin. The majority of previous work on ruin probabilities has assumed that claims occur according to a Poisson process. A notable departure is Asmussen et al. (1995) where other generalisations are referred to.

Since ruin occurs upon payment of claims, our approach here is to observe the surplus process embedded at claim instants only. We exploited this approach previously to obtain exact recursive formulae for the probability of ruin on

\textsuperscript{a} The graphs were prepared with the assistance of the late Lamia Abid. We are deeply moved and saddened by her untimely death, and she is remembered with great fondness. We dedicate this work to her memory.

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PII: S0167-6687(99)00047-5
a specified claim, in the case of a Poisson claim number process. Apart from being an exact method for calculating ruin probabilities, this approach has the advantage of revealing, in some sense, where the relative vulnerability to ruin lies over the duration of the process. In the limit, the recursive method produces ultimate probabilities of ruin, although other direct methods such as those of Asmussen (1992) and Asmussen and Rolski (1991) are better suited to that task. Finally, it is our hope that by concentrating on the risk process at claim instants, future research into related quantities such as the time and severity of ruin may benefit from insights gained from this new approach.

Caution has to be exercised when relating probabilities of ruin based at claim instants with classical finite-time ruin probabilities, as the former reflect in a different way the variation in the claim number process. There is no simple translation from the embedded probabilities to finite-time ruin probabilities, or vice versa. One might initially think of the \( n \)th claim instant as being a random quantity equal to the independent sum of \( n \) i.i.d. inter-claim times. However, ruin is more likely to occur if the inter-claim times are short, so the time of the \( n \)th claim and the likelihood of ruin on it are linked. Nonetheless, the embedded approach does afford an alternative view of the risk process.

Most efforts to date at determining the finite-time ruin probability (or its complement, the non-ruin probability) in the Poisson case can be classified into one of three groups. The solution to the integro-differential equation for the non-ruin probability (see Gerber (1979)) is typically stated in terms of Laplace transforms. Direct attempts at inverting the transform to obtain the probability of ruin have often encountered numerical difficulties (see, for instance, Janssen and Delfosse (1982) and Taylor (1978)). A second approach, followed by Garrido (1988) and others, is to employ diffusion approximations which retain the continuous-time aspect of the system under study and use an appropriate diffusion process to approximate the actual claims process. A third option, exploited by Dickson and Waters (1991) for instance, is to approximate the continuous-time process by one in discrete time.

The exact approach discussed here retains key aspects of both of these latter two approaches. The continuous-time process is retained, although ours is embedded at claim instants. In situations where it works in its entirety, there is no need to either discretise time or the claim size distribution. Thus questions on the accuracy of the discretisation and the size of the approximating discrete distribution are avoided. Similar to discrete time methods, integrals are replaced by recursive summations, however these are exact. Two of the three recursions for the coefficients in the predecessor to this paper involved only positive terms, with most if not all being between 0 and 1, and such recursions are known to be very accurate (see Grassmann (1990, p. 209)). It therefore is relevant to know how general a model can be accommodated to this approach, and conversely, what its limitations are.

The actual methodology consists of two parts: first, it is necessary to establish the basic recursion which indicates how the probability of ruin on the \( n \)th claim is related to the probability of ruin on the \((n-1)\)th claim. This recursion involves expressions indicating the level of the surplus process on a specified claim number when ruin does not occur. Second, an algorithm entailing Laplace transforms is given, and mathematical induction is used to establish a particular form for the probability of ruin on the \( n \)th claim. In the current work we address two questions relating to the utility of the method for non-Poisson claim processes. First, we identify one type of non-Poisson behaviour where both steps of the methodology extend readily to provide a stable, exact algorithm for the probability of ruin. Secondly, in circumstances where the algorithm is complicated beyond a reasonable degree, we illustrate a discrete time approximation that can be adapted to the basic underlying recursion from the first step of the process.

An abbreviated development of the underlying theory is presented here. The interested reader is directed to Stanford and Stroiński (1994) for an expanded treatment. The introduction to that paper provided an overview of various methods for determining the probability of ruin, and the review paper by Taylor (1985) provides numerous references to work prior to that date. For developments since that time, we suggest the following papers which provide a flavour of the diversity of methods available: Gerber et al. (1987), De Vylder and Goovaerts (1988), Shiu (1988), Garrido (1988), Dickson and Waters (1991), and Asmussen and Rolski (1991).

Following preliminary developments relating to phase-type distributions in Section 2, the algebraic results are presented in Section 3. In Section 4 a numerical algorithm which uses the De Vylder–Goovaerts method is presented. In Section 5 a number of examples which are based on the methods from Section 3 are discussed, as well as one example of the numerical approach.
2. Preliminaries

2.1. Phase-type distributions

Phase-type distributions have been used extensively in queueing theory and more recently in ruin theory to model continuous random variables (see, for instance, Asmussen and Rolski (1991)). The extensions to Stanford and Stroiński (1994) for the claim number process that we discuss in this paper belong to two subsets of phase-type distributions that are popular for modelling purposes: the Erlang distribution and mixtures of exponential distributions. The phase-type formulation of these is presented below. Neuts (1981) provides a thorough treatment of phase-type distributions.

Essentially, the phase-type family includes all distributions which can be modelled as the time to absorption in a continuous-time Markov chain with a single absorbing state. The resulting distribution function for the time to absorption has the form

\[ B(x) = 1 - \gamma, \exp(Tx)e, \quad x \geq 0, \]

where \( \gamma \) is the row vector of probabilities of starting in the various transient states, \( e \) a column vector of ones of appropriate size, and \( T \) is the matrix of transition rates among the various transient states. The corresponding density function is \( b(x) = \gamma \exp(Tx)t_0, \quad x \geq 0 \), where \( t_0 = -Te \). Its Laplace–Stieltjes transform (LST), which is denoted by \( \Phi_B(s) \), is \( \Phi_B(s) = E[e^{-sX}] = \gamma(sI - T)^{-1}t_0 \). We consider below two well known subsets

2.1.1. Mixtures of exponentials

Such a mixture can be used as a simple model of distributions that are more variable than the exponential in terms of the squared coefficient of variation (denoted SCV), defined as \( \text{SCV} = \frac{\text{Var}[X]}{\left(E[X]\right)^2} \). A mixture of exponentials has an SCV greater than or equal to 1. The phase-type formulation of a mixture of \( M \) exponentials has the following form:

\[ \gamma = [r_1, r_2, \ldots, r_M], \quad T = -\text{diag}[\theta_1, \theta_2, \ldots, \theta_M], \quad t_0 = [\theta_1, \theta_2, \ldots, \theta_M]^T, \]

\[ b(x) = \sum_{i=1}^{M} r_i \theta_i e^{-\theta_i x}, \quad x \geq 0, \quad E[X] = \sum_{i=1}^{M} \frac{r_i}{\theta_i}, \quad \Phi_B(s) = \sum_{i=1}^{M} \frac{r_i \theta_i}{(\theta_i + s)}. \]

We will use the mixture of exponentials to model inter-claim revenues that are more variable than the exponential.

2.1.2. Erlang-\( \alpha \)

The Erlang-\( \alpha \) distribution is the subset of gamma distributions with integer-valued shape parameter \( \alpha \). It can be viewed as an \( \alpha \)-fold convolution of the exponential distribution. It is often used as a simple model of distributions which are less variable than the exponential, since its SCV is given by \( \frac{1}{\alpha} \). We will use it to model inter-claim revenues that are less variable than the exponential.

The phase-type representation of the Erlang-\( \alpha \) can be visualised as a movement through \( \alpha \) states prior to absorption. Thus,

\[ \gamma = [1, 0, \ldots, 0], \quad T = \begin{bmatrix} -\theta & \theta & 0 \\ -\theta & \theta & 0 \\ 0 & \ddots & -\theta \end{bmatrix}, \quad t_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \]

Also,

\[ b(x) = \frac{\theta^\alpha x^{\alpha-1}}{(\alpha - 1)!} e^{-\theta x}, \quad x \geq 0, \quad E[X] = \alpha/\theta, \quad \Phi_B(s) = \left(\frac{\theta}{\theta + s}\right)^\alpha. \]
2.2. Notation and general formulae

We denote the distribution function of inter-claim revenue by $A(y)$, its density function by $a(y)$ and its LST by

$$
\Phi_A(s) = \int_0^\infty e^{-sy} dA(y).
$$

We define $B(y)$, $b(y)$, and $\Phi_B(s)$ as equivalent attributes of the claim size distribution. Let $p_n(y)$ be defined as the incomplete density for the surplus after the $n$th claim. (It is of course incomplete because ruin may have already occurred. Integration of $p_n(y)$ for all non-negative $y$ yields the probability of not being ruined by any of the first $n$ claims.) Also, let the Laplace transform of $p_n(y)$ be written as

$$
\mathcal{L}_n(s) = \int_0^\infty e^{-sy} p_n(y) \, dy. \quad (2.1)
$$

Evaluating (2.1) at $s = 0$ yields the probability of non-ruin up to the $n$th claim. Below, we develop a recursion between $\mathcal{L}_n(s)$ and $\mathcal{L}_{n-1}(s)$ which will enable us to determine the probability of ruin on the $n$th claim.

The increment between two consecutive claims is defined as the difference between the revenue earned and the subsequent claim amount. One can write the density function of the increment as

$$
g(y) = \begin{cases} 
\int_0^\infty a(y + t)b(t) \, dt, & y \geq 0, \\
\int_0^\infty a(t)b(t - y) \, dt, & y < 0,
\end{cases} \quad (2.2)
$$

and its Laplace transform is

$$
G(s) = \int_{-\infty}^\infty e^{-sy} g(y) \, dy = \Phi_A(s)\Phi_B(-s). \quad (2.3)
$$

Let $F(y) = \int_y^\infty g(x) \, dx$ denote the cumulative distribution function of the increment whose density function is defined by Eq. (2.2). The probability of ruin on the first claim is $F(-u)$, that is the probability that the first increment is so negative that it uses up the whole initial reserve.

Our departure point for the current work is the recursive formula (2.7) from Stanford and Stroiński (1994), which is equally valid here

$$
p_n(y) = \int_0^\infty p_{n-1}(x)g(y-x) \, dx, \quad y \geq 0. \quad (2.4)
$$

In words, this convolution says that the level of the surplus at the $n$th claim instant, assuming that ruin does not occur, is the sum of the level at the $(n - 1)$th claim instant plus the ensuing increment. When one takes Laplace transforms of (2.4) one obtains

$$
\mathcal{L}_n(s) = \mathcal{L}_{n-1}(s)G(s) - \int_{x=0}^\infty e^{-sx} p_{n-1}(x) \int_{y=x}^\infty e^{sy} g(-y) \, dy \, dx. \quad (2.5)
$$

Evaluating (2.5) at $s = 0$, we find the probability of ruin on the $n$th claim is

$$
P(n) = \text{Pr}[\text{ruin on the } n\text{th claim}] = \int_{x=0}^\infty p_{n-1}(x) \int_{y=x}^\infty g(-y) \, dy \, dx. \quad (2.6)
$$

Formulae (2.5) and (2.6) apply to arbitrary Sparre–Andersen claim number processes and (non-negative) claim sizes. Further development is presented in Section 3 for the two sets of non-Poisson claim number processes mentioned above: the Erlang-$\alpha$ distribution and mixtures of exponentials. The De Vylder–Goovaerts approach with a wider range of application is introduced in Section 4.
3. Algebraic formulae for selected non-Poisson claims processes

In this section, two simple extensions of the Poisson claim number process are considered. The Poisson process has exponential inter-event times whose variance equals the square of the mean. The first extension we consider is the Erlang-\(\alpha\) considered in Section 2 as an example of inter-claim times which are less variable than the exponential. The latter case considered is that of a mixture of exponentials, in order to consider a simple example of inter-claim times that are more variable than the exponential distribution. For the former, we will see that the methods of Stanford and Stroiński (1994) extend readily with a minor increase in complexity over the Poisson case. For the latter, we find that the extension becomes significantly more complicated even in its simplest case, although it is still tractable. In Section 4, we show that a pairing of our underlying recursion and the De Vylder–Goovaerts method can be used in these situations.

In each case considered, we first determine a difference equation involving the Laplace transforms \(L_n(s)\), and then obtain an explicit expression for \(L_n(s)\) via mathematical induction. Finally, corresponding expressions for the probability of ruin on the \(n\)th claim are found.

3.1. Recursions for Erlang-\(\alpha\) inter-claim revenues

When the inter-claim revenue follows an Erlang-\(\alpha\) distribution and the claim sizes have a mixture of \(N\) exponential distributions, a set of general formulae for \(L_n(s)\) and \(P(n)\) can be determined as follows.

**Theorem 3.1.** For Erlang-\(\alpha\) distributed inter-claim revenue and a mixture of \(N\) exponentials claim sizes, \(L_n(s), n \geq 1\), is given by

\[
L_n(s) = \sum_{i=1}^{N} \frac{p_i \mu_j}{\mu_i - s} \left[ \left( \frac{\lambda}{\lambda + s} \right)^\alpha L_{n-1}(s) - \left( \frac{\lambda}{\lambda + \mu_i} \right)^\alpha L_{n-1}(\mu_i) \right],
\]

(3.1)

and

\[
P(n) = \sum_{i=1}^{N} p_i \left( \frac{\lambda}{\lambda + \mu_i} \right)^\alpha L_{n-1}(\mu_i).
\]

**Proof.** Under the stated assumptions

\[
g(y) = \sum_{i=1}^{N} p_i \left( \frac{\lambda}{\lambda + \mu_i} \right)^\alpha \mu_i e^{\lambda y}, \quad y < 0, \quad G(s) = \left( \frac{\lambda}{\lambda + s} \right)^\alpha \sum_{i=1}^{N} \frac{p_i \mu_i}{\mu_i - s}.
\]

After substituting these expressions into (2.5) and after some manipulation one obtains the result (3.1). \(\square\)

**Theorem 3.2.** The general form of \(L_n(s), n \geq 1\), when the inter-claim revenue distribution is Erlangian-\(\alpha\)-distributed and the claim sizes follow a mixture of two exponential distributions, is given by

\[
L_n(s) = \sum_{j=1}^{N} \frac{\lambda}{\lambda + s}^j + \left( \frac{\lambda}{\lambda + s} \right)^{na} A^{(n)}(s),
\]

(3.2)

where

\[
A^{(n)}(s) = \sum_{k=n}^{\infty} \frac{\mu^{k-n}}{k} \sum_{m=1}^{2} P_m \left[ D_m^{(n)} (\mu_m - s)^{k-n} + \sum_{j=0}^{n-2} \sum_{\ell=0}^{k-n} \binom{k-n}{j} \binom{k-n-\ell}{j} (\mu_m - s)^\ell (\mu_m - \mu_{3-m})^{k-n-\ell} \right],
\]

(3.3)
and where the coefficients \( c_j^{(n)} \), \( D_m^{(n)} \) and \( f_{jm}^{(n)} \), \( n \geq 2 \), satisfy the recursions

\[
c_j^{(n)} = \sum_{i=1}^{2} \frac{p_i \mu_i}{\lambda + \mu_i} \left\{ \sum_{k=\max(1,j-\alpha)}^{(n-1)\alpha} c_k^{(n-1)} \left( \frac{\lambda}{\lambda + \mu_i} \right)^{k+j} + \left( \frac{\lambda}{\lambda + \mu_i} \right)^{j+\alpha-j} A_i^{(n-1)} \right\}, \tag{3.3}
\]

\[
D_m^{(n)} = D_m^{(n-1)} h_m, \tag{3.4}
\]

\[
f_{jm}^{(n)} = \begin{cases} h_m f_{jm}^{(n-1)} + h_{3-m} D_m^{(n-1)}, & j = 0, \\ h_m f_{jm}^{(n-1)} + h_{3-m} f_{j-1,m}^{(n-1)}, & j = 1, 2, \ldots, n-3, \\ h_{3-m} f_{j-3,m}^{(n-1)}, & j = n-2, \end{cases} \tag{3.5}
\]

where \( h_m = p_m \alpha \mu_m \) and \( A_i^{(n-1)} \equiv A_i^{(n-1)}(\mu_i) \).

The coefficients are initialised as follows:

\[
c_j^{(1)} = \sum_{i=1}^{2} \frac{p_i \mu_i}{\lambda + \mu_i} \left( \frac{\lambda}{\lambda + \mu_i} \right)^{\alpha-j} e^{-u \mu_i}, \quad j = 1, \ldots, \alpha,
\]

\[
D_i^{(1)} = e^{-u \mu_i}(u \mu_i), \quad i = 1, 2 \quad f_{0m}^{(2)} = h_{3-m} D_m^{(1)}, \quad m = 1, 2.
\]

**Proof.** The proof uses mathematical induction. When \( n = 1 \), (3.1) already satisfies the general form of \( L_n(s) \) in the theorem. We establish the remainder of the proof through the addition and subtraction of \( A_i^{(n-1)}(\lambda/(\lambda + s))^{\alpha} \) to facilitate the factoring of the root \((\mu_i - s)\). In particular, we obtain

\[
\left( \frac{\lambda}{\lambda + s} \right)^{\alpha} L_{n-1}(s) - \left( \frac{\lambda}{\lambda + \mu_i} \right)^{\alpha} L_{n-1}(\mu_i)
\]

\[
= \left( \frac{\mu_i - s}{\lambda + \mu_i} \right)^{\alpha} \sum_{j=1}^{j+\alpha-1} \sum_{l=0}^{(n-1)\alpha} C_j^{(n-1)} \left( \frac{\lambda}{\lambda + s} \right)^{l+1} \left( \frac{\lambda}{\lambda + \mu_i} \right)^{j+\alpha-l-1}
\]

\[
+ A_i^{(n-1)} \sum_{l=0}^{(n-1)\alpha} \left( \frac{\lambda}{\lambda + s} \right)^{l+1} \left( \frac{\lambda}{\lambda + \mu_i} \right)^{\alpha-l-1} + \left( \frac{\lambda}{\lambda + \mu_i} \right)^{\alpha} [A_i^{(n-1)}(s) - A_i^{(n-1)}].
\]

After factoring the term \((\mu_i - s)\) from the first bracketed term of the last expression, and after reversing the order of summation, it is readily seen that the coefficients \( c_j^{(n)} \) are related to the previous coefficients by (3.3). To complete the proof, we need to show that

\[
A^{(n)}(s) = \sum_{i=1}^{2} \frac{p_i \mu_i}{\mu_i - s} [A^{(n-1)}(s) - A^{(n-1)}].
\]

We observe that
\[ A^{(n-1)}(s) - A_i^{(n-1)} = (\mu_i - s) \sum_{k=n}^{\infty} \frac{u^{k+1-n}}{k!} \times \left\{ \sum_{m=1}^{2} \left[ \begin{array}{c} p_m \left( D_m^{(n-1)} \sum_{p=0}^{k-n} (\mu_m - \mu_l)^{k-n-p} (\mu_m - \mu_l)^{k-n-l} + \sum_{j=0}^{n-3} \sum_{l=0}^{k-n} \binom{k-n-l+j}{j} \right) \\
\sum_{p=0}^{l} (\mu_m - s)^p (\mu_m - \mu_l)^{l-p} (\mu_m - \mu_3)^{k-n-l} \right] \right\}. \]

After expanding this about indices \( c_n \), \( c_{n_i} \), and \( c_{n_i} \), the algorithm is initialised by setting 

\[ \mu_i/s, \]

we obtain

\[ \sum_{m=1}^{2} \left[ p_m \left( D_m^{(n-1)} \sum_{p=0}^{k-n} (\mu_m - \mu_l)^{k-n-p} (\mu_m - \mu_l)^{k-n-l} + \sum_{j=0}^{n-3} \sum_{l=0}^{k-n} \binom{k-n-l+j}{j} \right) \right\}. \]

The term multiplying \( (k - n - l + j) \) is \( f_{ji}^{(n)} \) from (3.5). Therefore the final expression can be recognised as \( A^{(n)}(s) \), which completes the proof.

For exponential claim sizes we can state the following algorithm. We state this result without proof as the derivation is straightforward but tedious.

\[ \mathcal{L}_\alpha(s) = e^{-\alpha s} \left[ \left( \frac{\mu}{\lambda + s} \right) \Phi_A(\mu) \sum_{j=0}^{n\alpha-1} c_j^{(n)} \left( \frac{\lambda + \mu}{\lambda + s} \right)^{j} + \Phi_A(s) \alpha \sum_{k=n}^{\infty} \frac{u(\mu - s)^{k-n}}{k!} \right], \]

where \( \Phi_A(s) = (\lambda/(\lambda + s))^\alpha \) and

\[ c_j^{(n)} = \begin{cases} 
  \frac{(\mu \lambda)^{n-1}}{(n-1)!} + \frac{\mu}{\lambda + \mu} \sum_{l=0}^{n-1} c_l^{(n-1)}, & 0 \leq j \leq \alpha - 1, \\
  \frac{(\mu \lambda)^{n-1}}{(n-1)!} + \frac{\mu}{\lambda + \mu} \sum_{l=\alpha}^{n-1} c_l^{(n-1)}, & \alpha \leq j \leq n\alpha - 1.
\end{cases} \] (3.6)

The algorithm is initialised by setting \( c_j^{(1)} = 1, j = 0, \ldots, \alpha - 1 \).

In summary, the algorithms for the computation of the coefficients \( c_j^{(n)} \) in the case of Erlang-\( \alpha \) revenues are only slightly more complex than their Poisson counterparts. We note in particular that with the exception of the \( A^{(n)}(\mu_i) \) term the algorithm consists of multiplications and additions of purely positive coefficients between 0 and 1. Numerical examples based on this algorithm are presented in Section 5.
3.2. Recursions for mixtures of exponentially distributed inter-claim revenues

**Theorem 3.3.** For inter-claim revenue having a mixture of $M$ exponential distributions with weights $r_1, \ldots, r_M$ and rates $\lambda_1, \ldots, \lambda_M$, and claim sizes having a phase-type distribution with representation $(\gamma, T)$, where $t_0 = -\mathcal{T}e$, $\mathcal{L}_n(s)$, $n \geq 1$, is given by

$$
\mathcal{L}_n(s) = \sum_{i=1}^{M} r_i \int_{x=0}^{\infty} p_{n-1}(x) \exp(Tx) \, dx - \mathfrak{y} \left( \frac{\lambda_i}{\lambda_i + s} \right) \mathcal{L}_{n-1}(s) \left( sI + T \right)^{-1} t_0,
$$

where $v_i = \lambda_i \mathfrak{y} (\lambda_i I - T)^{-1}$.

**Proof.** For the stated inter-claim revenue and claim size distributions, the density of the increment becomes

$$
g(y) = \begin{cases} 
\sum_{i=1}^{M} r_i \lambda_i e^{-\lambda_i y} \Phi_B(\lambda_i), & y \geq 0, \\
\sum_{i=1}^{M} r_i \lambda_i \int_{0}^{\infty} e^{-\lambda_i t} b(t - y) \, dt, & y < 0.
\end{cases}
$$

Substituting the expressions for phase-distributed claim sizes, and letting $v_i = \lambda_i \mathfrak{y} (\lambda_i I - T)^{-1}$, the following is obtained:

$$
g(y) = \begin{cases} 
\sum_{i=1}^{M} r_i v_i e^{-\lambda_i y} t_0, & y \geq 0, \\
\sum_{i=1}^{M} r_i v_i \exp(- Ty) t_0, & y < 0.
\end{cases}
$$

Therefore

$$
\int_{y=x}^{\infty} e^{s y} g(-y) \, dy = \sum_{i=1}^{M} r_i v_i \int_{y=x}^{\infty} e^{s y} \exp(T y) \, dy \, t_0 \\
= \sum_{i=1}^{M} r_i v_i \int_{y=x}^{\infty} \exp((s I + T) y) \, dy \, t_0 = -e^{sx} \sum_{i=1}^{M} r_i v_i \exp(Tx)(sI + T)^{-1} t_0.
$$

Substitution of (3.9) into the second term of (2.5) completes the proof. Evaluating (3.7) at $s = 0$, and recalling Eq. (2.10) of Stanford and Stroiński (1994) one obtains the probability of ruin on the $n$th claim.

$$
P(n) = \sum_{i=1}^{M} r_i v_i \int_{x=0}^{\infty} p_{n-1}(x) \exp(Tx) \, dx \, e.
$$

**Theorem 3.4.** The following general expressions for $P(n)$ and $\mathcal{L}_n(s)$ are obtained for mixtures of $M$ exponential inter-claim revenues and Erlang-\(a\) claim sizes:

$$
P(n) = \sum_{i=1}^{M} \sum_{k=0}^{a-1} r_i \mathcal{L}_{n-1}^{(k)}(\mu) \left\{ \frac{(-\mu)^k}{k!} \left[ 1 - \left( \frac{\mu}{\lambda_i + \mu} \right)^{a-k} \right] \right\},
$$

where $v_i = \lambda_i \mathfrak{y} (\lambda_i I - T)^{-1}$. □
\[
\mathcal{L}_n(s) = \sum_{i=1}^{M} r_i \left[ \mathcal{L}_{n-1}(s) \left( \frac{\lambda_i}{\lambda_i + s} \right), 0, \ldots, 0 \right] - \left[ \left( \frac{\lambda_i}{\lambda_i + \mu} \right) \mathcal{L}_{n-1}(\mu), \left( \frac{\lambda_i}{\lambda_i + \mu} \right) (-\mu) \mathcal{L}'_{n-1}(\mu) + \left( \frac{\lambda_i}{\lambda_i + \mu} \right) \left( \frac{\mu}{\lambda_i + \mu} \right) \mathcal{L}_{n-1}(\mu), \ldots \right].
\]

**Proof.** The proof is nearly identical to that presented in Section 2.3 of Stanford and Stroiński (1994), and is contained in Lee et al. (1994).

Since (3.12) involves the term \((\mu/(\mu - s))^{\alpha}\) it stands to reason that \((\mu - s)^{\alpha}\) must be factored from the bracketed terms so that these factors will cancel when finding \(\mathcal{L}_n(s)\). Therefore we hypothesise the following general form for \(\mathcal{L}_n(s)\) when claim sizes follow an Erlang-\(\alpha\) distribution:

\[
\mathcal{L}_n(s) = e^{-\alpha \mu} \sum_{j=1}^{n} \sum_{l=0}^{j} c_{jl}^{(n)} \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right)^l \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right)^{j-l} + \Phi_A(s)^n (\alpha \mu)^{\alpha n} \sum_{k=\alpha n}^{\infty} \frac{(\mu - s)^{k-\alpha n}}{k!}.
\]

The proof of the correctness of this hypothesis when \(\alpha = 1\), together with the recursion for the coefficients, is given in the following theorem. (The interested reader is directed to Lee et al. (1994) for the derivations and proofs for a comparable form when \(\alpha = 1\) and \(\alpha = 2\).)

**Theorem 3.5 (\(\alpha = 1\): Exponential claim sizes).** When the inter-claim revenue follows a mixture of two exponential distributions, and for exponentially distributed claim sizes, the Laplace transform of the distribution of the reserve following the \(n\)th claim, \(\mathcal{L}_n(s), n \geq 1\), is given by (3.13) where the coefficients \(c_{jl}^{(n)}\) at the \(n\)th stage are related to those at the \((n - 1)\)th stage by the following expressions for \(j = 1, 2, \ldots, n:\n
\[
c_{jl}^{(n)} = \begin{cases} 
\frac{(u \mu)^{n-1}}{(n-1)!} \Phi_A(\mu)^{n-j} + c_{j-1,0}^{(n-1)} \\
+ \Phi_A(\mu) \sum_{p=0}^{n-j} \sum_{q=0}^{p} c_{p,q}^{(n-1)} \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right)^q \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right)^{j-1-q} \\
\frac{(u \mu)^{n-1}}{(n-1)!} \Phi_A(\mu)^{n-j} \left[ \frac{\mu}{\lambda_1 + \mu} \left( \frac{j-1}{l-1} \right) + \frac{\mu}{\lambda_2 + \mu} \left( \frac{j-1}{l} \right) \right] \\
+ \frac{\mu}{\lambda_1 + \mu} c_{j-1,l-1}^{(n-1)} + \frac{\mu}{\lambda_2 + \mu} c_{j-1,l}^{(n-1)} + \frac{\mu}{\lambda_1 + \mu} \Phi_A(\mu) \sum_{p=0}^{n-j-1} c_{p,j-1,0}^{(n-1)} \left( \frac{r_1 \lambda_1}{\lambda_1 + \mu} \right)^p \\
\frac{(u \mu)^{n-1}}{(n-1)!} \Phi_A(\mu)^{n-j} + c_{j-1,l-1}^{(n-1)} + \Phi_A(\mu) \sum_{p=0}^{n-j-1} c_{p,j-1,0}^{(n-1)} \left( \frac{r_1 \lambda_1}{\lambda_1 + \mu} \right)^p \end{cases}, \quad l = 0,
\]

\[
\frac{(u \mu)^{n-1}}{(n-1)!} \Phi_A(\mu)^{n-j} + c_{j-1,l-1}^{(n-1)} + \Phi_A(\mu) \sum_{p=0}^{n-j-1} c_{p,j-1,0}^{(n-1)} \left( \frac{r_1 \lambda_1}{\lambda_1 + \mu} \right)^p, \quad 0 < l < j,
\]

where it is understood that sums of the form \(\sum_{l=0}^{j} (\cdot)\) equal 0, and that \(c_{ll}^{(n-1)} = 0\), for all \(l\) and \(n\). The algorithm is initialised by setting \(c_{10}^{(1)} = \mu/(\lambda_2 + \mu)\) and \(c_{11}^{(1)} = \mu/(\lambda_1 + \mu)\).

**Proof.** When \(\alpha = 1\), Eqs. (3.11) and (3.12) reduce to
\[ \mathcal{L}_n(s) = \left( \frac{\mu}{\mu - s} \right) \left[ \Phi_A(s) \mathcal{L}_{n-1}(s) - \Phi_A(\mu) \mathcal{L}_{n-1}(\mu) \right], \quad (3.15) \]

\[ P(n) = \Phi_A(\mu) \mathcal{L}_{n-1}(\mu), \quad (3.16) \]

where

\[ \Phi_A(s) = \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right) + \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right). \]

Assuming an initial reserve of \( u \), \( P(1) = \Phi_A(\mu) e^{-u\mu} \). Therefore \( \mathcal{L}_1(s) \) can be written as

\[ \mathcal{L}_1(s) = e^{-u\mu} \left[ \frac{\mu}{\mu - s} \right] \left[ \Phi_A(s) e^{u(\mu - s)} - \Phi_A(\mu) \right] \]

\[ = e^{-u\mu} \left[ \sum_{i=1}^{2} \left( \frac{r_i \lambda_i}{\lambda_i + s} \right) \left( \frac{\mu}{\lambda_i + \mu} \right) + \Phi_A(s) \left( u \mu \right) \sum_{k=1}^{\infty} \frac{(u(\mu - s))^{k-1}}{k!} \right]. \quad (3.17) \]

(3.17) satisfies the general form of \( \mathcal{L}_n(s) \), with \( c_1^{(1)} = \mu / (\lambda_2 + \mu) \) and \( c_1^{(2)} = \mu / (\lambda_1 + \mu) \). Now assume the general form is valid up to \( n = N - 1 \). Substituting into (3.15) for \( n = N \), one finds

\[ \mathcal{L}_N(s) = \left( \frac{\mu}{\mu - s} \right) e^{-u\mu} \left[ \sum_{j=0}^{N-1} \sum_{l=0}^{j} \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right)^j \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right)^{j-l} \Phi_A(s) \right. \]

\[ - \left( \frac{r_1 \lambda_1}{\lambda_1 + \mu} \right)^j \left( \frac{r_2 \lambda_2}{\lambda_2 + \mu} \right)^{j-l} \Phi_A(\mu) \]  

\[ + \frac{(u \mu)^{N-1}}{(N-1)!} \Phi_A(s)^N - \Phi_A(\mu)^N \right] + \Phi_A(s)^N (u \mu)^{N-1} u(\mu - s) \sum_{k=N}^{\infty} \frac{(u(\mu - s))^{k-N}}{k!} \]  

After several changes of indices, and simplifying and cancelling the term \( (\mu - s) \) one obtains

\[ \mathcal{L}_N(s) = e^{-u\mu} \left[ \sum_{j=0}^{N} \sum_{l=0}^{j} \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right)^j \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right)^{j-l} \Phi_A(s) \right. \]

\[ + \Phi_A(\mu) \sum_{j=1}^{N} \sum_{l=1}^{j} \left( \frac{r_1 \lambda_1}{\lambda_1 + s} \right)^l \left( \frac{r_2 \lambda_2}{\lambda_2 + s} \right)^{j-l} \sum_{p=0}^{N-j} \sum_{q=0}^{j-l} \left( \frac{r_1 \lambda_1}{\lambda_1 + \mu} \right)^p \left( \frac{r_2 \lambda_2}{\lambda_2 + \mu} \right)^q \frac{\mu}{\lambda_1 + \mu} \]  

\[ + \Phi_A(s)^N (u \mu)^{N-1} u(\mu - s) \sum_{k=N}^{\infty} \frac{(u(\mu - s))^{k-N}}{k!} \].
The proof is completed by collecting like terms and expressing the coefficients \( c_{jl}^{(N)} \) as in (3.14) for \( n = N \).

Thus, \( P(n), n \geq 2 \) can be explicitly determined from (3.16) with coefficients \( c_{jl}^{(n)} \) starting from \( c_{10}^{(1)} = \mu / (\lambda_2 + \mu) \) and \( c_{11}^{(1)} = \mu / (\lambda_1 + \mu) \).

Numerical examples for this case are presented in Section 5. By inspection of Eqs. (3.13) and (3.14), we see that the recursion involves no negative numbers. Furthermore, the multiplications involve terms between 0 and 1, except for two combinatorial terms in the middle line. Thus, while the form of (3.14) may be complicated, it can nonetheless be programmed to provide accurate results.

Theorem 3.5 shows that the recursion for the coefficients, while tractable, is somewhat cumbersome. The situation is even more cumbersome for \( D = 2 \). As an alternative to this exact method when the recursions are either too cumbersome or intractable, an approximate method by De Vylder and Goovaerts can be adopted. This is presented next.

4. A numerical approach using the De Vylder–Goovaerts method

The methodology that we have employed to determine the ruin probability up to this point and in our previous paper consists of the following steps. First, a recursion involving the Laplace transforms of the incomplete densities \( p_n(x) \) is determined in terms of the density function for the increment \( g(x) \). Next, an algebraic form for \( L_n(s) \) is obtained, and finally a recursion for the coefficients is developed. This procedure is tractable for the Poisson-arrival case, but can become an arduous task for certain arrival processes as we have seen in Section 3.2.

At the same time, it is possible to retain the advantages of working with the increment distribution and a recursion based at claim instants without having to determine explicit expressions for \( L_n(s) \). Adopting the algorithm in De Vylder and Goovaerts (1988, formulae (4) and (5)) we define \( \psi_n(u) = \sum_{i=1}^n P(i) \) to be the probability of ruin at or before the \( n \)th claim, given the initial surplus \( u \).

The probability of ruin on the first claim, as shown in Section 2, is obtained as \( \psi_1(u) = F(-u) \). The probability that ruin occurs at or before the \( n \)th claim is obtained recursively as the sum of the probability that ruin occurs on the first claim, plus the complementary probability that a surplus \( u + x > 0 \) resulted from the first increment \( x \), and the ruin occurs on or before the remaining \( n - 1 \) claims are paid. This yields

\[
\psi_n(u) = F(-u) + \int_{x=-u}^{\infty} g(x) \psi_{n-1}(u + x) \, dx. \tag{4.1}
\]

To obtain numerical results from the above formula (4.1) it was suggested by a referee to Lee et al. (1994) to discretise the density function of the increment and then to shorten the range of summation/integration by discarding small probabilities. The discretisation used will be based on formula (15) of De Vylder and Goovaerts (1988) rewritten as follows. Let \( f_k, -\infty < k < \infty \), be the probability mass function for the discretised increment distribution. Then

\[
f_k = \int_{x=k}^{k+1} F(x) \, dx - \int_{x=k-1}^{k} F(x) \, dx. \tag{4.2}
\]

One can obtain increasingly accurate discretisations by appropriate rescaling of the increment distribution.

The formulae for the cumulative distribution function \( F(-u) \) are given below for the cases considered in Section 3.

Case 1. Mixture of exponentials revenue, Erlang-\( N \) claims

\[
F(x) = \begin{cases} 
\sum_{i=1}^{M} q_i \left[ \sum_{n=0}^{N-1} \left( \frac{\lambda_i}{\lambda_i + \theta} \right) \left( \frac{\theta}{\lambda_i + \theta} \right)^n e^{i\theta} \sum_{l=0}^{N-n-1} \frac{(-x\theta)^l}{l!} \right], & x < 0, \\
\sum_{i=1}^{M} q_i \left[ 1 - \left( \frac{\theta}{\lambda_i + \theta} \right)^N e^{-\lambda_i x} \right], & x \geq 0.
\end{cases} \tag{4.3}
\]
Case 2. Erlang-$M$ revenue, mixture of exponentials claims

\[
F(x) = \begin{cases} 
\sum_{j=1}^N p_j \left( \frac{\lambda}{\lambda + \theta_j} \right)^M e^{\theta_j x}, & x < 0, \\
\sum_{j=1}^N p_j \left[ 1 - \sum_{m=0}^{M-1} \left( \frac{\theta_j}{\lambda + \theta_j} \right)^m \frac{\lambda^m}{m!} \sum_{l=0}^{M-m-1} \frac{\lambda^l x^l}{l!} \right], & x \geq 0.
\end{cases}
\]

Below we present the increment distribution for the following additional case:

Case 3. Mixture of exponentials revenue, mixture of exponentials claims

\[
F(x) = \begin{cases} 
\sum_{i=1}^M q_i \sum_{j=1}^N p_j \left( \frac{\lambda_i}{\lambda_i + \theta_j} \right) e^{\theta_j x}, & x < 0, \\
\sum_{i=1}^M q_i \sum_{j=1}^N p_j \left[ 1 - \left( \frac{\theta_j}{\lambda_i + \theta_j} \right) e^{-\lambda_i x} \right], & x \geq 0.
\end{cases}
\]

The determination of the $f_k$’s for each case is straightforward. Below the final results for Case 3 are given:

\[
F(-u) = \sum_{i=1}^M q_i \sum_{j=1}^N p_j \left( \frac{\lambda_i}{\lambda_i + \theta_j} \right) e^{-\theta_j u},
\]

and the discretised increment distribution $f_k, -\infty < k < \infty$, is given by

\[
f_k = \sum_{i=1}^M q_i \sum_{j=1}^N p_j f_{ijk},
\]

where

\[
f_{ijk} = \begin{cases} 
\left( \frac{\lambda_i}{\lambda_i + \theta_j} \right) e^{-\theta_j (k-1)} \left( 1 - e^{-\theta_j} \right)^2, & k < 0, \\
\left( \frac{\theta_j}{\lambda_i + \theta_j} \right) e^{-\lambda_i (k-1)} \left( 1 - e^{-\lambda_i} \right)^2, & k > 0.
\end{cases}
\]

and where

\[
f_{ij0} = 1 - \frac{\theta_j}{\lambda_i + \theta_j} \left( 1 - e^{-\lambda_i} \right) - \frac{\lambda_i}{\lambda_i + \theta_j} \frac{\lambda_i}{\theta_j} \left( 1 - e^{-\theta_j} \right).
\]

In the next section some numerical examples involving the exact method discussed in Section 3 are presented, as well as one example of the numerical approach discussed above.

5. Numerical results

In this section, selected numerical examples (Figs. 1–6) are presented to demonstrate the effects of the initial reserve ratio, relative security loading (RSL), and the variability of the inter-claim revenue (ICR) distribution (as measured by the SCV) on the probabilities of ruin.

Our principal goal is to assess the impact of these non-Poisson claim processes on the ruin probability. As such, we have used an exponential claim size in all of the examples, because the impacts of non-exponential claim sizes were considered previously.
Fig. 1. Effect of initial reserve ratio (IRR).

Fig. 2. Effect of relative security loading (RSL).
Fig. 3. Effect of inter-claim revenue variability (SCV).

Fig. 4. Effect of initial reserve ratio (IRR).
Fig. 5. Effect of relative security loading (RSL).

Fig. 6. Effect of inter-claim revenue variability (SCV).
The initial reserve ratio (IRR) is defined as the initial reserve divided by the expected claim size, where the expected revenue between claims is given by

\[ E[ICR] = (1 + \text{RSL}) E[CS]. \]

In the case of a mixture of two exponentials, the balanced means assumption \((r_1/\theta_1 = r_2/\theta_2 = E[X]/2)\) is used (see Stanford and Stroiński (1994, p. 248)).

Figs. 1–3 illustrate what happens when the ICR distribution is changed from exponential (corresponding to a Poisson claim arrival process) to a mixture of two exponentials. An exponential claim size is assumed in these examples. Fig. 1 illustrates how the probabilities of ruin decrease as the IRR increases. Here RSL = 0.2 and the SCV of the ICR is 6.25. It is readily observed that when there is no initial reserve the probabilities of ruin are much higher than in the Poisson case. It is well known that the probability of eventual ruin for the Poisson claim number process when there is no initial reserve is \(1 / (1 + \text{RSL})\) (see Bowers et al. (1986, p. 359)). For example, under the Poisson case the probability of eventual ruin would be 5/6 when RSL = 0.2. In contrast, the probability of ruin by the 20th claim for the mixture of exponentials with SCV = 6.25 and the same RSL is already higher (see the highest curve on the graph). For IRR = 0, it is obvious that the period of vulnerability starts immediately, while for IRR = 20, it is notably delayed.

The effect of the RSL is explored in Fig. 2 when IRR = 10 and SCV = 6.25. As expected, as the RSL increases, the probabilities of ruin on individual claims decrease. For smaller values of RSL, the probabilities decay more slowly. One also notes that the curves reach their largest values sooner for larger RSL. If the RSL were large, for the same IRR, one would expect either to be ruined very soon or else a considerable reserve would be built up. Fig. 3 illustrates even more dramatically the effect of replacing the usual Poisson claim arrival process with a process that is more variable. Here, the various curves correspond to different values of the SCV. Note that the probabilities of ruin increase sharply as the SCV increases. This effect is somewhat moderated after the “peak” probabilities. The lowest probabilities of ruin on this figure correspond to the Poisson claim arrival process.

To complement the case of highly variable inter-claim times, Figs. 4–6 show what effect a more regular claim number process has on ruin than the usual Poisson claim number process. The ICR distribution in Figs. 4 and 5 is Erlang-2, whereas the CS distribution in all three figures is assumed to be exponential.

Fig. 4, like Fig. 1, explores the impact of the IRR when the RSL = 0.2. If there is no initial reserve (the highest “curve” of probabilities in the graph), we can again compare with the previously determined ultimate probability of ruin for the Poisson case. It looks as though the cumulative probabilities of ruin stabilise and do not change much from, say, the 50th to the 100th claim. This probability is below 0.8 which is lower than the ultimate Poisson-case value of 5/6. The delayed onset of the period of relative vulnerability increases with the increase in IRR. However, even more visible from comparison of Figs. 1 and 4 is that the latter shows much reduced ruin probabilities due to the less variable claim arrival process. This difference is most pronounced for larger initial reserves.

Fig. 5 shows that the increase of the RSL both diminishes and shortens the period of vulnerability.

Fig. 6 explores the impact of \(\alpha\) on the ruin probability. It displays graphically the degree to which the ruin probability is reduced as the ICR variability \((1/\alpha)\) is decreased.

Table 1 presents a comparison of the exact probabilities of ruin obtained by using the algebraic approach with the corresponding values obtained from the numerical approach presented in Section 4. The layout of the table is adopted from Table 1 of Dickson and Waters (1991), and corresponds to a Poisson claims process with exponentially distributed claim amounts. The average claim size is 20, and relative security loadings of 10% and 20%, as well as initial reserve ratios of 0, 1, and 10 are considered. The discretised increment probabilities are likewise calculated until their sum exceeds \(1 - \epsilon = 1 - 10^{-6} / 3\).

The key difference is that Dickson and Waters consider discrete time instants, whereas Table 1 here is calculated for claim numbers. This Poisson/exponential pairing also has the advantage of being able to apply a simpler algebraic form and the simplest expressions for the numerical recursions (4.8) and (4.9).

The numerical approach is exact for the probability of ruin on the first claim, namely \(F(-u)\). The probability of ruin by the 10th claim has a comparable accuracy as in Dickson and Waters.
Table 1
A comparison of the algebraic (1), and the numerical (2) approaches

\begin{tabular}{|c|c|c|c|}
\hline
 & \multicolumn{2}{|c|}{n=1} & \multicolumn{1}{|c|}{n=10} \\
\hline
\hline
(a) Premium loading factor $\theta = 0.1$ & & & \\
$u=0$ & (1) \ 0.476190 & 0.782556 \\
 & (2) \ 0.476190 & 0.797214 \\
$u=1$ & (1) \ 0.175181 & 0.598495 \\
 & (2) \ 0.175181 & 0.609245 \\
$u=10$ & (1) \ 0.000022 & 0.014181 \\
 & (2) \ 0.000022 & 0.014354 \\
\hline
(b) Premium loading factor $\theta = 0.2$ & & & \\
$u=0$ & (1) \ 0.454545 & 0.741968 \\
 & (2) \ 0.454545 & 0.755167 \\
$u=1$ & (1) \ 0.167218 & 0.547779 \\
 & (2) \ 0.167218 & 0.557162 \\
$u=10$ & (1) \ 0.000021 & 0.010773 \\
 & (2) \ 0.000021 & 0.010901 \\
\hline
\end{tabular}

Determination of the numerical approach for the 100th claim was not possible from our PC-based Turbo Pascal programme, which encountered overflow errors.

Acknowledgements

This work has been financially supported by Canada’s Natural Sciences and Engineering Research Council through operating Grant No. OGP0041187 and through its summer scholarship programme. We wish to thank the referee, whose helpful comments have improved this paper.

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