Time stochastic $s$-convexity of claim processes

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Abstract

The purpose of this paper is to study the conditions on a stochastic process under which the $s$-convex ordering and the $s$-increasing convex stochastic ordering between two random instants is transformed into a stochastic ordering of the same type between the states occupied by this process at these moments. In this respect, the present work develops a previous study by Shaked and Wong (1995) [Probability in the Engineering and Informational Sciences 9, 563–580]. As an illustration, we show that the binomial and the Poisson processes, commonly used in actuarial sciences to model the occurrence of insured claims, possess this remarkable property. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Recently, Lefèvre and Utev (1996), Denuit and Lefèvre (1997), Denuit et al. (1999a,b,c,d) introduced and studied broad classes of stochastic order relations among real-valued random variables; they call them the $s$-convex, $s$-increasing convex, $s$-concave and $s$-increasing concave orderings. An original feature of these relations lies in the fact that they take into account the particular structure of the respective supports of the random variables to be compared. This leads so to stronger comparison results than those obtained by considering all the random variables as valued in the real line. Without going into details, we mention that these stochastic orderings can be seen as particular cases of the general Tchebycheff-type order relations introduced by Denuit et al. (1999d).

The problem investigated in this paper is as follows. Consider a stochastic process $X = \{X_t, \ t \in \mathbb{R}^+\}$ describing for instance the total amount of claims affecting an insurance company during $[0, t]$, or the number of insured claims occurring in $[0, t]$, for instance. Let $T_1$ and $T_2$ be two random instants and let $\preceq$ be some stochastic order relation. We wonder whether you can maintain that

$T_1 \preceq T_2 \Rightarrow X_{T_1} \preceq X_{T_2}$,

i.e. that an $\preceq$-ordering between two random instants $T_1$ and $T_2$ is transformed into an $\preceq$-ordering between the states $X_{T_1}$ and $X_{T_2}$ occupied by the process $X$ at these moments. We investigate here this problem by considering for the...
relation $\leq$ the $s$-increasing convex ordering (also called the stop-loss order of degree $s - 1$ in the actuarial literature) and the $s$-convex ordering. Applications of the results in risk theory are provided at the end of the article.

The paper is organized as follows. First, in Section 2, we present the $s$-convex and the $s$-increasing convex stochastic orderings. Then, in Section 3, we introduce the notion of time stochastic $s$-convexity and $s$-increasing convexity for stochastic processes. We mention that a similar problem has been previously discussed by Shaked and Wong (1995), though these authors restrict their study to point processes and usual stochastic order relations, such as the stop-loss order, the convex order or the stochastic dominance, for example. In Section 4, we show that the processes commonly used in actuarial sciences to model, on the one hand the occurrence of insured claims, and on the other hand the total claim amount (namely the binomial and the Poisson processes, as well as their compound versions), possess this remarkable property. Finally, in Section 5, we show how $s$-convex order lower bounds on the aggregate claim amount affecting an insurance company before ruin occurs (in the classical risk model) can be obtained.

2. Orderings of convex type

Many stochastic order relations $\preceq_s^S$ commonly used to compare two random variables $X$ and $Y$ valued in a subset $S$ of the real line $\mathbb{R}$ are defined (or, at least, can be characterized) by reference to a class $\mathcal{U}_s^S$ of measurable functions with domain $S$ (usually a convex cone) as follows: $X$ is said to be smaller than $Y$ in the $\preceq_s^S$-sense, which is denoted as $X \preceq_s^S Y$, when

$$E\phi(X) \leq E\phi(Y) \quad \forall \phi \in \mathcal{U}_s^S,$$

provided that the expectations exist. Whitt (1986) introduced the term of *integral stochastic ordering generated by $\mathcal{U}_s^S$* for $\preceq_s^S$ defined through (2.1). Such relations have been extensively studied by Marshall (1991) and Müller (1997).

In decision making context, the function $\phi$ involved in the definition (2.1) typically represents a utility function. The preferences shared by all the decision-makers whose utility function satisfies certain reasonable conditions (i.e. belongs to $\mathcal{U}_s^S$) constitute therefore a partial order $\preceq_s^S$ of all risks.

Among standard integral stochastic orderings used in actuarial sciences, the stochastic dominance (usually denoted by $\preceq_{st}$) is obtained when $\mathcal{U}_s^S = \mathcal{U}_{st}$, the class of the non-decreasing functions; $\preceq_{st}$ translates therefore the common preferences of all the actuaries thinking that more money is better (in the sense that $X \preceq_{st} Y$ means the financial loss modeled by the random variable $X$ is preferred over the one modeled by $Y$ all the profit-seeking insurers). The stop-loss order, also known in the statistical literature as the increasing convex order (usually denoted by $\preceq_{sl}$ or $\preceq_{icx}$) is obtained when $\mathcal{U}_s^S = \mathcal{U}_{cx}$, the class of the non-decreasing and convex functions; $\preceq_{sl}$ translates therefore the common preferences shared by all the risk-averse profit-seeking actuaries. Finally, the stop-loss order with equal means, better known in the statistical literature as the convex order (usually denoted by $\preceq_{sl} (= \preceq_{icx})$) is obtained when $\mathcal{U}_s^S = \mathcal{U}_{cx}$, the class of the convex functions. For more details about these standard stochastic order relations, the reader is referred, e.g., to Goovaerts et al. (1990) and Kaas et al. (1994). See also Denuit (1997) and Shaked and Shanthikumar (1994).

We are concerned here with the $s$-convex and $s$-increasing convex stochastic orderings. The latter are the integral stochastic orderings generated by the cones of the $s$-convex and of the $s$-increasing convex functions, respectively. More precisely, Popoviciu (1933) defined the $s$-convexity as follows: a real-valued function $\phi$ defined on $S \subseteq \mathbb{R}$ is said to be $s$-convex on $S$ if and only if

$$[x_0, x_1, \ldots, x_s] \phi \geq 0$$

for all choices of $s + 1$ distinct points $x_0, x_1, \ldots, x_s$ in $S$, where the divided difference $[\ldots] \phi$ is defined recursively by

$$[x_0, x_1, \ldots, x_k] \phi = \frac{[x_{k-1}, \ldots, x_1] \phi - [x_0, x_1, \ldots, x_{k-1}] \phi}{x_k - x_0}, \quad k = 1, 2, \ldots, s,$$
starting from \([x_k] \phi = \phi(x_k), k = 0, 1, \ldots, s\). In addition, \(\phi\) is said to be \(s\)-increasing convex when it is simultaneously \(k\)-convex for \(k = 1, 2, \ldots, s\). We denote by \(\mathcal{U}_s^{S_{-\text{ex}}} \) (resp. \(\mathcal{U}_s^{S_{-\text{icx}}} \)) the class of the \(s\)-convex (resp. \(s\)-increasing convex) functions on \(S\); obviously,

\[ \mathcal{U}_s^{S_{-\text{ex}}} = \bigcap_{k=1}^{s} \mathcal{U}_k^{S_{-\text{ex}}} \]

For more details about divided differences and convexity of higher degree, we refer the reader, e.g., to Roberts and Varberg (1973) (see also Chapters I and II in Denuit, 1997).

Now, given two random variables \(X\) and \(Y\) valued in \(\mathcal{S}\), \(X\) is said to be smaller than \(Y\) in the \(s\)-convex (resp. \(s\)-increasing convex) sense, denoted by \(X \preceq_{S_{-\text{ex}}} Y\) (resp. \(X \preceq_{S_{-\text{icx}}} Y\)) when (2.1) holds with \(\mathcal{U}_s^{S_{-\text{ex}}}=\mathcal{U}_s^{S_{-\text{ex}}}\) (resp. \(\mathcal{U}_s^{S}=\mathcal{U}_s^{S_{-\text{icx}}}\)). We mention that the 1-increasing convex and the 1-convex orders reduce to the standard stochastic dominance, while the 2-increasing convex order is the stop-loss order and the 2-convex order is the convex order, i.e.

\[ X \preceq_{1_{-\text{ex}}} Y \Leftrightarrow X \preceq_{1_{-\text{icx}}} Y \Leftrightarrow X \preceq_{s_{\text{tr}}} Y, \quad X \preceq_{2_{-\text{ex}}} Y \Leftrightarrow X \preceq_{s_{\text{tt}}} Y, \quad X \preceq_{2_{-\text{icx}}} Y \Leftrightarrow X \preceq_{s_{\text{tt}}} Y. \]

It is to be noted that, for \(s = 1\) and \(2\), the support \(\mathcal{S}\) of the risks \(X\) and \(Y\) to be compared has no influence on the resulting ranking, i.e.

\[ X \preceq_{S_{-\text{icx}}} Y \Leftrightarrow X \preceq_{S_{-\text{icx}}} Y \text{ for } s = 1 \text{ and } 2, \text{ and whatever } \mathcal{S} \subseteq \mathbb{R}^+ \text{ is}. \]

The above equivalence comes from the integral definition (2.1) of the stochastic orderings \(\preceq_{S_{-\text{ex}}}^{S_{-\text{ex}}}\) and \(\preceq_{S_{-\text{ex}}}^{S_{-\text{icx}}}\), together with the fact that any non-decreasing or convex function on \(\mathcal{S}\) can always be continued as a function with the same shape on the half-positive real line \(\mathbb{R}^+\). Quite surprisingly, as soon as \(s \geq 3\), the influence of \(\mathcal{S}\) on \(\preceq_{S_{-\text{ex}}}^{S_{-\text{ex}}}\) and \(\preceq_{S_{-\text{ex}}}^{S_{-\text{icx}}}\) is of primordial importance. Indeed, if instead of considering \(X\) and \(Y\) as valued in \(\mathcal{S}\), we see them as valued in a larger subset \(\mathcal{V}\) of the real line, it is easily seen that if \(\mathcal{S} \subseteq \mathcal{V}\) then

\[ X \preceq_{S_{-\text{ex}}} Y \Rightarrow X \preceq_{S_{-\text{ex}}} Y, \quad X \preceq_{S_{-\text{icx}}} Y \Rightarrow X \preceq_{S_{-\text{icx}}} Y, \]

but the reciprocal implication in (2.2) is not necessarily true. This particularity of the \(s\)-convex and \(s\)-increasing convex orderings has been extensively studied by Denuit et al. (1999c), where numerous counterexamples are provided for the reciprocal implication in (2.2).

When the support \(\mathcal{S}\) of the random variables to be compared possesses some structure (when it is a continuum or an arithmetic grid, for example), \(\preceq_{S_{-\text{ex}}}^{S_{-\text{ex}}}\) and \(\preceq_{S_{-\text{ex}}}^{S_{-\text{icx}}}\) can be characterized through (2.1) with for \(\mathcal{U}_s^{S}\) more usual classes of functions. For instance, when \(\mathcal{S} = [a, b]\), \(b\) possibly infinite, it suffices, in order to characterize \(\preceq_{S_{-\text{ex}}}^{[a,b]}\), to consider for \(\mathcal{U}_{[a,b]}^{S}\) the class of the regular \(s\)-convex functions given by

\[ \mathcal{U}_{[a,b]}^{S_{-\text{ex}}} \cap C^s ([a, b]) = \left\{ \phi : [a, b] \to \mathbb{R} | \phi^{(s)} \geq 0 \text{ on } [a, b] \right\}, \]

where \(C^s ([a, b])\) denotes the class of the functions \(\phi : [a, b] \to \mathbb{R}\) possessing a continuous \(s\)th derivative \(\phi^{(s)}\) on \([a, b]\). In fact, \(\mathcal{U}_{[a,b]}^{S_{-\text{ex}}} \cap C^s ([a, b])\) is weakly dense in \(\mathcal{U}_{[a,b]}^{S_{-\text{ex}}}\). In order to characterize \(\preceq_{[a,b]}^{[a,b]}\), it suffices to consider for \(\mathcal{U}_{[a,b]}^{S_{-\text{ex}}}\) the class of the regular \(s\)-increasing convex functions given by

\[ \mathcal{U}_{[a,b]}^{S_{-\text{ex}}} \cap C^s ([a, b]) = \left\{ \phi : [a, b] \to \mathbb{R} | \phi^{(k)} \geq 0 \text{ on } [a, b] \text{ for } k = 1, 2, \ldots, s \right\}. \]

On the other hand, when \(\mathcal{S} = \mathcal{D}_n\equiv\{0, 1, \ldots, n\}, n \in \mathbb{N}_0\equiv\{1,2,3,\ldots\}\), it can be shown (see, e.g., Denuit et al., 1999c) that

\[ \mathcal{U}_{S_{-\text{ex}}}^{\mathcal{D}_n} = \left\{ \phi : \mathcal{D}_n \to \mathbb{R} | \Delta^s \phi \geq 0 \text{ on } \mathcal{D}_{n-s} \right\}, \]

where \(\Delta^k\) denotes the usual \(k\)th degree forward difference operator defined recursively by

\[ \Delta^k \phi (i) = \Delta^{k-1} \phi (i+1) - \Delta^{k-1} \phi (i), \quad i \in \mathcal{D}_{n-k}, \]
with the convention that $\Delta^0\phi \equiv \phi$. Moreover,
\[ U_{s-icx}^{D_n} = \left\{ \phi : D_n \to \mathbb{R} | \Delta^k\phi \geq 0 \text{ on } D_{n-k} \text{ for } k = 1, 2, \ldots, s \right\}. \quad (2.6) \]

The orderings $\preceq_{s-icx}^{D_n}$ and $\preceq_{s-icx}^{D_n}$ have been studied in detail by Denuit and Lefèvre (1997) and Denuit et al. (1999b).

To end with, let us mention that the $s$-convex and the $s$-increasing convex orderings are closely related. More precisely, given two risks $X$ and $Y$ valued in $S$, the following equivalence holds:
\[ X \preceq_{s-icx}^S Y \iff EX^k = EY^k \text{ for } k = 1, 2, \ldots, s - 1, \quad (2.7) \]

The $s$-convex ordering can thus be seen as a strengthening of the $s$-increasing convex ordering obtained by requiring in addition that the first $s - 1$ moments of the random variables to be compared are equal. We mention that the aforementioned strengthening has been previously studied in the actuarial literature.

The reader interested in a deep study of the $s$-convex and $s$-increasing convex orderings is referred to Denuit (1997).

3. Time stochastic $s$-convexity

The idea underlying the time stochastic $s$-convexity is very similar to the philosophy of the stochastic $s$-convexity. The latter concept has been recently introduced and investigated in details by Denuit and Lefèvre (1998) and Denuit et al. (1999d). We begin with a brief description of it.

Let us consider a family of random variables $\{X_\theta, \theta \in \Theta\}$ valued in a subset $S \subseteq \mathbb{R}$ indexed by a single parameter $\theta \in \Theta \subseteq \mathbb{R}$. Now, given a function $\phi : S \to \mathbb{R}$, let us construct the new function $\phi^*$ defined as
\[ \phi^* : \Theta \to \mathbb{R}, \quad \theta \mapsto \phi^*(\theta) \equiv E\phi(X_\theta). \quad (3.1) \]

provided that the expectation exists. A natural question is to which extent some properties of the function $\phi$ can be transmitted to the function $\phi^*$. Such a question is rather general and has already been discussed in probability and statistics, for instance by Shaked and Shanthikumar (1988); see also Chapter VI of the book of Shaked and Shanthikumar (1994). Denuit et al. (1999d) defined the stochastic $s$-convexity and the stochastic $s$-increasing convexity as follows.

**Definition 3.1.** The family $\{X_\theta, \theta \in \Theta\}$ of random variables valued in $S$ and indexed by $\theta \in \Theta$ is said to be $s$-convex when, given any function $\phi \in U_{s-icx}^S$, we have that $\phi^* \in U_{s-icx}^S$ with $\phi^*$ defined in (3.1). Similarly, the family $\{X_\theta, \theta \in \Theta\}$ is said to be stochastically $s$-increasing convex when, given any function $\phi \in U_{s-icx}^S$, we have that $\phi^* \in U_{s-icx}^S$.

Most parametric families of probability distributions possess the stochastic $s$-convexity and $s$-increasing convexity properties, as Denuit and Lefèvre (1998) and Denuit et al. (1999d) showed, so that this notion possesses many applications, namely in relation with mixture models and compound sums.

Now, let $X = \{X_t, t \in T\}$ be a stochastic process with time space $T \subseteq \mathbb{R}^+$ and state space $S \subseteq \mathbb{R}$, say. Analogous to (3.1), let us define for $\phi : S \to \mathbb{R}$ the associated function $\phi^*$ as
\[ \phi^* : T \to \mathbb{R}, \quad t \mapsto \phi^*(t) \equiv E\phi(X_t), \quad (3.2) \]

provided that the expectation exists. The time stochastic $s$-convexity and $s$-increasing convexity can then be defined as follows.

**Definition 3.2.** A stochastic process $X = \{X_t, t \in T\}$ with time space $T$ and state space $S$ is said to possess the time stochastic $s$-convexity property when, given any function $\phi \in U_{s-icx}^T$, we have that $\phi^* \in U_{s-icx}^T$, with $\phi^*$ defined
Similarly, \( X \) is said to possess the time stochastic \( S \)-increasing convexity property when, given any function \( \phi \in \mathcal{U}^S_{S-icx} \), we have that \( \phi^* \in \mathcal{U}^T_{S-icx} \).

Let us now examplify the practical use of this notion. Let \( T_1 \) and \( T_2 \) be two random variables valued in \( T \) and independent of \( X \); \( X_{T_1} \) and \( X_{T_2} \) are the states occupied by \( X \) at the instants \( T_1 \) and \( T_2 \), respectively. We study here conditions on \( X \) under which an \( S \)-convex ordering or an \( S \)-increasing convex ordering between \( T_1 \) and \( T_2 \) is transformed into an ordering of the same type between \( X_{T_1} \) and \( X_{T_2} \), i.e. under which

\[
T_1 \preceq^S_{S-icx} T_2 \Rightarrow X_{T_1} \preceq^S_{S-icx} X_{T_2}, \quad T_1 \preceq^T_{S-icx} T_2 \Rightarrow X_{T_1} \preceq^S_{S-icx} X_{T_2}. \tag{3.3}
\]

Let us examine the \( S \)-convex case, for instance. It is easily seen that

\[
E\phi(X_{T_i}) = E\phi^*(T_i) \quad \text{for } i = 1, 2.
\]

From the integral definition (2.1) of the \( S \)-convex ordering, we then deduce that a sufficient condition for (3.3) to hold is that

\[
\phi \in \mathcal{U}^S_{S-icx} \Rightarrow \phi^* \in \mathcal{U}^T_{S-icx}, \tag{3.4}
\]

i.e. that \( X \) possesses the time stochastic \( S \)-convexity property. Indeed, when (3.4) is satisfied, \( T_1 \preceq^T_{S-icx} T_2 \) implies that for any \( \phi \in \mathcal{U}^S_{S-icx} \),

\[
E\phi(X_{T_1}) = E\phi^*(T_1) \leq E\phi^*(T_2) = E\phi(X_{T_2}),
\]

so that

\[
X_{T_1} \preceq^S_{S-icx} X_{T_2}
\]

holds. Of course, the above reasoning is still valid with the \( S \)-increasing convex order substituted for the \( S \)-convex one.

The problem (3.3) was examined by Shaked and Wong (1995) with different classes of order relations. Moreover, in Remark 2.3, these authors considered the Rolski orderings and obtained results similar to some of those discussed in Section 4. Nevertheless, as mentioned in Denuit et al. (1998), the \( S \)-increasing convex ordering and the Rolski ordering are close but mathematically distinct.

4. Claim processes

4.1. Binomial and compound binomial processes

In the compound binomial risk process, time is measured in discrete time units \( t \in \mathbb{N} = \{0, 1, 2, \ldots \} \) and the number of insured claims is governed by a binomial process \( \{N_t, \ t \in \mathbb{N}\} \) with parameter \( p \in ]0, 1[ \) (i.e. in any time period, there occurs 1 or 0 claim with probabilities \( p \) and \( 1 - p \), respectively, and occurrences of claims in different time intervals are independent events). Instead of establishing directly that the binomial process owns the time stochastic \( S \)-convexity and \( S \)-increasing convexity properties, we prove the next general result stating that any random walk possesses this remarkable property.

**Proposition 4.1.** Let \( \{Y_k, \ k \in \mathbb{N}_0\} \) be a sequence of independent and identically distributed (i.i.d., in short) random variables valued in \( \mathbb{R}^+ \) and consider the random walk process \( X = \{X_t, \ t \in \mathbb{N}\} \) defined by

\[
X_0 = 0 \ \text{a.s.,} \quad X_t = \sum_{k=1}^t Y_k, \quad t \in \mathbb{N}_0.
\]

Then, (i) \( X \) possesses the time stochastic \( S \)-convexity property and (ii) \( X \) possesses the time stochastic \( S \)-increasing convexity property.
Proof. Part (i) directly follows from Property 4.6 in Denuit et al. (1999d). To get (ii), it suffices to apply the reasoning provided in Denuit et al. (1999d), (Property 4.6) successively for \( k = 1 \) to \( s \).

**Corollary 4.2.** The binomial process owns the time stochastic \( s \)-convexity and the time stochastic \( s \)-increasing convexity properties.

**Proof.** It suffices to notice that if \( \{N_t, \ t \in \mathbb{N}\} \) is a binomial process, then

\[
N_0 = 0 \text{ a.s., } \quad N_t = \sum_{k=1}^{t} Y_k, \quad t \in \mathbb{N}_0,
\]

where the \( Y_k \)'s are i.i.d. Bernoulli random variables. The announced result then follows from Proposition 4.1.

As a consequence, we have that when the occurrence of the insured claims is described by a binomial process \( \{N_t, \ t \in \mathbb{N}\} \),

\[
T_1 \preceq_{s-icx} T_2 \Rightarrow N_{T_1} \preceq_{s-icx} N_{T_2}, \quad T_1 \preceq_{s-icx} T_2 \Rightarrow N_{T_1} \preceq_{s-icx} N_{T_2}.
\]

The aggregate claim process \( \{S_t, \ t \in \mathbb{N}\} \) is modeled by a compound binomial process of the form

\[
S_t = 0 \text{ as long as } N_t = 0, \quad S_t = \sum_{k=1}^{N_t} Z_k \text{ when } N_t \geq 1, \quad t \in \mathbb{N},
\]

where \( \{N_t, \ t \in \mathbb{N}\} \) is a binomial process with parameter \( p \in \]0,1[ and \( \{Z_k, \ k \in \mathbb{N}_0\} \) is a sequence of non-negative i.i.d. random variables, independent of the occurrence process \( \{N_t, \ t \in \mathbb{N}\} \); \( Z_k \) represents the amount of claim during the period \([k-1,k)\), \( k \in \mathbb{N}_0 \).

**Proposition 4.3.** Let \( \mathcal{T} = \mathbb{N} \) or \( \mathbb{R}^+ \). Let \( \{Z_k, \ k \in \mathbb{N}_0\} \) be a sequence of non-negative iid random variables and let \( \{N_t, \ t \in \mathcal{T}\} \) be an integer-valued stochastic process possessing the time stochastic \( s \)-increasing convexity property. Define the compound process \( X = \{S_t, \ t \in \mathcal{T}\} \) as in (4.2). Then, \( X \) still possesses the time stochastic \( s \)-increasing convexity property.

**Proof.** This result is immediate from Property 4.8 in Denuit et al. (1999d).

We mention that the latter result only holds with the time stochastic \( s \)-increasing convexity property, and not with the \( s \)-convex one. The reason is the proof uses the fact that a composition of two \( s \)-increasing functions is itself an \( s \)-increasing function, but this is no more valid for \( s \)-convex functions.

From Proposition 4.3, a compound process of the form (4.2) built with a counting process \( \{N_t, \ t \in \mathcal{T}\} \) possessing the time \( s \)-increasing convexity property will itself own this interesting property. As a consequence, we have from Corollary 4.2 that when the aggregate claim is described by a compound binomial process of the form (4.2),

\[
T_1 \preceq_{s-icx} T_2 \Rightarrow S_{T_1} \preceq_{s-icx} S_{T_2}.
\]

4.2. Poisson and the compound Poisson processes

Another classical model for the aggregate claim is the compound Poisson process. In this case, the number of insured claims is governed by a Poisson process \( \{N_t, \ t \in \mathbb{R}^+\} \) with intensity rate \( \lambda > 0 \). Let us prove the following result.

**Proposition 4.4.** The Poisson process \( \{N_t, \ t \in \mathbb{R}^+\} \) owns the time stochastic \( s \)-convexity property, as well as the time stochastic \( s \)-increasing convexity property.

**Proof.** In order to prove the result, we need to show that the function \( \phi^* \) given by

\[
\phi^*: \mathbb{R}^+ \to \mathbb{R}, \quad t \mapsto \phi^*(t) = \sum_{i \in \mathbb{N}} \phi(i) e^{-\lambda t} \frac{(\lambda t)^i}{i!}
\]
is $s$-convex whenever $\phi : \mathbb{N} \to \mathbb{R}$. By Leibniz formula, we get

$$\frac{d^s}{dt^s} \phi^*(t) = \lambda^s \sum_{i \in \mathbb{N}} \phi(i) \sum_{j=0}^{\min(i,s)} \binom{s}{j} (-1)^{s-j} \frac{e^{-\lambda t} (\lambda t)^{i-j}}{(i-j)!} = \lambda^s \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} \sum_{i=j}^{+\infty} \phi(i) e^{-\lambda t} \frac{(\lambda t)^i}{i!}.$$

The well-known Newton Binomial formula (see, e.g., Agarwal, 1992) ensures that

$$D_s \sum_{i \in \mathbb{N}} \phi(i) \frac{e^{-\lambda t} (\lambda t)^i}{i!} = \sum_{j=0}^{s} \binom{s}{j} (-1)^{s-j} \sum_{i=j}^{+\infty} \phi(i) e^{-\lambda t} \frac{(\lambda t)^i}{i!}.$$

so that it is easily checked that

$$\frac{d^s}{dt^s} \phi^*(t) = \lambda^s \sum_{i \in \mathbb{N}} \Delta^s \phi(i) \frac{e^{-\lambda t} (\lambda t)^i}{i!} \geq 0,$$

whence $\phi^* \in \mathcal{U}_{s-conv}^{\mathbb{R}^+}$ (see (2.3) together with (2.5)) follows. This achieves the first part of the proof. The second part follows similarly by applying the same reasoning successively for $k = 1$ to $s$.

As a consequence, we have that when the occurrence of the claims is described by a Poisson process $\{N_t, \ t \in \mathbb{R}^+\}$,

$$T_1 \leq_{s-conv} T_2 \Rightarrow N_{T_1} \leq_{s-conv} N_{T_2}, \quad T_1 \leq_{s-conv} T_2 \Rightarrow N_{T_1} \leq_{s-conv} N_{T_2}.$$

(4.4)

Now, the aggregate claim process $\{S_t, \ t \in \mathbb{R}^+\}$ is described by a compound Poisson process of the form

$$S_t = 0 \ as \ long \ as \ N_t = 0, \quad S_t = \sum_{k=1}^{N_t} Z_k \ when \ N_t \geq 1, \quad t \in \mathbb{R}^+,$$

(4.5)

where $\{N_t, \ t \in \mathbb{R}^+\}$ is a Poisson process with parameter $\lambda > 0$ and $\{Z_k, \ k \in \mathbb{N}_0\}$ a sequence of non-negative i.i.d. random variables, independent of the occurrence process $\{N_t, \ t \in \mathbb{R}^+\}$; $Z_k$ represents the amount of the $k$th claim affecting the insurance company, $k \in \mathbb{N}_0$. By Proposition 4.3, we have that the compound Poisson process also owns the time stochastic $s$-increasing convexity property. As a consequence, we have that when the aggregate claim is described by a compound Poisson process $\{S_t, \ t \in \mathbb{R}^+\}$,

$$T_1 \leq_{s-conv} T_2 \Rightarrow S_{T_1} \leq_{s-conv} S_{T_2}.$$

(4.6)

5. $s$-Convex order lower bounds on aggregate claims

Assume that $\{S_t, \ t \in \mathbb{R}^+\}$ is a compound Poisson process of the form (4.5) modeling the total amount of claims affecting an insurance company. Let $T$ represent the first time when $\{S_t, \ t \in \mathbb{R}^+\}$ hits the linear upper barrier $u + ct$, $u \geq 0$, $c > 0$, modeling the premium income of the company; $T$ can be interpreted as the time of ruin and $S_T$ as the aggregate claims affecting the company during $[0, T]$, i.e., before ruin occurs.

Nevertheless, the distribution of $T$ is often complicated, so that the exact distribution of $S_T$ can be difficult to obtain. Our purpose here is to show how to get lower bounds on $S_T$ in the $s$-convex sense when a few moments of the $Z_k$’s are known.

It is well known that if $c < \lambda E Z_1$, then $T < +\infty$ a.s., i.e. ruin occurs with probability 1. This corresponds for instance to a situation when one of the subsidiaries of an insurance company runs to ruin with probability 1, but the
whole company has a positive non-ruin probability. When the \( Z_k \)'s are valued in \( N_0 \) and \( u = 0 \), Picard and Lefèvre (1998) have shown that

\[
\mu_1 = ET = \frac{1}{cQ_0} \quad (5.1)
\]

and

\[
\mu_2 = ET^2 = \frac{2\mu_1^2}{1 - \exp(-Q_0)g'[\exp(-Q_0)]/c}, \quad (5.2)
\]

where \( Q_0 \) is the positive solution of the equation \( \lambda - cQ_0 = \lambda E\exp(-Q_0 Z_1) \) and \( g(s) = \lambda \sum_{j=1}^{\infty} s^j P[Z_1 = j] \).

When the moments of \( T_k = ET^k \) say, are known for \( k = 1, 2, \ldots, s - 1 \), it is possible to construct a random variable \( T_{(s)}^{(s)} \) with the same moments as \( T \) and such that

\[
T_{(s)}^{(s)} \leq_{s-\text{cx}} T. \quad (5.3)
\]

It is not possible to bound \( T \) from above since the support of \( T \) is the whole half-positive real line (the maxima in the \( s \)-convex sense always put a positive probability mass on the upper bound of the support). Explicit expressions of the bounds involved in (5.3) are available in Denuit et al. (1998, 1999a). As an illustration, for \( s = 1 \) and \( 2 \), using (5.1) and (5.2), we have that \( T_{(s)}^{(s)} \) in a.s. and

\[
T_{(s)}^{(s)} = \begin{cases} 
0 & \text{with probability } (\mu_2 - \mu_1^2)/\mu_2, \\
\mu_1 + (\mu_2 - \mu_1^2)/\mu_1 & \text{with probability } \mu_1^2/\mu_2.
\end{cases} \quad (5.3)
\]

Now, from (4.6), we have that

\[
S_{T_{(s)}^{(s)}} \leq_{s-\text{icx}} S_T, \quad (5.4)
\]

so that (5.4) provides a lower bound for the total amount of claim affecting an insurance company before ruin occurs. In particular, from the integral definition (2.1) of the \( s \)-increasing convex ordering, (5.4) provides lower bounds on \( E\phi_0(S_T) \) for any \( \phi_0 \in \mathcal{U}_{s-\text{icx}}^S \).

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