Insurer’s optimal reinsurance strategies

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Received May 1999; accepted December 1999

Abstract

The problem of finding an optimal insurer’s strategy of purchasing reinsurance is considered under the standard deviation calculation principles. It is assumed that the strategy must satisfy several kinds of constraints. The optimal reinsurance contract is found out. © 2000 Elsevier Science B.V. All rights reserved.

MSC: 90A46, 62P05; 62C25, 49J55

Keywords: Principles of premium calculation; Optimal reinsurance

1. Introduction

Let $Y$ denote the total claim in a given time period and $P$ be an amount of money which the insurer is ready to spend on the reinsurance. We shall assume that $Y$ is a nonnegative random variable defined on a given probability space $(\Omega, S, Pr)$. A reinsurer offers to cover a random quantity $R$ of $Y$, which is a measurable function of $Y$ for a premium $Pr$. So the total claim $Y$ would split into a part paid by the reinsurer, $R(Y)$, and the one paid by the insurer, say $Q(Y)$, in such a way that $Y = R(Y) + Q(Y)$. The premium paid by the insurer for the reinsurance arrangement is calculated according to the standard deviation principle (with a safety loading parameter $\beta > 0$) and we will consider the set of all plausible reinsurance arrangements whose reinsurance premium is less than or equal to $P$, i.e. all the arrangements such that

$$ P \geq Pr = ER(Y) + \beta DR(Y), \quad (1) $$

where $DR(Y) = \sqrt{Var(R(Y))}$. Throughout the paper we assume that $EY^2 < \infty$, otherwise the premium for reinsurance would be infinite except for the case $\beta = 0$, corresponding to the pure risk premium, which is of least interest to us. As pointed out by many authors (see, e.g. Bühlmann, 1970; Daykin et al., 1993; Deprez and Gerber, 1985) the standard deviation premium calculation principle (1) takes into account the variability of the reinsurer’s share so it is less problematic in practice than, e.g. pure risk premium. On the other hand, the insurer may still ask a question as to how to choose the reinsurance rule $R$ in a convenient way. Assume that the insurer is interested in minimizing the variance of the retained risk, measured by the variance of $\tilde{R}$, in his own share. Additionally, the insurer is assumed to admit only those reinsurance arrangements which satisfy the following condition:

$$ 0 \leq \tilde{R}(Y) \leq Y, \quad (2a) $$

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or equivalently,
\[ 0 \leq R(Y) \leq Y. \quad (2b) \]
Condition (2a) means that the insurer’s share is never larger than the total claim amount ((2a) is assumed, e.g. when finding the optimal reinsurance arrangement under the pure risk premium; see Daykin et al., 1993, p. 191). So the problem of finding the optimal insurer’s strategy of reinsuring relies on minimizing \( \text{Var}(Y - R(Y)) \) over the space of all measurable functions \( R : \mathbb{R}_+ \to \mathbb{R} \), where \( \mathbb{R}_+ = [0, \infty) \), subject to the constraints (1) and (2b).

In Section 2 we prove that this minimization problem has a solution \( R_1 \) of the form
\[
R_1(y) = \begin{cases} 
0 & \text{if } Y \leq M \equiv \text{change loss point}, \\
(1 - r)(Y - M) & \text{else},
\end{cases}
\quad (3)
\]
where \( r(0 \leq r < 1) \) and \( M \) depend on \( P, \beta \) and truncated moments of \( Y \). Since the rule is somewhat similar to the stop loss reinsurance we shall call it \textit{change loss reinsurance}. Our optimality proof is based on the Lagrange multipliers method and uses Gâteaux derivatives of the variance and constraint functionals. A numerical example is provided to illustrate the advantage of using change loss reinsurance over the quota share reinsurance rule.

Another interesting result which concerns optimal purchasing of reinsurance was proven by Deprez and Gerber (1985). They called a reinsurance arrangement \( R \) optimal if it maximized the expectation of a given (risk averse) utility function, without imposing any restrictions on \( R \), however. Strictly speaking, Deprez and Gerber (1985) considered a nonconstrained optimization problem with a target function different than ours.

2. Optimality of the change loss reinsurance contract

Let \( F \) denote the distribution function of a total claim. We may assume that \( \mathbb{E}R^2(Y) < \infty \) (otherwise the premium \( P \) could not be finite because of (1)) and let us denote the set of all measurable functions \( R : \mathbb{R}_+ \to \mathbb{R} \) satisfying this condition by \( \mathcal{R} \). Clearly \( \mathcal{R} \) is a Hilbert space with the norm \( \| R \| = \sqrt{\mathbb{E}R^2(Y)} \). Given a fixed premium \( P > 0 \) and the safety loading coefficient \( \beta > 0 \), let us define functionals \( g : \mathcal{R} \to \mathbb{R} \) and \( V : \mathcal{R} \to \mathbb{R} \) by
\[
g(R) = \mathbb{E}R(Y) + \beta \mathbb{D}(R(Y)) - P, \quad (4)
\]
\[
V(R) = \text{Var}(Y - R(Y)). \quad (5)
\]
Both functionals \( g \) and \( V \) are convex (see Deprez and Gerber, 1985) and Gâteaux differentiable. We recall that the Gâteaux derivative of a mapping \( Q : \mathcal{R} \to \mathbb{R} \) (here \( \mathcal{R} \) and \( \mathbb{R} \) stand for the Hilbert space defined above and for the set of real numbers, respectively) at \( R \in \mathcal{R} \) is any linear continuous functional \( x^* \) from the dual of \( \mathcal{R} \) such that
\[
\lim_{t \to 0} t^{-1} (Q(R + th) - Q(R)) = (x^*, H)
\]
for any \( H \in \mathcal{R} \). In the sequel we write \( \nabla Q(R)(H) \) instead of \( (x^*, H) \). It is not difficult to check that for every \( R_1, H \in \mathcal{R} \), we have
\[
\nabla V(R_1)(H) = -2\mathbb{E}(Y - R_1(Y))H(Y) + 2\mathbb{E}(Y - R_1(Y))EH(Y), \quad (6)
\]
and by the chain rule
\[
\nabla g(R_1)(H) = EH(Y) + \beta \frac{\mathbb{E}R(Y)H(Y) - \mathbb{E}R_1(Y)EH(Y)}{\mathbb{D}R_1(Y)}, \quad (7)
\]
whenever \( \mathbb{D}R_1(Y) > 0 \) and \( V, g \) are as in (4) and (5), respectively. Let us define a subset \( C \subseteq \mathcal{R} \) by
\[
C = \{ R \in \mathcal{R} \mid 0 \leq R(Y) \leq y \text{ a.e., } [F] \text{ on } \mathbb{R}_+ \}.
\]
Clearly \( C \) is a convex set.
Our goal is to minimize \( V(\cdot) \) over sets
\[
\mathcal{M}_\leq = \{ R \in \mathcal{R} | g(R) \leq 0 \text{ and } R \in C \},
\]
and
\[
\mathcal{M}_= = \{ R \in \mathcal{R} | g(R) = 0 \text{ and } R \in C \},
\]
respectively. These are nonlinear constrained optimization problems (the former one is convex) which we shall solve using a Lagrangian function
\[
L_\lambda(R) = V(R) + \lambda g(R) + \Psi_C(R),
\]
where \( \lambda \) is a Lagrange multiplier and \( \Psi_C: \mathcal{R} \to \mathbb{R} \cup \{+\infty\} \) is defined by
\[
\Psi_C(R) = \begin{cases} 
0 & \text{if } R \in C, \\
+\infty & \text{otherwise}.
\end{cases}
\]
Under suitable assumptions minimizing \( V \) over the set \( \mathcal{M}_\leq \) is equivalent to a nonconstrained minimization of \( L_\lambda \) with some parameter \( \lambda \geq 0 \). This follows from the Karush–Kuhn–Tucker theorem (see, e.g. Peressini et al., 1988 or Ioffe and Tikhomirov, 1974). Since we are interested only in a sufficient condition for constrained optimality, the following simple lemma will be useful.

**Lemma 2.1.** Let \( R_1 \in C \) and \( \lambda \geq 0 \) be such that
\( \lambda g(R_1) = 0 \),
\( L_\lambda(R) \geq L_\lambda(R_1) \) for every \( R \in \mathcal{R} \).

Then \( R_1 \) minimizes \( V \) over the set \( \mathcal{M}_\leq = \{ R \in \mathcal{R} | g(R) \leq 0 \text{ and } R \in C \} \). It also minimizes \( V \) over \( \mathcal{M}_= \) whenever \( g(R_1) = 0 \) (since it minimizes \( V \) over \( \mathcal{M}_\leq \)).

**Proof.** Let us notice that for every \( R \in \mathcal{R} \)
\[
V(R_1) = L_\lambda(R_1) \leq L_\lambda(R)
\]
due to (i) and (ii). If \( R \in C \) and \( g(R) \leq 0 \), \( L_\lambda(R) \leq V(R) \) so \( V(R_1) \leq V(R) \). \( \square \)

**Remark 2.2.** It should be pointed out at this moment that the necessary condition for the existence of minimum over \( \mathcal{M}_\leq \) does not guarantee that \( \lambda \) is nonnegative (see, e.g. Ioffe and Tikhomirov, 1974). However if we can find a real \( \lambda \) for which (ii) holds, then of course \( R_1 \) becomes the minimizer over \( \mathcal{M}_= \), since \( L_\lambda(R) = V(R) \) on \( \mathcal{M}_= \) in this case. When we are dealing with the convex problem (we want to minimize \( V \) over \( \mathcal{M}_\leq \)), then the Karush–Kuhn–Tucker theorem (the necessary part) guarantees the existence of a nonnegative \( \lambda \) satisfying (i) and (ii) of Lemma 2.1.

Though Lemma 2.1 seems to be very simple, the main difficulty when applying it relies on finding \( \lambda \geq 0 \) for which (ii) holds. In order to preserve the existence of such lambdas we assume that the following conditions are satisfied:
\[
EY - M - \int_{(M,\infty)} (y - M) \, dF + \frac{r}{\beta} \int_{(M,\infty)} (y - M)^2 \, dF \leq 0, 
\]
(8)
\[
(1 - r) \int_{(M,\infty)} (y - M) \, dF + \beta \int_{(M,\infty)} (y - M)^2 \, dF - \left( \int_{(M,\infty)} (y - M) \, dF \right)^2 \leq 0, 
\]
(9)
An important question arises when the solutions \( M \) and \( \beta \) to the Eqs. (8) and (9) exist. It is easy to notice that if the solutions exist, \( EY + \beta DY \geq P \) must hold; see (9). Lemma 2.3 shows that it is also a sufficient condition for
solvability of (8) and (9). What is more, $P \geq EY + \beta DY$ means that the insurer is ready, and has enough money, to transfer the whole portfolio to the reinsurer, which is the least interesting situation from both theoretical and practical points of view.

Lemma 2.3. Assume that $P > 0$, $\beta DY > 0$ and

$$EY + \beta DY > P.$$  \hfill (10)

Then there exist real numbers $M > 0$ and $r \in [0, 1)$ satisfying Eqs. (8) and (9).

Proof. Let us notice that the set

$$A = \left\{ M \geq 0 \left| \int_{(M, \infty)} (y - M)^2 \, dF > \left[ \int_{(M, \infty)} (y - M) \, dF \right]^2 \right. \right\}$$

is nonempty (because $0 \in A$). Hence $M_1 = \sup A$ is well defined though $M_1$ might be $+\infty$. Obviously $M_1 > 0$ because otherwise $DY$ would be 0, a contradiction. We derive from (8) that

$$r = \frac{\beta \int_{[0, M]} (M - y) \, dF}{\sqrt{\int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2}},$$  \hfill (11)

which is well defined for every $M \in [0, M_1)$. Let us define a function $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(M) = \left[ 1 - \beta \frac{\int_{[0, M]} (M - y) \, dF}{\sqrt{\int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2}} \right] \times \left( \int_{(M, \infty)} (y - M) \, dF + \beta \int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2 \right).$$  \hfill (12)

From (9) and (11) it follows that every $M \in [0, M_1)$ such that $h(M) = P$ is a solution to (8) and (9), together with a corresponding $r$ defined by (11). Let us observe that $h(0) = EY + \beta DY > P$ by assumption (10). On the other hand, $\lim \inf_{M \nearrow M_1} h(M) \leq 0$. Indeed, if $M_1 = \infty$ then the first factor in the right-hand side of (12) tends to $-\infty$ as $M \nearrow M_1$, while the second one is always nonnegative. If $M_1 < \infty$, then

$$\int_{(M, \infty)} (y - M)^2 \, dF \geq \Pr(Y \geq M) \int_{(M, \infty)} (y - M)^2 \, dF \geq \left( \int_{(M, \infty)} (y - M) \, dF \right)^2$$  \hfill (13)

by the Cauchy–Schwartz inequality, and for all $M > M_1$ there must hold equalities in (13) by the definition of $M_1$. This implies however that either $F$ has one atom not smaller than $M_1$, a contradiction, or $\Pr(Y > M_1) = 0$. In the latter case

$$\lim_{M \nearrow M_1} \left[ \int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2 \right] = 0,$$

and again the first factor in the right-hand side of (12) tends to $-\infty$ as $M \nearrow M_1$. Eventually, there must exist $M \in (0, M_1)$ such that $h(M) = P$ because $h(\cdot)$ is continuous on $[0, M_1)$. For this very $M$ the corresponding $r$ is smaller than 1 by the definition of $h(M)$ and (11).

Now we are ready to prove the following result.
Theorem 2.4. Assume that $P > 0$ and $\beta D Y > 0$. Let $M \geq 0$ and $r \in [0, 1)$ be such numbers that

\[
EY - M - \int_{(M, \infty)} (y - M) \, dF + \frac{r}{\beta} \int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2 = 0,
\]

(14)

\[
(1 - r) \left[ \int_{(M, \infty)} (y - M) \, dF + \beta \int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2 \right] = P.
\]

(15)

Then the function

\[
R_1(y) = \begin{cases} 
0 & \text{if } 0 \leq y < M, \\
(1 - r)(y - M) & \text{otherwise}, 
\end{cases}
\]

is the optimal reinsurance arrangement for the insurer under restrictions (1) and (2b) (i.e. it gives the minimum value of $V$ over the set $\mathcal{M}_\leq$). The function $V$ attains also its minimum over $\mathcal{M}_\leq$ at $R_1$, since $P = E R_1 + \beta D R_1$, by (15).

Proof. First let us observe that $Pr(Y > M) > 0$ because otherwise (15) leads to a contradiction. Now let us define

\[
\lambda = \frac{2r}{\beta} \sqrt{\int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2} \geq 0.
\]

(16)

We shall check if (i) of Lemma 2.1 holds. By (15),

\[
g(R_1) = (1 - r) \left[ \int_{(M, \infty)} (y - M) \, dF + \beta \int_{(M, \infty)} (y - M)^2 \, dF - \left( \int_{(M, \infty)} (y - M) \, dF \right)^2 \right] - P = 0,
\]

therefore $\lambda g(R_1) = 0$. In order to check if (ii) holds, let us consider $R \in C$ (otherwise $\Psi_C(R) = +\infty$ and (ii) holds in an obvious way). By the definition of $R_1$,

\[
R(y) - R_1(y) \geq 0 \quad \text{for all } 0 \leq y < M.
\]

(17)

Since $R_1 \in C$ and $\lambda \geq 0$, $L_\lambda$ is differentiable at $R_1$ in the direction of any point $R \in C$ with the directional derivative equal to $[\nabla V(R_1) + \lambda \nabla g(R_1)](R - R_1)$, where $\nabla V(R_1)$ and $\nabla g(R_1)$ denote the Gâteaux derivatives of $V$ and $g$, respectively. Keep in mind here that $\int_{[0, \infty)} R_2^2(y) \, dF > (\int_{[0, \infty)} R_1(y) \, dF)^2$ because otherwise $Pr(Y > M) = 1$ and consequently $Pr(Y = \text{const.}) = 1$, which contradicts the assumption. It follows from (6) and (7) that

\[
\nabla V(R_1)(H) = -2 \int_{[0, \infty)} [y - R_1(y)] H(y) \, dF + 2 \int_{[0, \infty)} [y - R_1(y)] \, dF \int_{[0, \infty)} H(y) \, dF,
\]

\[
\nabla g(R_1)(H) = \int_{[0, \infty)} H(y) \, dF + \beta \frac{\left( \int_{[0, \infty)} R_1^2(y) \, dF - \left( \int_{[0, \infty)} R_1(y) \, dF \right)^2 \right)}{\sqrt{\int_{[0, \infty)} R_2^2(y) \, dF - (\int_{[0, \infty)} R_1(y) \, dF)^2}}.
\]

As a sum of two convex functions, $L_\lambda$ is convex on $C$, therefore for every $R \in C$, we have
\[ L_2(R) - L_2(R_1) \geq \nabla V(R_1)(R - R_1) + \lambda \nabla g(R_1)(R - R_1) \]
\[ = 2 \left\{ - \int_{[0,\infty)} [y - R_1(y)](R - R_1) dF + \int_{[0,\infty)} [y - R_1(y)] dF \right\} \]
\[ + \lambda \int_{[0,\infty)} [R(y) - R_1(y)] dF + \beta \frac{\int_{[0,\infty)} [R(y) - R_1(y)] dF \int_{[0,\infty)} [R(y) - R_1(y)] dF}{\sqrt{\int_{[0,\infty)} R_1^2(y) dF - \left[ \int_{[0,\infty)} R_1(y) dF \right]^2}} \]
\[ = -2 \int_{[0,M]} y[R(y) - R_1(y)] dF - 2 \int_{[M,\infty)} [y - (1 - r)(y - M)][R(y) - R_1(y)] dF(y) \]
\[ + 2 \int_{[0,M]} y dF + \int_{[M,\infty)} [y - (1 - r)(y - M)] dF \int_{[0,\infty)} [R(y) - R_1(y)] dF \]
\[ + \lambda \int_{[0,\infty)} [R(y) - R_1(y)] dF + \lambda \beta \frac{\int_{[M,\infty)} (1 - r)(y - M) dF \int_{[0,\infty)} [R(y) - R_1(y)] dF}{\sqrt{\int_{[0,\infty)} R_1^2(y) dF - \left[ \int_{[0,\infty)} R_1(y) dF \right]^2}} \]
\[ = \left\{ -2M + 2 \left[ \int_{[0,M]} y dF + \int_{[M,\infty)} [y - (1 - r)(y - M)] dF \right] \right\} \]
\[ - \lambda \beta \frac{\int_{[M,\infty)} (1 - r)(y - M) dF}{\sqrt{\int_{[0,\infty)} R_1^2(y) dF - \left[ \int_{[0,\infty)} R_1(y) dF \right]^2}} + \lambda \int_{[0,\infty)} [R(y) - R_1(y)] dF \]
\[ + 2 \int_{[0,M]} (M - y)[R(y) - R_1(y)] dF + \int_{[M,\infty)} \left\{ 2[M - y + (1 - r)(y - M)] \right\} \]
\[ + \lambda \beta \frac{(1 - r)(y - M)}{\sqrt{\int_{[0,\infty)} R_1^2(y) dF - \left[ \int_{[0,\infty)} R_1(y) dF \right]^2}} \int_{[0,\infty)} [R(y) - R_1(y)] dF \]
\[ = 2 \int_{[0,M]} (M - y)[R(y) - R_1(y)] dF, \quad (18) \]

where the last equality holds because of (14) and (16). Now by (17) and (18) we eventually get
\[ L_2(R) - L_2(R_1) \geq 0 \quad \text{for all } R \in \mathcal{R}, \]
so the assumption (ii) of Lemma 2.1 is satisfied and the proof is complete. \hfill \Box

**Remark 2.5.** Let us notice that the solutions \( M \) and \( \beta \) of Eqs. (14) and (15) depend on \( \beta \) in such a way that \( M \) is bounded when \( \beta \to 0 \). Indeed, under assumptions of Theorem 2.4 if \( M \to \infty \), the left-hand side of (15) would tend to 0, a contradiction. But then it follows from (14) that \( r \to 0 \) as \( \beta \to 0 \), which implies that \( R_1 \), defined by (3), tends to a stop loss reinsurance contract. It is known that stop loss is an optimal reinsurance arrangement under the pure risk premium calculation (see Daykin et al., 1993) so our solution coincides with this special case optimality result.

**Remark 2.6.** For a given \( \beta > 0 \) the optimal strategy could be treated as a combination of the quota share and stop loss reinsurance strategy. Namely, the insurer company retains a certain amount of risk \( Y \leq M \) and the rest above
M is reinsured proportionally (i.e. the reinsurer company reimburses a constant percentage of the claim above M, exactly \((1 - r)(Y - M)\)).

**Remark 2.7.** A careful verification of the proof of Theorem 2.4 shows that we can admit \(r = 0\). In this case we have \(\lambda = 0\) (see (16)). However, if \(\lambda = 0\), then \(L_\lambda = V\) on \(C\), thus the explicit form of (ii) in Lemma 2.1 is: \(V(R) \geq V(R_1)\) for every \(R \in M_\leq\). Hence getting (ii) we prove directly that \(R_1\) is a minimizer.

Let \(P\) be fixed. Below we compare the optimal strategies described in Theorem 2.4 with the quote share reinsurance arrangement. Let us define the reinsurance risk reduction function, \(\phi(R)\), as follows:

\[
\phi(R) = DY - D(Y - R(Y)).
\]

It is interesting to see how many times \(\phi(R_1)\) is larger than \(\phi(R_2)\), where \(R_1\) the optimal reinsurance contract described in (3), while \(R_2\) is the quota share reinsurance rule. Of course the larger the \(\phi(R_1)/\phi(R_2)\), the better \(R_1\) is compared with \(R_2\).

For the quote share reinsurance we have

\[
\phi(R_2) = DY - D(\tilde{R}_2(Y)) = (1 - \alpha)DY.
\]

Now using (1) let us calculate \(\alpha\) (we assume here that the whole sum of money is spent on the reinsurance for this type of reinsurance strategy). We have

\[
P = ER_2(Y) + \beta DR_2(Y) = (1 - \alpha)(EY + \beta DY).
\]

Hence

\[
1 - \alpha = \frac{P}{EY + \beta DY},
\]

which implies

\[
\phi(R_2) = \frac{PDY}{EY + \beta DY} = \frac{P}{\beta} \frac{DY}{DY + EY/\beta}.
\]

Example 2.8. Assume that \(Y\) has a normal distribution with \(EY = 10^9\) USD and \(DY = 10^8\) USD. Take \(\beta = 1.64\) and \(P = 2.91 \times 10^8\) USD. Then \(M = 8.506 \times 10^8\) USD and \(r = 0.052\) are the solutions to (8) and (9). Hence \(E\tilde{R}_1 = 8.556 \times 10^8\) USD and \(D\tilde{R}_1 = 0.125 \times 10^8\) USD and therefore

\[
\phi(R_1) = 0.875 \times 10^8\ USD.
\]

Paying the same premium \(P = 2.91 \times 10^8\) USD the insurer can buy a quota share contract \(R_2\) with \(\alpha = 0.75\) (keep in mind (19)). Then by (20) the risk reduction function amounts to

\[
\phi(R_2) = 0.250 \times 10^8\ USD.
\]

Hence \(\phi(R_1)/\phi(R_2) = 3.5\).
References