The moments of the time of ruin, the surplus before ruin, and the deficit at ruin

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Abstract

In this paper we extend the results in Lin and Willmot (1999 Insurance: Mathematics and Economics 25, 63–84) to properties related to the joint and marginal moments of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin. We use an approach developed in Lin and Willmot (1999), under which the solution to a defective renewal equation is expressed in terms of a compound geometric tail, to derive explicitly the joint and marginal moments. This approach also allows for the establishment of recursive relations between these moments. Examples are given for the cases when the claim size distribution is exponential, combinations of exponentials and mixtures of Erlangs. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and background

There has been considerable research interest in the analysis of the distributions of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin. The identification of these distributions is difficult since analytic expressions do not exist in most cases. Further, there are in general no closed form solutions for the moments of these distributions. Various approaches have been utilized in order to study the probabilistic properties of these distributions. In Gerber et al. (1987) and Dufresne and Gerber (1988) the investigations focus on the analytic expression of the probability of ruin and the distribution function of the surplus before the time of ruin for the case that the individual claim size is a combination of exponential distributions or a combination of gamma distributions. Dickson (1992), Dickson and Waters (1992), Dickson and Dos Reis (1996), Willmot and Lin (1998) and Schmidli (1999), discuss analytic properties of the distribution of the surplus before the time of ruin for the case that the individual claim size is a combination of exponential distributions or a combination of gamma distributions. Dickson (1992), Dickson and Waters (1992), Dickson and Dos Reis (1996), Willmot and Lin (1998) and Schmidli (1999), discuss analytic properties of the distribution of the surplus before the time of ruin, the distribution of the deficit at the time of ruin and their relationship. Dickson and Waters (1991) and Dickson et al. (1995) consider recursive calculation for these distributions when the individual claim size is fully discrete. Di Lorenzo and Tessitore (1996) propose a numerical approximation for calculation of the distribution of the surplus before the time of ruin. The moment properties of the time of ruin are considered in Delbaen (1990) and in Picard and Lefevre (1998, 1999).
(also see Gerber (1979, Chapter 9, Section 3)). Recently, Gerber and Shiu (1997, 1998) study the joint distribution of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin by considering an expected discounted penalty involving these three random variables. They show that this expected discounted penalty as a function of the initial surplus is the solution of a certain (defective) renewal equation. This approach allows for the use of existing renewal theoretic techniques as well as new techniques developed for renewal equations.

In Lin and Willmot (1999) and Willmot and Lin (2000), a different approach is proposed for solving defective renewal equations. Under this approach the solution of a defective renewal equation is expressed in terms of a compound geometric tail. This has the advantage of allowing for the use of analytic properties of a compound geometric distribution and for the implementation of exact and approximate results which have been developed for the tail. Tails of compound geometric distributions have been studied extensively both analytically and numerically. In particular, recursive formulas (e.g., Panjer and Willmot (1992)) are available, as are upper and lower bounds (e.g., Lin (1996)). Exact solutions are sometimes available (e.g., Dufresne and Gerber (1988) and Tijms (1994)). Also, the well known Cramer–Lundberg asymptotic formula (e.g., Gerber (1979)) is available. Second and more importantly, for a large class of claim size distributions, the tail of the associated compound geometric distribution can be approximated very nicely by a combination of two exponential distributions. For example, if the claim size distribution is new worse than used in convex ordering (NWUC) or new better than used in convex ordering (NBUC), a combination of two exponential distributions with proper parameter values will match the probability mass at zero and the mean, and has the same asymptotic formula. This combination is referred to as the Tijms approximation (see Tijms (1994) and Willmot (1997)). Thus, we can take advantage of the analytic form of the Tijms approximation.

In this paper we extend the approach in Lin and Willmot (1999) to properties related to the joint and marginal moments of the time of ruin, the surplus before the time of ruin, and the deficit at the time of ruin. Although these problems have been considered by various authors, the present approach allows one to express the joint and marginal moments in terms of compound geometric tails and to reveal recursive relations between these moments. The advantage of this approach becomes apparent when applied to the recursive calculation of the higher moments at the time of ruin. In the rest of this section, we present the classical continuous time risk model, Gerber and Shiu’s expected discounted penalty function and its associated renewal equation, and the solution of the renewal equation in terms of a compound geometric tail which is obtained in Lin and Willmot (1999). Section 2 discusses equilibrium distributions and reliability classifications. Section 3 introduces auxiliary functions which involve the compound geometric tail associated with Gerber and Shiu’s renewal equation and the related equilibrium distributions. The auxiliary functions are necessary for computation of the moments of the deficit and the moments of the time of ruin. Section 4 considers properties related to the moments of the deficit at ruin and Section 5 considers the moments of the surplus immediately before ruin. In Section 6, we derive the moments at the time of ruin.

We now begin with the classical continuous time risk model. Let $N_t$ be the number of claims from an insurance portfolio. It is assumed that $N_t$ follows a Poisson process with mean $\lambda$. The individual claim sizes $X_1, X_2, \ldots$ are independent of $N_t$, are positive, independent and identically distributed random variables with common distribution function (DF) $P(x) = 1 - \hat{P}(x) = \Pr(X \leq x)$, moments $p_j = \int_0^\infty x^j dP(x)$ for $j = 0, 1, 2, \ldots$ and Laplace–Stieltjes transform $\hat{p}(s) = \int_0^\infty e^{-st} dP(x).$ The aggregate claims process is $\{S_t; t \geq 0\}$, where $S_t = X_1 + X_2 + \cdots + X_{N_t}$ (with $S_t = 0$ if $N_t = 0$). The insurer’s surplus process is $\{U_t; t \geq 0\}$ with $U_t = u + ct - S_t$, where $u \geq 0$ is the initial surplus, $c = \lambda p_1(1 + \theta)$ the premium rate per unit time, and $\theta > 0$ the relative security loading.

Let $T = \inf\{t; U(t) < 0\}$ be the first time that the surplus becomes negative and is called the time of ruin. The probability $\psi(u) = \Pr[T < \infty]$ is called the probability of (ultimate) ruin. Two nonnegative random variables in connection with the time of ruin are the surplus immediately before the time of ruin $U(T^-)$, where $T^-$ is the left limit of $T$ and the deficit at the time of ruin $[U(T)]$. For details, see Bowers et al. (1997, Chapter 13), Gerber (1979, Chapters 8 and 9) or De Vylder (1996, Part I, Chapter 7).
Let \( w(x_1, x_2) \), \( 0 \leq x_1, x_2 < \infty \), be a nonnegative function. For \( \delta \geq 0 \), define

\[
\phi(u) = E\{e^{-\delta T} w(U(T), |U(T)|) I(T < \infty)\},
\]

where \( I(T < \infty) = 1 \), \( T < \infty \), and \( I(T < \infty) = 0 \) otherwise. The quantity \( w(U(T), |U(T)|) \) can be interpreted as the penalty at the time of ruin, where the surplus is \( U(T) \) and the deficit is \( |U(T)| \). Thus, \( \phi(u) \) is the expected discounted penalty when \( \delta \) is interpreted as a force of interest. Since

\[
\psi(u) = E\{I(T < \infty)\},
\]

\( \phi(u) \) reduces to \( \psi(u) \) if \( \delta = 0 \) and \( w(x_1, x_2) = 1 \). The function \( \phi(u) \) is useful both for unifying and generalizing known results in connection with joint and marginal distributions of \( T, U(T) \), and \( |U(T)| \) (see, e.g., Gerber and Shiu (1997, 1998)). Moreover, (1.1) may be viewed in terms of Laplace transforms with \( \delta \) the argument. Gerber and Shiu (1998) show that the function \( \phi(u) \) satisfies the defective renewal equation

\[
\phi(u) = \frac{\lambda}{c} \int_0^u \phi(u - x) \int_x^\infty e^{-\rho(y-x)} dP(y) dx + \frac{\rho}{c} e^{\rho u} \int_u^\infty e^{-\rho y} \int_0^\infty w(x, y-x) dP(y) dx.
\]

In (1.3), \( \rho = \rho(\delta) \) is the unique nonnegative solution of the equation

\[
cp - \delta = \lambda - \lambda \tilde{p}(\rho).
\]

We assume the function \( w(x_1, x_2) \) to be such that the second term of (1.3) is finite. It is easy to see that (1.3) has a unique solution by identifying the Laplace transforms of the equation or applying the principle of contraction maps. Further discussion of (1.4) and its discrete analogue can be found in Cheng et al. (2000) and Gerber and Shiu (1999).

For the purpose of our analysis, the renewal equation (1.3) may be rewritten (Lin and Willmot (1999)) as follows:

\[
\phi(u) = \frac{1}{1 + \beta} \int_0^u \phi(u - x) dG(x) + \frac{1}{1 + \beta} H(u), \quad u \geq 0,
\]

where the DF \( G(x) = 1 - \tilde{G}(x) \) is given by

\[
\tilde{G}(x) = \frac{\tilde{P}(x) - e^{\rho x} \int_x^\infty e^{-\rho y} dP(y)}{\rho \int_0^\infty e^{-\rho y} P(y) dy},
\]

\[
\beta = \frac{(1 + \theta) \rho_1}{\int_0^\infty e^{-\rho y} P(y) dy},
\]

\[
H(u) = \frac{e^{\rho u} \int_u^\infty e^{-\rho y} \int_0^\infty w(x, y-x) dP(y) dx}{\int_0^\infty e^{-\rho y} P(y) dy}.
\]

The solution of (1.5) exists uniquely and can be proved using Laplace transformation or the principle of contraction maps.

An alternative expression for \( G(x) \) is given by

\[
\tilde{G}(x) = \frac{\int_0^\infty e^{-\rho y} \tilde{P}(x + y) dy}{\int_0^\infty e^{-\rho y} P(y) dy}, \quad x \geq 0.
\]

Furthermore, \( G(x) \) is differentiable and

\[
G'(x) = \frac{e^{\rho x} \int_x^\infty e^{-\rho y} dP(y)}{\int_0^\infty e^{-\rho y} P(y) dy}, \quad x \geq 0.
\]
Define now the associated compound geometric DF \( K(u) = 1 - \tilde{K}(u) \) by

\[
\tilde{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1 + \beta} \left( \frac{1}{1 + \beta} \right)^n \tilde{G}^*(u), \quad u \geq 0,
\]  

(1.11)

where \( \tilde{G}^*(u) \) is the tail of the \( n \)-fold convolution of \( G(u) \). It is easy to see that \( \tilde{K}(u) \) is the solution to the following renewal equation:

\[
\tilde{K}(u) = \frac{1}{1 + \beta} \int_0^u \tilde{K}(u - x) \, dG(x) + \frac{1}{1 + \beta} \tilde{G}(u), \quad u \geq 0.
\]  

(1.12)

\( \tilde{K}(u) \) may also be viewed as the Laplace transform at the time of ruin \( T \). To see this, we let \( w(x_1, x_2) = 1 \). It follows immediately from (1.8) and (1.9) that \( H(u) = \tilde{G}(u) \). Thus, with \( w(x_1, x_2) = 1, \phi(u) \) is the Laplace transform of \( T \) and \( \tilde{K}(u) = \phi(u) \).

We now look at the special case when \( \delta = 0 \). It is clear that

\[
G(x) = \frac{1}{p_{1}} \int_0^x \tilde{P}(y) \, dy.
\]  

(1.13)

Thus, in this case,

\[
\tilde{K}(u) = \psi(u).
\]  

(1.14)

We now state a theorem which shows that the solution to (1.5) can be expressed in terms of \( \tilde{K}(u) \).

**Theorem 1.1** (Lin and Willmot (1999)). The solution \( \phi(u) \) to (1.5) may be expressed as

\[
\phi(u) = \frac{1}{\beta} \int_0^u H(u - x) \, dK(x) + \frac{1}{1 + \beta} H(u),
\]  

(1.15)

\[
\phi(u) = -\frac{1}{\beta} \int_0^u \tilde{K}(u - x) \, dH(x) - \frac{H(0)}{\beta} \tilde{K}(u) + \frac{1}{\beta} H(u).
\]  

(1.16)

If \( H(u) \) is differentiable, then \( \phi(u) \) may be expressed as

\[
\phi(u) = -\frac{1}{\beta} \int_0^u \tilde{K}(u - x) H'(x) \, dx - \frac{H(0)}{\beta} \tilde{K}(u) + \frac{1}{\beta} H(u), \quad u \geq 0.
\]  

(1.17)

The proof of Theorem 1.1 and a detailed discussion of \( G(x), H(u) \) and \( K(u) \) are given in Lin and Willmot (1999).

2. Equilibrium distributions and reliability classifications

Equilibrium distributions associated with a given distribution function and reliability classifications play an important role in our analysis. Many results in the following sections heavily depend on equilibrium distributions. Reliability classifications enable us to derive upper and lower bounds for functions of the deficit.

In what follows, we briefly discuss equilibrium distributions and some reliability classes. Let \( P(x), x > 0 \), be a distribution function. The equilibrium DF \( P_1(x) = 1 - P_1(x) \) of \( P(x) \) is defined by \( P_1(x) = \int_0^x \tilde{P}(y) \, dy / p_1 \). In the present context \( P_1(x) \) may be interpreted as the DF of the amount of the drop in the surplus level, given that there is a drop (e.g., Bowers et al. (1997, Chapter 12)). The DF \( P(x) \) is said to be decreasing (increasing) failure rate or DFR (IFR) if \( \tilde{P}(x + y) / \tilde{P}(x) \) is nondecreasing (nonincreasing) in \( x \) for fixed \( y \geq 0 \). A larger class of distributions is the new worse (better) than used or NWU (NBU) class with \( \tilde{P}(x + y) \geq (\leq) \tilde{P}(x) \tilde{P}(y) \) for all \( x \geq 0 \) and \( y \geq 0 \).
The mean residual lifetime of \( P(x) \) is defined by
\[
\frac{\int_{x}^{\infty} \bar{P}(y) \, dy}{\bar{P}(x)} = r_P(x) = \frac{r_P(0) \bar{P}_1(x)}{\bar{P}(x)}, \quad x \geq 0.
\] (2.1)

Another class larger than the DFR (IFR) class is the increasing (decreasing) mean residual lifetime or IMRL (DMRL) class for which \( r_P(x) \) is nondecreasing (nonincreasing) for \( x \geq 0 \). Similarly, \( P(x) \) is new worse (better) than used in expectation or NWUE (NBUE) if \( r_P(x) \geq (\leq)r_P(0) \), or equivalently \( \bar{P}_1(x) \geq (\leq)\bar{P}(x) \), \( x \geq 0 \). See Fagiuoli and Pellerey (1994) and Lin and Willmot (1999) and references therein for further details on these classifications.

The notion of higher order equilibrium DFs is also needed in what follows. Let \( \bar{P}_n(x) = 1 - \bar{P}_n(x) \) be the equilibrium DF of \( P_1(x) \). The DF \( P_2(x) \) is called the second-order equilibrium DF of \( P(x) \). Similarly, we define the \( n \)th order equilibrium DF of \( P(x) \) by
\[
P_n(x) = 1 - \bar{P}_n(x) = \int_{0}^{\infty} \frac{\pi_{n-1}(y)}{\int_{0}^{\infty} \pi_{n-1}(y) \, dy} \, dy, \quad n = 1, 2, \ldots ,
\] (2.2)

where \( \bar{P}_0(x) = 1 - P_0(x) = \bar{P}(x) \). It can be shown that the mean of the DF \( P_n(x) \), \( n = 1, 2, \ldots \), is given by
\[
\int_{0}^{\infty} \bar{P}_n(x) \, dx = \frac{p_{n+1}}{(n+1)p_n},
\] (2.3)

and that the relation between the DFs \( P_n(x) \) and \( P(x) \) is given by
\[
\bar{P}_n(x) = \frac{1}{p_n} \int_{0}^{\infty} (y - x)^n \, dP(y).
\] (2.4)

See Hesselager et al. (1998) and references therein for further details.

The connection of the equilibrium DFs \( P_n(x) \) of \( P(x) \) and the equilibrium DFs \( G_n(x) \) of \( G_0(x) = G(x) \), where \( P(x) \) is the individual claim size DF and \( G(x) \) is the associated “claim size” DF to the renewal equation (1.5) can now be established.

It is easy to see from (1.9) that in the special case when \( \delta = 0 \), \( G(x) = P_1(x) \) since \( \rho = 0 \). Thus, \( G_n(x) = P_{n+1}(x) \) in this case. Let \( \mu_n(\rho) \) be the \( n \)th moment of \( G(x) \), i.e.
\[
\mu_n(\rho) = \int_{0}^{\infty} x^n \, dG(x).
\] (2.5)

Thus,
\[
\mu_n(0) = \int_{0}^{\infty} x^n \, dP_1(x) = \frac{1}{p_1} \int_{0}^{\infty} x^n \bar{P}(x) \, dx = \frac{p_{n+1}}{(n+1)p_1}.
\] (2.6)

For \( \rho > 0 \), we have
\[
\mu_n(\rho) = \mu_n(0) \frac{\int_{0}^{\infty} e^{-\rho x} \, dP_{n+1}(x)}{\int_{0}^{\infty} e^{-\rho x} \, dP_1(x)}.
\] (2.7)

The equilibrium DF \( G_n(x) \) of \( G(x) \) can be expressed in terms of the equilibrium DF \( P_n(x) \). In fact we have, analogous to (1.9),
\[
\bar{G}_n(x) = \frac{\int_{0}^{\infty} e^{-\rho y} \bar{P}_n(x+y) \, dy}{\int_{0}^{\infty} e^{-\rho y} \bar{P}_n(y) \, dy}.
\] (2.8)

The derivation of (2.7) and (2.8) can be found in Section 3 of Lin and Willmot (1999).
3. Convolution of equilibrium distributions and compound geometric tails

In this section we introduce two auxiliary functions which involve the convolution of the compound geometric tail \( \tilde{K}(u) \) and the \( n \)th equilibrium tail \( \tilde{G}_n(x) \) of the “claim size” distribution \( G(x) = G_0(x) \). Both functions are necessary in computing the moments of the (discounted) deficit and the moments at the time of ruin as well as other important quantities in classical ruin theory, as will become clear in Sections 4 and 6.

Define \( g_{-1}(\rho) = p_1 \), \( \mu_0(\rho) = 1 \), and \( g_n(\rho) \) to be the mean of the \( n \)th equilibrium DF \( G_n(x) \) of \( G(x) \). It follows from (2.3) that

\[
g_n(\rho) = \frac{\mu_{n+1}(\rho)}{(n+1)\mu_n(\rho)}, \quad n = 0, 1, 2, \ldots \quad (3.1)
\]

Also define

\[
\alpha_n(u, \rho) = g_n(\rho) \left\{ \int_0^u \tilde{K}(u-x) \, dG_{n+1}(x) + \tilde{G}_{n+1}(u) \right\}, \quad n = -1, 0, 1, 2, \ldots \quad (3.2)
\]

Evidently, \( \alpha_{-1}(u, \rho) = p_1(1 + \beta)\tilde{K}(u) \) from (1.12), and

\[
\alpha_n(u, \rho) = \int_0^u \tilde{K}(u-x)\tilde{G}_n(x) \, dx + \int_0^\infty \tilde{G}_n(x) \, dx, \quad n = 0, 1, 2, \ldots \quad (3.3)
\]

We will derive an alternative representation for \( \alpha_n(u, \rho) \), beginning with the following recursive relation.

**Lemma 3.1.**

\[
\alpha_{n+1}(u, \rho) = \frac{1}{g_n(\rho)} \int_u^\infty \alpha_n(t, \rho) \, dt - \int_u^\infty \tilde{K}(t) \, dt, \quad n = -1, 0, 1, 2, \ldots \quad (3.4)
\]

**Proof.** For \( n = -1, 0, 1, 2, \ldots \), one has from (3.2) that

\[
\frac{1}{g_n(\rho)} \int_u^\infty \alpha_n(t, \rho) \, dt = \int_u^\infty \int_0^t \tilde{K}(t-x) \, dG_{n+1}(x) \, dt + \int_0^\infty \tilde{G}_{n+1}(t) \, dt.
\]

By interchanging the order of integration, we have

\[
\int_u^\infty \int_0^t \tilde{K}(t-x) \, dG_{n+1}(x) \, dt = \int_0^\infty \int_u^t \tilde{K}(t-x) \, dG_{n+1}(x) \, dt + \int_u^\infty \int_x^\infty \tilde{K}(t-x) \, dG_{n+1}(x) \, dt
\]

\[
= \int_0^\infty \int_0^u \tilde{K}(t) \, dt \, dG_{n+1}(x) + \tilde{G}_{n+1}(u) \int_0^\infty \tilde{K}(t) \, dt.
\]

Integration by parts yields

\[
\int_0^u \tilde{K}(t) \, dt \, dG_{n+1}(x) = -\tilde{G}_{n+1}(x) \int_0^u \tilde{K}(t) \, dt \bigg|_{t=0}^{t=u} + \int_0^u \tilde{K}(u-x) \tilde{G}_{n+1}(x) \, dx
\]

\[
= -\tilde{G}_{n+1}(u) \int_0^\infty \tilde{K}(t) \, dt + \int_0^\infty \tilde{K}(t) \, dt + \int_0^u \tilde{K}(u-x) \tilde{G}_{n+1}(x) \, dx.
\]

Thus

\[
\frac{1}{g_n(\rho)} \int_u^\infty \alpha_n(t, \rho) \, dt = \int_u^\infty \tilde{K}(t) \, dt + \int_0^u \tilde{K}(u-x) \tilde{G}_{n+1}(x) \, dx + \int_0^\infty \tilde{G}_{n+1}(t) \, dt
\]

\[
= \int_u^\infty \tilde{K}(t) \, dt + \alpha_{n+1}(u, \rho)
\]

using (3.3).
We have the following alternative representation for $\alpha_n(u, \rho)$ from the lemma.

**Theorem 3.1.**

\[
\alpha_0(u, \rho) = \beta \int_{u}^{\infty} \tilde{K}(x) \, dx \tag{3.5}
\]

and for $n = 1, 2, 3, \ldots$,

\[
\alpha_n(u, \rho) = \frac{\beta}{\mu_n(\rho)} \int_{u}^{\infty} (x-u)^n \tilde{K}(x) \, dx - \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mu_{n-j}(\rho)}{\mu_n(\rho)} \int_{u}^{\infty} (x-u)^j \tilde{K}(x) \, dx. \tag{3.6}
\]

**Proof.** Since $\alpha_{-1}(u, \rho) = p_1(1 + \beta) \tilde{K}(u)$, (3.4) yields with $n = -1$,

\[
\alpha_0(u, \rho) = \frac{p_1(1 + \beta)}{p_1} \int_{u}^{\infty} \tilde{K}(t) \, dt - \int_{u}^{\infty} \tilde{K}(t) \, dt = \beta \int_{u}^{\infty} \tilde{K}(t) \, dt,
\]

proving (3.5). For $j \geq 0$, one has by interchanging the order of integration

\[
\int_{u}^{\infty} \sum_{j=0}^{\infty} (y-t)^j \tilde{K}(y) \, dy \, dt = \frac{1}{j+1} \int_{u}^{\infty} (y-u)^{j+1} \tilde{K}(y) \, dy. \tag{3.7}
\]

With the help of (3.7) and Lemma 3.1, it is straightforward to verify that (3.6) holds by induction on $n$. □

In the special case when $\delta = 0$, it is convenient to define

\[
\tau_n(u) = \int_{0}^{u} \psi(u-x) \tilde{P}_n(x) \, dx + \int_{u}^{\infty} \tilde{P}_n(x) \, dx, \quad n = 1, 2, 3, \ldots, \tag{3.8}
\]

and so

\[
\tau_n(u) = \frac{p_{n+1}}{(n+1)p_n} \left\{ \int_{0}^{u} \psi(u-x) \, dP_{n+1}(x) + \tilde{P}_{n+1}(u) \right\}, \quad n = 1, 2, 3, \ldots \tag{3.9}
\]

We have the following corollary.

**Corollary 3.1.**

\[
\tau_1(u) = \theta \int_{u}^{\infty} \psi(x) \, dx \tag{3.10}
\]

and for $n = 2, 3, 4, \ldots$,

\[
\tau_n(u) = \frac{np_1 \theta}{p_n} \int_{u}^{\infty} (x-u)^{n-1} \psi(x) \, dx - \sum_{j=0}^{n-2} \binom{n}{j} \frac{p_{n-j}}{p_n} \int_{u}^{\infty} (x-u)^j \psi(x) \, dx. \tag{3.11}
\]

**Proof.** When $\delta = 0$, $\rho = 0$, $G(x) = P_1(x)$, and (2.6) holds. Thus $\tilde{P}_n(x) = \tilde{G}_{n-1}(x)$. Since $\tilde{K}(u) = \psi(u)$ in this case, it follows from (3.3) and (3.8) that

\[
\tau_n(u) = \alpha_{n-1}(u, 0), \quad n = 1, 2, 3, \ldots \tag{3.12}
\]

Since $\beta = \theta$ when $\rho = 0$, (3.5) becomes (3.10). Then (3.11) follows directly from (2.6) and (3.6). □
It is evident in the proofs of the above theorems that the existence of certain moments of the claim size distribution \( P(x) \) is required. For instance, if \( p_{n+2} \) is finite, then \( \alpha_n(u, \rho) \) and \( \tau_{n+1}(u) \) are both finite. In order to focus on analytical representation of functions of interest, we always assume that required moments are finite throughout the rest of the paper.

In the following examples we demonstrate how to evaluate \( \alpha_n(u, \rho) \) and \( \tau_n(u) \) when \( P(x) \) is a combination of exponentials or a mixture of Erlangs. The case when \( P(x) \) is a combination of exponentials is considered in Gerber et al. (1987) and Dufresne and Gerber (1988). From a formal standpoint, a combination of two exponentials is very important since it may be viewed as providing a Tijms approximation to the relevant quantity (see Willmot (1997) and references therein). In particular, Tijms approximations for \( \alpha_n(u, \rho) \) and \( \tau_n(u) \) allow for corresponding approximations for moments at the time of ruin and the deficit of ruin, as is discussed in later sections. In what follows, a combination of two exponentials may be viewed in this context.

**Example 3.1 (Combinations of exponentials).** The case where \( P(x) \) is a combination of exponentials is first considered in Gerber et al. (1987) and Dufresne and Gerber (1988) in which a closed form expression for the probability of ruin and the distribution of the deficit at the time of ruin are derived.

We begin with an exponential claim size DF \( P(x) = e^{-\mu x} \). In this case, for all \( n = 0, 1, \ldots \), \( \tilde{P}_n(x) = e^{-\mu x} \). It follows from (2.8) that

\[
\bar{G}_n(x) = e^{-\mu x}.
\]

Thus, \( \mu_n(\rho) = \mu_n(0) = n! / \mu^n \) from (2.7). Moreover,

\[
\bar{K}(u) = \frac{1}{1 + \beta} e^{-Ru},
\]

where

\[
\frac{1}{1 + \beta} = \frac{\mu}{\rho + \mu} \frac{1}{1 + \theta},
\]

\[
R = \frac{\beta \mu}{1 + \beta} = \frac{\theta \mu}{1 + \theta} + \frac{\rho}{\rho + \mu} \frac{\mu}{1 + \theta}
\]

(see Example 4.1 of Lin and Willmot (1999)). A direct computation from (3.2) yields

\[
\alpha_n(u, \rho) = \frac{1}{\mu} e^{-Ru}.
\]

We now assume that the function \( K(u) \) is of the form

\[
\tilde{K}(u) = C_1 e^{-R_1 u} + C_2 e^{-R_2 u}.
\]

This is the case when \( P(x) \) is a combination of two exponentials or a Tijms approximation is used. A detailed discussion of this is also given in Example 4.1 of Lin and Willmot (1999).

Since

\[
\int_u^\infty (x - u) \hat{K}(x) \, dx = C_1 \int_0^\infty x \, e^{-R_1 x} \, dx + C_2 \int_0^\infty x \, e^{-R_2 x} \, dx = C_1 j! R_1^{-j-1} e^{-R_1 u} + C_2 j! R_2^{-j-1} e^{-R_2 u},
\]

Theorem 3.1 yields

\[
\alpha_n(u, \rho) = C_{1,n} e^{-R_1 u} + C_{2,n} e^{-R_2 u},
\]

(3.15)
where

\[
C_{1,n}^* = C_1 \left[ \frac{\beta n! R_1^{n-1}}{\mu_n(\rho)} - \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mu_{n-j}(\rho) j! R_1^{j-1}}{\mu_n(\rho)} \right],
\]

\[
C_{2,n}^* = C_2 \left[ \frac{\beta n! R_2^{n-1}}{\mu_n(\rho)} - \sum_{j=0}^{n-1} \binom{n}{j} \frac{\mu_{n-j}(\rho) j! R_2^{j-1}}{\mu_n(\rho)} \right].
\]

With \( \rho = 0 \), we obtain from (2.6) that

\[
\tau_n(u) = C_{1,n}^{**} e^{-R_1 u} + C_{2,n}^{**} e^{-R_2 u},
\]

(3.16)

\[
C_{1,n}^{**} = \frac{n! C_1 R_1^{n-1}}{p_n} \left[ p_1 \theta - \sum_{j=1}^{n-1} \frac{P_{j+1} R_1^j}{(j+1)!} \right],
\]

\[
C_{2,n}^{**} = \frac{n! C_2 R_2^{n-1}}{p_n} \left[ p_1 \theta - \sum_{j=1}^{n-1} \frac{P_{j+1} R_2^j}{(j+1)!} \right].
\]

For a general linear combination of exponentials, the derivation is similar and is omitted here.

Mixtures of Erlangs are an important distributional class in modeling insurance losses for the reason that any continuous distribution on \((0, \infty)\) may be approximated arbitrarily accurately by a distribution of this type (see Tijms (1994, pp. 162–164)).

**Example 3.2 (Mixtures of Erlangs).** The density function of a mixture of Erlangs is given by

\[
P'(x) = \sum_{k=1}^{r} q_k \frac{\mu(\mu x)^{k-1} e^{-\mu x}}{(k-1)!}, \quad x \geq 0,
\]

where \(\{q_1, q_2, \ldots, q_r\} \) is a probability distribution. Example 4.2 of Lin and Willmot (1999) shows that

\[
K(u) = e^{-\mu u} \sum_{j=0}^{\infty} \tilde{C}_j \left( \frac{(\mu u)^j}{j!} \right), \quad u \geq 0,
\]

where

\[
\tilde{C}_j = \frac{1}{1 + \beta} \sum_{k=1}^{j} q_k \tilde{C}_{j-k} + \frac{1}{1 + \beta} \sum_{k=j+1}^{\infty} q_k^*, \quad j = 1, 2, 3, \ldots
\]

Here it is assumed that \( \tilde{C}_0 = (1 + \beta)^{-1} \), and

\[
q_k = \frac{\sum_{j=k}^{r} q_j \left( \frac{\mu}{\mu + \rho} \right)^{j-k}}{\sum_{j=1}^{r} q_j \sum_{i=0}^{j-1} \left( \frac{\mu}{\mu + \rho} \right)^i},
\]

for \( k = 1, 2, \ldots, r \), and \( q_k^* = 0 \), for \( k > r \).
Now, for any \(i = 0, 1, \ldots\), and \(j = 0, 1, \ldots\),
\[
\int_0^\infty (x - u)^j \frac{(\mu x)^i}{i!} e^{-\mu x} \, dx = \frac{\mu^i}{i!} \int_0^\infty x^i (x + u)^j e^{-\mu x} \, dx = \frac{\mu^i}{i!} e^{-\mu u} \sum_{k=0}^i \binom{i}{k} u^k \int_0^\infty x^{j+i-k} e^{-\mu x} \, dx
\]
\[
= \mu^{-j-1} e^{-\mu u} \sum_{k=0}^i \frac{(j + i - k)! (\mu u)^k}{(i - k)! k!}.
\]
Thus,
\[
\alpha_n(u, \rho) = e^{-\mu u} \sum_{k=0}^\infty \tilde{C}_{k,n}^* \frac{(\mu u)^k}{k!},
\]
where
\[
\tilde{C}_{k,n}^* = \sum_{i=0}^\infty \tilde{C}_{k+i}^* \left[ \frac{\beta \mu_{n-i}(\rho)}{\mu_n(\rho)} \frac{(n+i)!}{i!} - \sum_{j=0}^{n-i} \binom{n}{j} \frac{\mu_{n-j}(\rho) \mu_{j-1}(\rho)}{\mu_n(\rho)} \frac{(j+i)!}{l!} \right].
\]
In the rest of this paper, we turn to evaluation of \(\phi(u)\) for a particular weight function \(w(x, y)\).

**4. The deficit at the time of ruin**

In this section we consider in more detail the (possibly discounted) deficit at the time of ruin beginning with the moments. We are focusing on their analytic solutions. For numerical algorithms, see Dickson et al. (1995) and the references therein.

Consider the special case \(w(x_1, x_2) = x_2^k\), where \(k\) is a positive integer. A relatively simple representation exists for \(\phi(u)\) in (1.1).

**Theorem 4.1.** For \(k = 1, 2, 3, \ldots\),
\[
E[|e^{-\delta T}U(T)^k| \mid T < \infty] = \frac{1}{\beta \psi(u)} \left[ k \mu_{k-1}(\rho) \alpha_{k-1}(u, \rho) - \mu_k(\rho) \tilde{K}(u) \right],
\]
where \(\mu_n(\rho) = \int_0^\infty x^n \, dG(x)\) is given in (2.7) and \(\alpha_n(u, \rho)\) is given in Theorem 3.1.

**Proof.** Using (2.4) and (2.8), Eq. (1.8) becomes, with \(w(x_1, x_2) = x_2^k\),
\[
H(u) = \frac{e^{\mu u} \int_0^\infty e^{-\rho x} P(x) \, dx}{\int_0^\infty e^{-\rho x} P(y) \, dy} = \frac{p_k \int_0^\infty e^{-\rho y} \tilde{P}_k(y) \, dx}{\int_0^\infty e^{-\rho y} \tilde{P}_y(y) \, dy} = \frac{p_k \int_0^\infty e^{-\rho y} \tilde{P}_k(y) \, dy}{\int_0^\infty e^{-\rho y} \tilde{P}_y(y) \, dy} \tilde{G}_k(u).
\]
Furthermore, using (2.3),
\[
H(u) = \frac{p_{k+1} \int_0^\infty e^{-\rho y} P_{k+1}(y) \, dy}{(k + 1) \int_0^\infty e^{-\rho y} P_1(y) \, dy} \tilde{G}_k(u),
\]
and from (2.6) and (2.7), we obtain the simple result
\[
H(u) = \mu_k(\rho) \tilde{G}_k(u).
\]
Then, using (3.1), since \( \frac{G_k(u)}{g_{k-1}(\rho)} = -k \mu_{k-1}(\rho) G_{k-1}(u) \), we obtain

\[
H'(u) = -\frac{\mu_k(\rho)}{g_{k-1}(\rho)} G_{k-1}(u) = -k \mu_{k-1}(\rho) G_{k-1}(u),
\]

and from (1.17),

\[
\begin{align*}
\phi(u) &= \frac{k \mu_{k-1}(\rho)}{\beta} \int_0^u \tilde{K}(u-x) G_{k-1}(x) \, dx + \frac{\mu_k(\rho)}{\beta} \int_u^\infty \frac{G_{k-1}(x)}{g_{k-1}(\rho)} \, dx - \frac{\mu_k(\rho)}{\beta} \tilde{K}(u) \\
&= \frac{k \mu_{k-1}(\rho)}{\beta} \left\{ \int_0^u \tilde{K}(u-x) G_{k-1}(x) \, dx + \int_u^\infty \tilde{G}_{k-1}(x) \, dx \right\} - \frac{\mu_k(\rho)}{\beta} \tilde{K}(u) \\
&= \frac{k \mu_{k-1}(\rho)}{\beta} \alpha_{k-1}(u, \rho) - \frac{\mu_k(\rho)}{\beta} \tilde{K}(u),
\end{align*}
\]

using (3.3). That is,

\[
E\{e^{-\delta T} | U(T) |^k I(T < \infty) \} = \frac{k \mu_{k-1}(\rho)}{\beta} \alpha_{k-1}(u, \rho) - \frac{\mu_k(\rho)}{\beta} \tilde{K}(u),
\]

and division by \( \beta \) gives (4.1).

When \( k = 1 \), with the help of (3.5), (4.3) reduces to

\[
E\{e^{-\delta T} | U(T) | I(T < \infty) \} = \int_u^\infty \tilde{K}(x) \, dx - \frac{\mu_1(\rho)}{\beta} \tilde{K}(u).
\]

When \( \delta = 0 \) we have the following important corollary.

**Corollary 4.1.** For \( k = 1, 2, 3, \ldots \),

\[
E\{|U(T)| | T < \infty \} = \frac{p_k}{p_1 \psi(u)} \tau_k(u) - \frac{p_{k+1}}{(k+1) p_1 \psi(u)},
\]

where \( \tau_k(u) \) is given in Corollary 3.1.

**Proof.** When \( \delta = 0 \), \( \tilde{K}(u) = \psi(u) \), \( \alpha_{k-1}(u, 0) = \tau_k(u) \), \( \mu_n(0) \) is given by (2.6), and \( \beta = \theta \). Then (4.1) reduces to (4.5).

When \( k = 1 \), (4.5) reduces to (using (3.10))

\[
E\{|U(T)| | T < \infty \} = \int_u^\infty \frac{\psi(x)}{\psi(u)} \, dx - \frac{p_2}{2 p_1 \theta}.
\]

The second moment of the deficit, hence the variance of the deficit, can also easily be evaluated from (4.5).

As demonstrated in Examples 3.1 and 3.2, \( \psi(u) \) and \( \tau_k(u) \) can be computed explicitly when \( P(x) \) is a combination of exponentials, a mixture of Erlangs, or approximated using a Tijms approximation. Thus, with the help of (4.5) the moments of the deficit \( |U(T)| \) are also obtainable as shown in the following examples.

**Example 4.1 (Combinations of exponentials).** We again begin with an exponential. In this case,

\[
E\{|U(T)| | T < \infty \} = \frac{p_k}{p_1 \psi(u)} \tau_k(u) - \frac{p_{k+1}}{(k+1) p_1 \psi(u)} = \frac{k! \mu^{-k}}{(k+1) \mu^{-1} \theta (1 + \theta)^{-1} e^{-Ru}} - \frac{(k+1)! \mu^{-k-1}}{(k+1) \mu^{-1} \theta} = k! \mu^{-k},
\]

which agrees with the fact that in the exponential case the deficit is also exponentially distributed and it is independent of the time of ruin.
Next, we examine a combination of two exponentials. This is also the case when a Tijms approximation is used. It follows from (3.16) that

$$E[f_j U(T) \mid T < \infty] = \frac{p_k}{p_1 \theta} \frac{T_k(u)}{\psi(u)} - \frac{p_{k+1}}{(k+1)p_1 \theta} \frac{\hat{C}_{1,k}}{C_1} e^{-R_1 u} + \frac{\hat{C}_{2,k}}{C_2} e^{-R_2 u},$$

(4.7)

where

$$\hat{C}_{1,k} = \frac{k! C_1 R_1^{-k}}{p_1 \theta} \left[ p_1 - \sum_{j=1}^{k} \frac{p_{j+1} R_j^j}{(j+1)!} \right],$$

$$\hat{C}_{2,k} = \frac{k! C_2 R_2^{-k}}{p_1 \theta} \left[ p_1 - \sum_{j=1}^{k} \frac{p_{j+1} R_2^j}{(j+1)!} \right].$$

**Example 4.2 (Mixtures of Erlangs).** For a mixture of Erlangs claim size distribution, it follows immediately from Example 3.2 that the $k$th unconditional discounted moment is

$$E[e^{-\delta^T} |U(T)|^k I(T < \infty)] = e^{-\mu u} \sum_{j=0}^{\infty} \hat{c}_{j,k} \frac{(\mu u)^j}{j!},$$

(4.8)

where

$$\hat{c}_{j,k} = \frac{1}{\beta} [k \mu_{k-1}(\rho) \hat{C}_{j,k-1}^\ast - \mu_k(\rho) \hat{C}_j].$$

The constants $\hat{c}_{j,k-1}$ and $\hat{C}_j$ are given in Example 3.1.

We remark that when the individual claim size is a combination of exponentials or a mixture of Erlangs, the density function of the deficit is of the same type but with new weights. These weights are expressed as functions of the initial surplus $u$. As a result, the moments of the deficit can be calculated accordingly (for details, see Willmot (2000)). The method we employed in this section provides a different approach in calculating the moments of the deficit. Under this method, the moments of the deficit are calculated directly. The coefficients in (4.7) and (4.8) are independent of the initial surplus $u$, which allows for the calculation of the moments of the deficit for different values of $u$ without reevaluating the distribution of the deficit. Another advantage of this approach is that we are able to evaluate the expected discounted deficit using (4.1), which the approach in Willmot (2000) cannot. Next, we consider bounds for functions of the deficit.

**Theorem 4.2.** If $P(x)$ is IMRL (DMRL), then for $k = 1, 2, 3, \ldots$,

$$E[e^{-\delta T} |U(T)|^k I(T < \infty)] \geq (\leq) \mu_k(\rho) \tilde{K}(u).$$

(4.9)

**Proof.** By Theorem 3.2(b) of Lin and Willmot (1999), $G(x)$ is IMRL (DMRL). This implies (Fagioli and Pellerey, 1994) that the $j$th equilibrium DF $G_j(x)$ is NWUE (NBUE), i.e., $G_{j+1}(x) \geq (\leq) G_j(x)$. Then by (4.2),

$$H(u) = \mu_k(\rho) \tilde{G}_k(u) \geq (\leq) \mu_k(\rho) \tilde{G}_{k-1}(u) \geq (\leq) \cdots \geq (\leq) \mu_k(\rho) \tilde{G}(u).$$

Thus, Theorem 4.1 of Lin and Willmot (1999) applies, which shows that if $H(u) \leq (\geq) c \tilde{G}(u)$, then $\phi(u) \leq (\geq) c \tilde{K}(u)$, and the result is proved.

To derive a simple bound for more general functions of the deficit, we recall a result in Lin and Willmot (1999).
Lemma 4.1. If
\[ \eta(x) = \frac{1}{P(x)} \int_x^\infty w(x, y - x) dP(y) \leq (\geq) c, \] (4.10)

where 0 < c < \infty, then \( \phi(u) \leq (\geq) e^{\tilde{K}(u)} \).

With the help of Lemma 4.1, we have the following simple bound.

Theorem 4.3. Suppose that \( A(y) = 1 - \tilde{A}(y) \) is a DF on \((0, \infty)\) satisfying
\[ \frac{\tilde{P}(x + y)}{P(x)} \geq (\leq) \tilde{A}(y), \quad x \geq 0, \quad y \geq 0. \] (4.11)
Then if \( w(x) \) is nondecreasing in \( x \),
\[ E[e^{-\delta T} w(|U(T)|) I(T < \infty)] \geq (\leq) \int_0^\infty w(y) dA(y) \int_0^\infty \tilde{K}(u) \] (4.12)

Proof. Let \( Z_x \) for \( x \geq 0 \) be a random variable such that \( \Pr(Z_x > y) = \tilde{P}(x + y) / \tilde{P}(x) \). Then with \( w(x_1, x_2) = w(x_2), \) (4.10) may be expressed as \( \eta(x) = E[w(Z_x)] \). Then by (4.11) and Ross (1996, p. 405), if \( w(x) \) is nondecreasing, then \( \eta(x) \geq (\leq) \int_0^\infty w(y) dA(y) \). \( \square \)

It is clear from the proof of Theorem 4.3 that if (4.11) holds and \( w(x) \) is nonincreasing in \( x \), then inequality (4.12) is reversed.

The key to the application of Theorem 4.3 is to identify the DF \( A(y) \) for a given \( P(x) \). If \( P(x) \) is NWU (NBU), then (4.11) holds with \( A(y) = P(y) \). For more details on the choice of \( P(x) \), see Willmot and Lin (1997).

It is worth noting that Eq. (4.12) may be written when \( \delta = 0 \) as
\[ E[w(|U(T)|) \mid T < \infty] \geq (\leq) \int_0^\infty w(y) dA(y). \] (4.13)

In the special case when \( w(x_1, x_2) = x_1^k \) and \( P(x) \) is DFR (IFR), from Ross (1996, p. 405) and Theorem 3.5, \( \mu_k(\rho) \geq (\leq) p_k \), and Theorem 4.2 gives a tighter bound than Theorem 4.3.

If \( P(x) = 1 - \exp(-x/p_1) \) and \( w(x_1, x_2) = w(x_2) \), then \( \eta(x) = \int_0^\infty w(y) dP(y) \) independent of \( x \) and \( H(u) = \{ \int_0^\infty w(y) dP(y) \} \tilde{G}(u) \), implying that \( \phi(u) = \{ \int_0^\infty w(y) dP(y) \} \tilde{K}(u) \). Thus, with \( w(y) = e^{-xy} \),
\[ E[\exp(-\delta T - s|U(T)|) I(T < \infty)] = (1 + p_1 s)^{-1} E[e^{-s T} I(T < \infty)]. \] (4.14)

Eq. (4.14) may be viewed as the joint Laplace transform of the (defective) distribution of \( T \) and \(|U(T)|\), implying that \( T \) and \(|U(T)|\) are independent (conditional on \( T < \infty \)), and that \(|U(T)|\) is exponential with mean \( p_1 \). This result is given by Gerber (1979, p. 138). The Laplace transform of the deficit for arbitrary claim size DF can also be evaluated in a similar manner. In that case, we simply let \( w(x_1, x_2) = e^{-x_2}. \) However, the resulting formula will be complicated, except when the claim size is exponentially distributed as discussed above.

5. Surplus before the time of ruin

We now consider joint moments of the surplus \( U(T-) \) before ruin and the deficit \(|U(T)|\) at ruin may be obtained with \( \delta = 0 \) and \( w(x, y) = x^k y^k \). Then (1.8) becomes, using (2.4),
\[ H(u) = \frac{1}{p_1} \int_u^\infty x^k \int_x^\infty (y - x)^k dP(y) dx = \frac{p_2}{p_1} \int_u^\infty x^k \tilde{P}(x) dx, \quad u \geq 0. \] (5.1)
Thus,
\[ H'(u) = -\frac{P_k}{p_1} u^j \bar{P}_k(u), \]
and
\[ H(0) = \frac{P_{k+1}}{(k+1)p_1} \int_0^\infty x^j \, dP_{k+1}(x) = \frac{k! j! p_{k+j+1}}{(k+j+1)! p_1} \]
using Hesselager et al. (1998). Therefore, from (1.17)
\[
E\{[U(T-)]^j | U(T) = k \, I(T < \infty)\} = \frac{p_k}{p_1 \theta} \left\{ \int_0^u \psi(u-x) x^j \, dP_1(x) + \int_u^\infty x^j \, dP_1(x) \right\} - \frac{k! j! p_{k+j+1}}{(k+j+1)! p_1 \theta} \psi(u). \tag{5.2}
\]
Moments of \(U(T-)\) are obtainable from (5.2) with \(k = 0\). That is, for \(j = 1, 2, 3, \ldots\)
\[
E\{[U(T-)]^j | T < \infty\} = \frac{1}{\theta \psi(u)} \left\{ \int_0^u \psi(u-x) x^j \, dP_1(x) + \int_u^\infty x^j \, dP_1(x) \right\} - \frac{p_{j+1}}{(j+1) p_1 \theta}. \tag{5.3}
\]
The integrals in (5.3) are easily evaluated when the claim size distribution is a combination of exponentials or a mixture of Erlangs. We omit the details.

The defective density of the surplus \(U(T-)\) is also easily obtainable. Let \(\delta = 0, w(x_1, x_2) = e^{-sx_1} \) in (1.8). Thus,
\[
H(u) = \int_u^\infty e^{-sx} \, dP_1(x),
\]
and (1.17) yields
\[
E\{e^{-sU(T-)} I(T < \infty)\} = \frac{1}{\theta} \int_0^u \psi(u-x) e^{-sx} \, dP_1(x) + \frac{1}{\theta} \int_u^\infty e^{-sx} \, dP_1(x) - \frac{\psi(u)}{\theta} \int_0^\infty e^{-sx} \, dP_1(x) = \frac{1}{\theta} \int_0^u [\psi(u-x) - \psi(u)] e^{-sx} \, dP_1(x) + \frac{1}{\theta} \int_u^\infty [1 - \psi(u)] e^{-sx} \, dP_1(x). \tag{5.4}
\]
Hence, by the uniqueness of the Laplace transform, the defective density of the surplus \(U(T-)\) before the time of ruin is
\[
f(x) = \begin{cases} \frac{1}{\theta} [(1-P(x))/p_1][\psi(u-x) - \psi(u)], & x \leq u, \\ \frac{1}{\theta} [(1-P(x))/p_1][1 - \psi(u)], & x > u, \end{cases} \tag{5.5}
\]
which is in agreement with Dickson (1992).

6. Moments involving the time of ruin

In this section we employ the interpretation of \(\phi(u)\) in (1.1) as a Laplace transform with argument \(\delta\) to find moments involving \(T\). Our approach is to differentiate (1.5) with respect to \(\delta\) and show that the corresponding moment satisfies another defective renewal equation. However, to be able to differentiate (1.5), we need the following lemma.

**Lemma 6.1.** Suppose that \(\phi(u, \delta)\) is the solution of the following renewal equation:
\[
\frac{e}{\lambda} \phi(u, \delta) = \int_0^u \phi(u-x, \delta)m(x, \rho) \, dx + M(u, \delta), \tag{6.1}
\]
where $\rho = \rho(\delta)$ is the nonnegative solution of (1.4) and
\[
m(x, \rho) = \int_x^{\infty} e^{-\rho(y-x)} \, dP(y), \quad x \geq 0.
\] (6.2)

If the second moment $p_2$ of $P(x)$ is finite, $M(u, \delta)$ is a differentiable function with respect to $\delta$ on $[0, \delta_0)$ for some $\delta_0 > 0$, and $M_1(u, \delta) = \partial M(u, \delta)/\partial \delta$ is continuous in $u$, then $\phi(u, \delta)$ is differentiable with respect to $\delta$ on $[0, \delta_0)$ and $\phi_1(u, \delta) = \partial \phi(u, \delta)/\partial \delta$ is the solution of the following renewal equation:
\[
\frac{c}{\lambda} \phi_1(u, \delta) = \int_0^u \phi_1(u-x, \delta)m(x, \rho) \, dx + \int_0^u \phi(u-x, \delta)m_1(x, \delta) \, dx + M_1(u, \delta),
\] (6.3)

where $m_1(x, \rho) = \partial m(x, \rho)/\partial \delta$.

**Proof.** Obviously, (6.3) is a defective renewal equation and hence has a unique solution which we denote as $\phi_1(u, \delta)$. Define
\[
B(u, \delta, \Delta \delta) = \frac{\phi(u, \delta + \Delta \delta) - \phi(u, \delta)}{\Delta \delta} - \phi_1(u, \delta).
\]

Then, we have
\[
\frac{c}{\lambda} B(u, \delta, \Delta \delta) = \int_0^u B(u-x, \delta, \Delta \delta)m(x, \rho(\delta + \Delta \delta)) \, dx + C(u, \delta, \Delta \delta),
\]

where
\[
C(u, \delta, \Delta \delta) = \int_0^u \phi(u-x, \delta) \left[ \frac{m(x, \rho(\delta + \Delta \delta)) - m(x, \rho)}{\Delta \delta} - m_1(x, \delta) \right] \, dx
\]
\[
+ \frac{M(u, \delta + \Delta \delta) - M(u, \delta)}{\Delta \delta} - M_1(u, \delta).
\]

Let
\[
J(u, \delta, \Delta \delta) = \sup_{0 \leq x \leq u} |B(u, \delta, \Delta \delta)|.
\]

Then,
\[
\frac{c}{\lambda} J(u, \delta, \Delta \delta) \leq J(u, \delta, \Delta \delta) \int_0^u m(x, \rho(\delta + \Delta \delta)) \, dx + |C(u, \delta, \Delta \delta)|
\]
\[
\leq J(u, \delta, \Delta \delta) \int_0^u P(x) \, dx + |C(u, \delta, \Delta \delta)| \leq J(u, \delta, \Delta \delta)p_1 + |C(u, \delta, \Delta \delta)|.
\]

Thus,
\[
J(u, \delta, \Delta \delta) \leq \frac{1}{\theta p_1} |C(u, \delta, \Delta \delta)|.
\]

Hence, $\lim_{\Delta \delta \to 0} C(u, \delta, \Delta \delta) = 0$ implies that $\lim_{\Delta \delta \to 0} J(u, \delta, \Delta \delta) = 0$. The latter implies that
\[
\lim_{\Delta \delta \to 0} \frac{\phi(u, \delta + \Delta \delta) - \phi(u, \delta)}{\Delta \delta} = \phi_1(u, \delta),
\]
and we will prove the lemma.
It is obvious that
\[
\lim_{\Delta \delta \to 0} \frac{M(u, \delta + \Delta \delta) - M(u, \delta) - M_1(u, \delta)}{\Delta \delta} = 0.
\]

To show that the first term of \( C(u, \delta, \Delta \delta) \) goes to 0, we first show that
\[
\frac{m(x, \rho(\delta + \Delta \delta)) - m(x, \rho)}{\Delta \delta} - m_1(x, \delta)
\]
is bounded. It is easy to check that
\[
\left| \frac{m(x, \rho(\delta + \Delta \delta)) - m(x, \rho)}{\Delta \delta} - m_1(x, \delta) \right|
\]
\[
= \left| \int_x^\infty \exp[-\rho(\delta + \Delta \delta)(y - x)] - \exp[-\rho(\delta)(y - x)] \frac{\rho' \Delta \delta \exp[-\rho(\delta)(y - x)]}{\Delta \delta} \right| \, dP(y)
\]
\[
= \left| \Delta \delta \int_x^\infty [(\rho'(\delta_1))^2 \exp[-\rho(\delta_1)(y - x)](y - x)^2 - \rho''(\delta_1) \exp[-\rho(\delta_1)(y - x)](y - x)] \, dP(y) \right|
\]
\[
\leq \Delta \delta \left\{ p_2 \max_{0 \leq \delta_1 \leq \delta_0} [\rho'(\delta_1)]^2 + p_1 \max_{0 \leq \delta_1 \leq \delta_0} [\rho''(\delta_1)] \right\} < \infty.
\]
The second equality comes from the Taylor expansion of \( e^{-\rho(\delta + \Delta \delta)(y - x)} \) up to the second derivative. Since \( \phi(u - x, \delta) \) is continuous and hence is bounded, applying the bounded convergence theorem yields the result.

We now consider moments involving the time of ruin. It is easy to see that with (6.2), (1.5) can be rewritten as
\[
\frac{\phi(u)}{\lambda} = \int_0^u \frac{\phi(u - x)m(x, \rho)}{\lambda} \, dx + \int_x^\infty \exp[-\rho(\delta)(x - u)] \, w(x, y - x) \, dP(y) \, dx.
\]

Thus, Lemma 6.1 yields
\[
\frac{\phi(u)}{\lambda} \frac{\partial}{\partial \delta} \phi(u) = \int_0^u \left\{ \frac{\partial}{\partial \delta} \phi(u - x) \right\} m(x, \rho) \, dx + \rho'(\delta) \int_0^u \phi(u - x) \frac{\partial}{\partial \rho} m(x, \rho) \, dx
\]
\[
- \rho'(\delta) \int_x^\infty (x - u) \exp[-\rho(\delta)(x - u)] \, w(x, y - x) \, dP(y) \, dx.
\]

Next, define
\[
H(u, 0) = \frac{1}{p_1} \int_0^\infty \int_x^\infty w(x, y - x) \, dP(y) \, dx,
\]
\[
\phi_1(u) = E[Tw(U(T) - 1)](T < \infty).
\]

We then have the following theorem.

**Theorem 6.1.** The function \( \phi_1(u) \) satisfies the defective renewal equation
\[
\phi_1(u) = \frac{1}{1 + \theta} \int_0^u \phi_1(u - x) \, dP_1(x) + \frac{1}{1 + \theta} H_1(u),
\]
where
\[
H_1(u) = \frac{1}{\lambda p_1 \theta} \int_0^u \psi(u - x) \, H(x, 0) \, dx + \frac{1}{\lambda p_1 \theta} \int_x^\infty (x - u) \int_u^\infty w(x, y - x) \, dP(y) \, dx.
\]
Proof. It follows from (1.1) that \( \phi_1(u) = -\frac{\partial}{\partial \delta} \phi(u) \big|_{\delta=0} \). Also, from (6.2), \( m(x, 0) = \tilde{P}(x) \), and
\[
\frac{\partial}{\partial \rho} m(x, \rho) \big|_{\rho=0} = -\int_{x}^{\infty} (y - x) \, dP(y) = -p_1 \tilde{P}_1(x).
\] (6.10)

Then, differentiating (1.4) with respect to \( \delta \), one has
\[
\rho'(\delta)(c + \lambda \tilde{p}(\rho)) = 1,
\] (6.11)
which implies that \( \rho'(0) = 1/(\lambda p_1 \theta) \) since \( \rho(0) = 0 \). Then put \( \delta = 0 \) into (6.5) to obtain
\[
\frac{c}{\lambda} \phi_1(u) = \int_{0}^{u} \phi_1(u - x) \tilde{P}(x) \, dx + \frac{1}{\lambda \theta} \int_{0}^{u} \phi(u - x, 0) \tilde{P}_1(x) \, dx
\]
\[+ \frac{1}{\lambda \rho_1 \theta} \int_{u}^{\infty} (x - u) \int_{x}^{\infty} w(x, y - x) \, dP(y) \, dx.
\] (6.12)

In (6.12), \( \phi(u, 0) = \phi(u) \big|_{\delta=0} \), and thus from (1.5) with \( \delta = 0 \),
\[
\phi(u, 0) = \frac{1}{1 + \theta} \int_{0}^{u} \phi(u - x, 0) \, dP(x) + \frac{1}{1 + \theta} H(u, 0)
\] (6.13)
with \( H(u, 0) \) given by (6.6). Now, one has (e.g., Panjer and Willmot (1992))
\[
\tilde{\psi}(s) = \int_{0}^{\infty} e^{-sx} \psi(x) \, dx = \frac{1}{1 + \theta} \left\{ 1 - \frac{\tilde{p}_1(s)}{s} \right\} \left\{ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right\}^{-1},
\] (6.14)
where \( \tilde{p}_1(s) = \int_{0}^{\infty} e^{-sx} \, dP_1(x) \). Then from (6.13),
\[
\tilde{\phi}(s, 0) = \int_{0}^{\infty} e^{-su} \phi(u, 0) \, du = \frac{1}{1 + \theta} \tilde{H}(s, 0) \left\{ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right\}^{-1}
\] with \( \tilde{H}(s, 0) = \int_{0}^{\infty} e^{-sx} H(u, 0) \, du \). Therefore, \( s^{-1} [1 - \tilde{p}_1(s)] \tilde{\phi}(s, 0) = \tilde{H}(s, 0) \tilde{\psi}(s) \), implying that
\[
\int_{0}^{u} \phi(u - x, 0) \tilde{P}_1(x) \, dx = \int_{0}^{u} \psi(u - x) H(x, 0) \, dx,
\] (6.15)
and substitution into (6.12) yields (6.8) after multiplication by \( \lambda/c \).

Next, consider an important special case. Define
\[
\phi_{1,k}(u) = E\{T|U(T)\leq k|T < \infty\}, \quad k = 0, 1, 2, \ldots.
\] (6.16)
We have the following result.

Theorem 6.2. For \( k = 0, 1, 2, \ldots \), the function \( \phi_{1,k}(u) \) satisfies the defective renewal equation
\[
\phi_{1,k}(u) = \frac{1}{1 + \theta} \int_{0}^{u} \phi_{1,k}(u - x) \, dP_1(x) + \frac{p_{k+1}}{(k + 1) \lambda p_1 \theta} \tau_{k+1}(u),
\] (6.17)
and is given explicitly by
\[
\phi_{1,k}(u) = \frac{p_k}{\lambda p_1 \theta^2} \int_{0}^{u} \psi(u - x) \tau_k(x) \, dx + \frac{p_{k+1}}{(k + 1) \lambda p_1 \theta^2} \left\{ \tau_{k+1}(u) - \int_{0}^{u} \psi(u - x) \psi(x) \, dx \right\}
\]
\[+ \frac{p_{k+2}}{(k + 1)(k + 2) \lambda p_1 \theta^2} \psi(u),
\] (6.18)
with \( \tau_0(u) = p_1(1 + \theta) \psi(u) \), \( \tau_1(u) \) given by (3.10), and \( \tau_k(u) \) given by (3.11) for \( k = 2, 3, 4, \ldots \).
Proof. With $\delta = 0$ and $w(x, y) = y^k$, Eqs. (2.4) and (6.6) yield $H(u, 0) = [p_{k+1}/(k + 1) p_1] \tilde{P}_{k+1}(u)$, and

$$\int_{-\infty}^u (x - u) \int_{-\infty}^x (y - x)^k dP(y) dx = p_k \int_{-\infty}^x (x - u) \tilde{P}_k(x) dx$$
$$= \frac{p_{k+1}}{k + 1} \int_{-\infty}^u (x - u) dP_{k+1}(x) = \frac{p_{k+1}}{k + 1} \int_{-\infty}^u \tilde{P}_{k+1}(x) dx.$$

Then substitution into (6.9) yields

$$H_1(u) = \frac{p_{k+1}}{(k + 1) \lambda p_1^2 \theta} \left\{ \int_0^u \psi(u - x) \tilde{P}_{k+1}(x) dx + \int_{-\infty}^u \tilde{P}_{k+1}(x) dx \right\},$$

and from (3.8) we obtain

$$H_1(u) = \frac{p_{k+1}}{(k + 1) \lambda p_1^2 \theta} \tau_{k+1}(u). \quad (6.19)$$

Then (6.17) follows from Theorem 6.1. Also,

$$H_1(0) = \frac{p_{k+1}}{(k + 1) \lambda p_1^2 \theta} \int_0^\infty \tilde{P}_{k+1}(x) dx = \frac{p_{k+2}}{(k + 1)(k + 2) \lambda p_1^2 \theta}$$

using (2.3). Now, from (3.4) with $\rho = 0$ and (3.12),

$$\tau_{k+1}(u) = \frac{(k + 1) p_k}{p_{k+1}} \int_u^\infty \tau_k(t) dt - \int_u^\infty \psi(t) dt,$$  

(6.20)

which holds for $k = 0, 1, 2, \ldots$, with $\tau_0(u) = p_1(1 + \theta) \psi(u)$. Hence

$\tau_{k+1}'(u) = -\left(\frac{(k + 1) p_k}{p_{k+1}} \tau_k(u) + \psi(u)\right),$  

and from (6.19), $H_1'(u) = -\left(\frac{p_k/\lambda p_1^2 \theta \tau_k(u) - [p_{k+1}/((k + 1) \lambda p_1^2 \theta)] \psi(u)}{p_{k+1}}\right).$ Then from (1.17),

$$\phi_{1,k}(u) = \frac{1}{\theta} \int_0^u \psi(u - x) \left\{ \frac{p_k}{\lambda p_1^2 \theta} \tau_k(x) - \frac{p_{k+1}}{(k + 1) \lambda p_1^2 \theta} \psi(x) \right\} dx$$
$$+ \frac{p_{k+1}}{(k + 1) \lambda p_1^2 \theta} \tau_{k+1}(u) - \frac{p_{k+2}}{(k + 1)(k + 2) \lambda p_1^2 \theta^2} \psi(u),$$

which is (6.18).  

The choice $k = 0$ allows one to obtain the mean time to ruin.

Corollary 6.1. The mean time to ruin, given that ruin has occurred, is given by

$$E(T | T < \infty) = \frac{\psi_1(u)}{\psi(u)}, \quad (6.21)$$

where $\psi_1(u)$ satisfies the defective renewal equation

$$\psi_1(u) = \frac{1}{1 + \theta} \int_0^u \psi_1(u - x) dP_1(x) + \frac{1}{c \psi} \int_u^\infty \psi(x) dx \quad (6.22)$$

and is given explicitly by

$$\psi_1(u) = \frac{1}{\lambda p_1 \theta} \left\{ \int_0^u \psi(u - x) \psi(x) dx + \int_u^\infty \psi(x) dx - \frac{p_2}{2p_1 \theta} \psi(u) \right\}. \quad (6.23)$$
Proof. With the help of (3.10), (6.17) becomes (6.22) with $k = 0$ and $\psi_1(u) = \phi_{1,0}(u)$. Similarly, (6.18) reduces to (6.23) using (3.10) and $\tau_0(u) = p_1(1 + \theta)\psi(u)$. Then dividing by $\psi(u)$ yields (6.21).

We remark that (6.23) was obtained earlier by Schmidli (1994) with an alternative approach.

The conditional covariance between $T$ and $|U(T)|$ is obtained with $k = 1$.

Corollary 6.2.

$$E[T|U(T)| \mid T < \infty] = \frac{\phi_{1,1}(u)}{\psi(u)},$$

where $\phi_{1,1}(u)$ satisfies the defective renewal equation

$$\phi_{1,1}(u) = \frac{1}{1 + \theta} \int_0^u \phi_{1,1}(u - r) \, dP_1(r) + \frac{1}{c} \int_0^u (u - r) \psi(x) \, dx - \frac{p_2}{2cp_1\theta} \int_u^\infty \psi(x) \, dx$$

and is given explicitly by

$$\phi_{1,1}(u) = \frac{1}{\lambda p_1\theta} \int_0^u (u - r) \psi(x) \, dx - \frac{p_2}{2\lambda p_1^2\theta^2} \int_0^u \psi(x) \, dx$$

Proof. When $k = 1$, (6.19) reduces to

$$H_1(u) = \frac{1}{\lambda p_1\theta} \int_0^u (u - r) \psi(x) \, dx - \frac{p_2}{2\lambda p_1^2\theta^2} \int_0^u \psi(x) \, dx$$

with the help of (3.11). Thus (6.17) becomes (6.25). Similarly, when $k = 1$, (6.18) reduces to (6.26) using (3.10) and (3.11).

The covariance of $T$ and $|U(T)|$ follows from Corollaries 6.1 and 6.2, and (4.6). In general, $\phi(u, 0)$ satisfying the defective renewal equation (6.13) satisfies

$$\phi(u, 0) = E[w(U(T) -), |U(T)|I(T < \infty)],$$

which yields the mean of $w$ (e.g., from (1.17)). Thus (at least in principle) the covariance of $T$ and $w$ may be obtained using Theorem 6.1 for arbitrary $w(x_1, x_2)$. We now turn to evaluation of higher moments at the time of ruin $T$.

Theorem 6.3. The $k$th moment at the time of ruin, given that ruin has occurred, is given by

$$E(T^k|T < \infty) = \frac{\psi_k(u)}{\psi(u)},$$

where $\psi_k(u)$ satisfies a sequence of defective renewal equations

$$\psi_k(u) = \frac{1}{1 + \theta} \int_0^u \psi_k(u - r) \, dP_1(r) + \frac{k}{c} \int_u^\infty \psi_{k-1}(x) \, dx$$

for $k = 1, 2, \ldots$, with $\phi_0(u) = \psi(u)$, the probability of ruin. Moreover, each $\psi_k(u)$ is given recursively by

$$\psi_k(u) = \frac{k}{\lambda p_1\theta} \left[ \int_0^u (u - r) \psi_{k-1}(x) \, dx + \int_u^\infty \psi_{k-1}(x) \, dx - \psi(u) \int_0^\infty \psi_{k-1}(x) \, dx \right]$$

in terms of $\psi_{k-1}(u)$. 

Proof. Recalling that $\tilde{K}(u) = E[e^{-sT} I(T < \infty)]$, the Laplace transform of $T$, we have

$$\psi_k(u) = E[T^k I(T < \infty)] = (-1)^k \frac{d^k}{ds^k} \tilde{K}(u)|_{s=0}.$$ 

To prove the theorem we begin with the defective renewal equation

$$\frac{c}{\lambda} \tilde{K}(u) = \int_0^u \tilde{K}(u-x)m(x, \rho) \, dx + \int_u^\infty m(x, \rho) \, dx,$$

(6.30)

where $m(x, \rho)$ is defined in (6.2). Noting that

$$m(x, \rho) = G'(x) \int_0^\infty e^{-\rho y} \tilde{P}(y) \, dy,$$

this equation is obtained from (1.8), (1.9) and (6.4) with $w(x_1, x_2) = 1$. We note that since $\int_0^\infty m(x, \rho) \, dx$ has the $n$th derivative with respect to $\delta$, assuming that $p_n$ is finite. By applying Lemma 6.1, we are able to differentiate (6.30) $n$ times.

Define the Laplace transforms

$$\tilde{K}(s, \rho) = \int_0^\infty e^{-sx} \tilde{K}(x) \, dx, \quad \tilde{m}(s, \rho) = \int_0^\infty e^{-sx} m(x, \rho) \, dx, \quad \tilde{g}(s) = \int_0^\infty e^{-sx} dG(x).$$

(6.31)

Then we have from (6.30),

$$\frac{c}{\lambda} \tilde{K}(s, \rho) = \tilde{K}(s, \rho) \tilde{m}(s, \rho) + \frac{\tilde{m}(0, \rho) - \tilde{m}(s, \rho)}{s},$$

(6.32)

and from direct calculation

$$\tilde{g}(s) = \frac{\rho}{\rho - s} \left\{ \frac{\tilde{p}(s) - \tilde{p}(\rho)}{1 - \tilde{p}(\rho)} \right\},$$

(6.33)

where $\tilde{p}(s) = \int_0^\infty e^{-sx} \, dP(x)$.

It follows from (1.4) and (6.33) that

$$\tilde{m}(s, \rho) = \frac{\tilde{p}(s) - \tilde{p}(\rho)}{\rho - s} = \frac{\lambda \tilde{p}(s) + \rho \delta - \lambda}{\lambda(\rho - s)}.$$ 

(6.34)

With the help of (6.34), (6.32) becomes

$$cs \left[ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right] - \delta \rho \tilde{K}(s, \rho) = \left[ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right] \rho \delta - \delta,$$

(6.35)

where

$$\tilde{p}_1(s) = \int_0^\infty e^{-sx} \, dP_1(x) = \frac{1 - \tilde{p}(s)}{\rho_1 s}.$$ 

Also, define the Laplace transform

$$\tilde{\psi}_k(s) = \int_0^\infty e^{-sx} \psi_k(x) \, dx.$$ 

An obvious relation is

$$\tilde{\psi}_k(s) = (-1)^k \frac{d^k}{ds^k} \tilde{K}(s, 0).$$

(6.36)
Eq. (6.28) is equivalent to
\[ s \left[ 1 - \frac{1}{1 + \theta \bar{p}_1(s)} \right] \tilde{\psi}_k(s) = \frac{k}{c} \left[ \tilde{\psi}_{k-1}(0) - \tilde{\psi}_{k-1}(s) \right]. \] (6.37)

We now prove this identity by induction.

For \( k = 1 \), this is a restatement of Corollary 6.1. Assume that for \( n = 1, \ldots, k - 1 \), identity (6.37) holds. Differentiate Eq. (6.35) \( k + 1 \) times at \( \delta = 0 \). The right-hand side is simply
\[ c \left[ 1 - \frac{1}{1 + \theta \bar{p}_1(s)} \right] \rho^{(k+1)}(0), \] (6.38)
where \( \rho^{(n)} \) denotes the \( n \)th derivative of \( \rho \), while the left-hand side equals
\[ c \frac{d}{d\delta} \left[ \rho(\delta) \tilde{K}(s, \rho) \right]_{\delta=0} = (k + 1) \frac{d}{d\delta} \left[ \rho(\delta) \tilde{K}(s, \rho) \right]_{\delta=0}. \] (6.39)

Since \( \rho(0) = 0 \),
\[ \frac{d^n}{d\delta^n} \left[ \rho(\delta) \tilde{K}(s, \rho) \right] \bigg|_{\delta=0} = \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} \rho^{(n-j)}(0) \tilde{\psi}_j(s) \]
for \( n = k, k + 1 \). Equating (6.38) and (6.39) yields
\[ s \left[ 1 - \frac{1}{1 + \theta \bar{p}_1(s)} \right] \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} \rho^{(k-j+1)}(0) \tilde{\psi}_j(s) \]
\[ = \frac{k + 1}{c} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \rho^{(k-j)}(0) \tilde{\psi}_j(0) + \left[ 1 - \frac{1}{1 + \theta \bar{p}_1(s)} \right] \rho^{(k+1)}(0). \] (6.40)

Let \( s = 0 \) in (6.40). We have
\[ \frac{k + 1}{c} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \rho^{(k-j)}(0) \tilde{\psi}_j(0) + \left[ 1 - \frac{1}{1 + \theta} \right] \rho^{(k+1)}(0) = 0. \]

Thus, (6.40) can be rewritten as
\[ s \left[ 1 - \frac{1}{1 + \theta \bar{p}_1(s)} \right] \sum_{j=0}^{k} (-1)^j \binom{k+1}{j} \rho^{(k-j+1)}(0) \tilde{\psi}_j(s) \]
\[ = -\frac{k + 1}{c} \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \rho^{(k-j)}(0) \tilde{\psi}_j(0) + \frac{1}{1 + \theta} \left[ 1 - \bar{p}_1(s) \right] \rho^{(k+1)}(0). \] (6.41)

With the fact that
\[ \tilde{\psi}_0(s) = \tilde{\psi}(s) = \frac{(1 - \bar{p}_1(s))/s(1 + \theta)}{1 - (1/(1 + \theta)) \bar{p}_1(s)}, \]
the first term in the left-hand side of (6.41) is equal to
\[ \frac{1}{1 + \theta} \left[ 1 - \bar{p}_1(s) \right] \rho^{(k+1)}(0). \]
Hence,
\[
s \left[ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right] \sum_{j=1}^{k} (-1)^j \left( \frac{k+1}{j} \right) \rho^{(k-j+1)}(0) \tilde{\psi}_j(s) = -\frac{k+1}{c} \sum_{j=0}^{k-1} (-1)^j \left( \frac{k}{j} \right) \rho^{(k-j)}(0) [\tilde{\psi}_j(0) - \tilde{\psi}_j(s)]. \tag{6.42}
\]

By the induction assumption,
\[
s \left[ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right] (-1)^{j+1} \left( \frac{k+1}{j+1} \right) \rho^{(k-j)}(0) \tilde{\psi}_{j+1}(s) = -\frac{k+1}{c} (-1)^j \left( \frac{k}{j} \right) \rho^{(k-j)}(0) [\tilde{\psi}_j(0) - \tilde{\psi}_j(s)]
\]
for \( j = 0, 1, \ldots, k - 2 \). All but the last terms in both sides of (6.42) are canceled out. Hence (6.42) becomes
\[
s \left[ 1 - \frac{1}{1 + \theta} \tilde{p}_1(s) \right] (-1)^k (k+1) \rho'(0) \tilde{\psi}_k(s) = -\frac{k+1}{c} (-1)^{k-1} k \rho'(0) [\tilde{\psi}_{k-1}(0) - \tilde{\psi}_{k-1}(s)]. \tag{6.43}
\]
which is the same as (6.37).

Eq. (6.29) is a direct application of Theorem 1.1 to (6.28), where \( G(x) = P_1(x) \) and \( H(u) = (k/\lambda p_1) \int_u^\infty \psi_{k-1}(x) \, dx \).

**Corollary 6.3.** The second moment of the time to ruin, given that ruin has occurred, is given by
\[
E(T^2 \mid T < \infty) = \frac{\psi_2(u)}{\psi(u)}, \tag{6.44}
\]
where \( \psi_2(u) \) satisfies the defective renewal equation
\[
\psi_2(u) = \frac{1}{1 + \theta} \int_0^u \psi_2(u - x) \, dP_1(x) + \frac{1}{1 + \theta} H_2(u) \tag{6.45}
\]
with
\[
H_2(u) = \frac{2}{\lambda^2 p_1^3 \theta^2} \left\{ \int_0^u \psi(u - x) \int_x^\infty \psi(t) \, dt \, dx + \int_u^\infty (x - u) \psi(x) \, dx \right\} \tag{6.46}
\]
and is given explicitly by
\[
\psi_2(u) = \frac{2}{\lambda^2 p_1^5 \theta^2} \int_0^u \psi(u - x) \int_0^x \psi(x - t) \psi(t) \, dt \, dx + \frac{4}{\lambda^2 p_1^3 \theta^2} \int_0^u \psi(u - x) \int_x^\infty \psi(t) \, dt \, dx
\]
\[+ \int_u^\infty (x - u) \psi(x) \, dx - \frac{p_2}{\lambda^2 p_1^3} \int_0^u \psi(u - x) \psi(x) \, dx - \frac{2 p_1 p_3 \theta + 3 p_2^2}{6 \lambda^2 p_1^4 \theta^4} \psi(u). \tag{6.47}
\]

**Proof.** Since
\[
H_2(u) = \frac{2}{\lambda p_1} \int_u^\infty \psi_1(x) \, dx,
\]
replacing \( \psi_1(x) \) by (6.23) and employing integration by parts on each term yields (6.46). The fact that \( \int_0^\infty \psi(x) \, dx = p_2/2 p_1 \theta \) is used in the computation. Eq. (6.47) is a direct application of Theorem 1.1. Noting that
\[
H_2(u) = \frac{2}{\lambda^2 p_1^5 \theta^2} \left\{ -\int_0^u \psi(u - x) \psi(x) \, dx - \int_u^\infty \psi(x) \, dx + \frac{p_2}{2 p_1 \theta} \psi(u) \right\},
\]
we have (6.47).
We now evaluate the moments at the time of ruin in the case when the claim size is either a combination of exponentials or a mixture of Erlangs. Gerber (1979, p. 138) considers the mean time to ruin in the exponential case. In the next example, formulas are given for higher moments at the time to ruin in the exponential case. We wish to point out that if the claim size distribution is neither a combination of exponentials nor a mixture of Erlangs, the calculation of higher \( n \) moments could be complicated. However, we may use the Tijms approximation and in this case, the approach in Example 6.1 applies.

**Example 6.1** (Combinations of exponentials). First, let \( \tilde{P}(x) = e^{-\mu x} \). Then, from (3.13) with \( \delta = 0 \), \( \psi(u) = C e^{-\theta u} \), where \( R = \theta \mu / (1 + \theta) \). It is not hard to see from (6.29) by induction that

\[
\psi_k(u) = e^{-Ru} \sum_{j=0}^{k} \tilde{C}_{j,k} \frac{(Ru)^j}{j!}.
\]

(6.48)

We now derive a recursive formula for the coefficients \( \tilde{C}_{j,k} \).

From (6.29),

\[
\begin{align*}
\int_{u}^{\infty} \psi(u-x)\psi_{k-1}(x)\, dx &= \frac{1}{1+\theta} e^{-Ru} \sum_{j=0}^{k-1} \tilde{C}_{j,k-1} \int_{0}^{u} (Rx)^j \frac{dx}{j!} e^{-Ru} \sum_{j=1}^{k} \tilde{C}_{j-1,k-1} \frac{(Ru)^j}{j!}, \\
\int_{0}^{\infty} \psi_k(x)\, dx &= R^{-1} \sum_{j=0}^{k-1} \tilde{C}_{j,k-1} \int_{0}^{\infty} \frac{R^{j+1} x^j e^{-Rx}}{j!} \, dx = R^{-1} e^{-Ru} \sum_{j=0}^{k} \left( \sum_{i=j}^{k-1} \tilde{C}_{i,k-1} \right) \frac{(Ru)^j}{j!}, \\
\int_{0}^{\infty} \psi_k(x)\, dx &= R^{-1} \sum_{j=0}^{k} \tilde{C}_{i,k-1}.
\end{align*}
\]

Thus, by equating the coefficients of \( (Ru)^j / j! \), we have

\[
\tilde{C}_{0,k} = \frac{k(1+\theta)}{c\mu \theta} \sum_{i=0}^{k-1} \tilde{C}_{i,k-1}
\]

(6.49)

and for \( j = 1, 2, \ldots, k \),

\[
\tilde{C}_{j,k} = \frac{k(1+\theta)^2}{c\mu \theta^2} \left[ \frac{1}{1+\theta} \tilde{C}_{j-1,k-1} + \sum_{i=j}^{k-1} \tilde{C}_{i,k-1} \right],
\]

(6.50)

with \( \sum_{i=k}^{k-1} = 0 \) and \( \tilde{C}_{0,0} = C \). The first two moments can be easily derived using (6.49) and (6.50).

\[
\begin{align*}
\tilde{C}_{0,1} &= \frac{1+\theta}{c\mu \theta} \tilde{C}_{0,0} = \frac{1}{c\mu \theta}, \\
\tilde{C}_{1,1} &= \frac{(1+\theta)^2}{c\mu \theta^2} \frac{1}{1+\theta} \tilde{C}_{0,0} = \frac{1}{c\mu \theta^2}.
\end{align*}
\]

Thus,

\[
E\{TI(T < \infty)\} = \left[ \frac{1}{c\mu \theta} + \frac{1}{c\mu \theta^2} (Ru) \right] e^{-Ru},
\]

which is in agreement with Gerber (1979, p.138).
Also,

\[
\tilde{C}_{0,2} = \frac{2(1 + \theta)}{c \mu \theta} \left[ \tilde{C}_{0,1} + \tilde{C}_{1,1} \right] = \frac{2(1 + \theta)^2}{c^2 \mu^2 \theta^3},
\]

\[
\tilde{C}_{1,2} = \frac{2(1 + \theta)^2}{c \mu \theta^2} \frac{1}{1 + \theta} \tilde{C}_{0,1} + \frac{2(1 + \theta)(1 + 2 \theta)}{c^2 \mu^2 \theta^4},
\]

\[
\tilde{C}_{2,2} = \frac{2(1 + \theta)^2}{c \mu \theta^2} \frac{1}{1 + \theta} \tilde{C}_{1,1} = \frac{2(1 + \theta)}{c^2 \mu^2 \theta^4}.
\]

\[
E[T^2 \mathbb{I}(T < \infty)] = \frac{2}{c^2 \mu^2 \theta^3} \left[ (1 + \theta)^2 + \frac{(1 + \theta)(1 + 2 \theta)}{\theta} (Ru) + \frac{1 + \theta (Ru)^2}{2!} \right] e^{-Ru}.
\]

For a combination of two exponentials,

\[
\psi(u) = C_1 e^{-Ru} + C_2 e^{-R_2u},
\]

as shown in Example 3.1. The general formula for \(\psi_k(u)\) is

\[
\psi_k(u) = \sum_{j=0}^{k} \left[ A_{j,k} e^{-R_1u} + B_{j,k} e^{-R_2u} \right] u^j / j!.
\]

(6.51)

The coefficients \(A_{j,k}\) and \(B_{j,k}\) are obtainable recursively, similar to the method used in the exponential case above. We omit the details but give the mean of the time of ruin in this case. It is easy to see that

\[
\int_0^u \psi(u-x)\psi(x) \, dx = C_1^2 \mu^2 e^{-Ru} + C_2^2 \mu^2 e^{-R_2u} + \frac{2C_1 C_2}{R_2 - R_1} [e^{-Ru} - e^{-R_2u}],
\]

\[
\int_u^\infty \psi(x) \, dx = \frac{C_1}{R_1} e^{-Ru} + \frac{C_2}{R_2} e^{-R_2u}.
\]

Thus,

\[
E[T^2 \mathbb{I}(T < \infty)] = \frac{1 + \theta}{c \theta} \left[ C_1^2 \mu^2 e^{-Ru} + C_2^2 \mu^2 e^{-R_2u} + \frac{C_1}{R_1} e^{-Ru} + \frac{2C_1 C_2}{R_2 - R_1} \left( \frac{C_1}{R_1} + \frac{C_2}{R_2} \right) C_1 e^{-Ru} \right]
\]

\[
= \left[ \frac{C_2}{R_2} - \frac{2C_1 C_2}{R_2 - R_1} \left( \frac{C_1}{R_1} + \frac{C_2}{R_2} \right) C_2 e^{-R_2u} \right].
\]

Example 6.2 (Mixtures of Erlangs). In the case when a claim size is a mixture of Erlangs, noting that \(\psi(u) = K(u)_{u=0}\), we have (Example 4.2 of Lin and Willmot (1999) or Example 3.2 this paper)

\[
\psi(u) = e^{-\mu u} \sum_{j=0}^{\infty} C_j (\mu u)^j / j!.
\]

The \(k\)th moment of the time of ruin is again of this form, i.e.,

\[
\psi_k(u) = e^{-\mu u} \sum_{j=0}^{\infty} \overline{C}_{j,k} (\mu u)^j / j!,
\]

(6.52)

where \(\overline{C}_{j,0} = \overline{C}_j\).
A recursive algorithm for \( C_{j,k} \) is now derived. Since

\[
\int_0^u \psi(u - x) \psi_{k-1}(x) \, dx \\
= e^{-\mu u} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} C_j C_{i,k-1} \frac{\mu^{j+i} x^i}{j!i!} \int_0^u (u - x)^i x^i \, dx \\
= e^{-\mu u} \sum_{j=0}^{\infty} \sum_{i=0}^{j} C_j C_{i,k-1} \frac{\mu^{j+i} x^i}{j!(j+i)!} \\
= e^{-\mu u} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left( \sum_{i=1}^{j} C_j C_{i-1,k-1} \right) \frac{\mu^{j+i} x^i}{j!},
\]

\[
\int_0^\infty \psi_{k-1}(x) \, dx = \mu^{-1} \sum_{j=0}^{\infty} C_{j,k-1} \int_0^\infty \frac{\mu^{j+1} x^j e^{-\mu x}}{j!} \, dx \\
= \mu^{-1} e^{-\mu u} \sum_{j=0}^{\infty} C_{j,k-1} \sum_{i=0}^{j} \frac{\mu^{j+i} x^i}{j!} \\
= \mu^{-1} e^{-\mu u} \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left( \sum_{i=1}^{j} C_{j,k-1} \right) \frac{\mu^{j+i} x^i}{j!}.
\]

\[
\int_0^\infty \psi_{k-1}(x) \, dx = \mu^{-1} \sum_{i=0}^{\infty} C_{i,k-1}.
\]

Thus, from (6.29) we obtain

\[
C_{0,k} = k(1 + \theta) \left( 1 - C_0 \right) \sum_{i=0}^{\infty} C_{i,k-1} = \frac{k}{\lambda \theta} (1 - C_0) \sum_{i=0}^{\infty} C_{i,k-1},
\]

(6.53)

and for \( j = 1, 2, \ldots \),

\[
C_{j,k} = \frac{k}{\lambda \theta} \left[ \sum_{i=1}^{j} C_{j-i} C_{i,k-1} \right] + \sum_{i=0}^{\infty} C_{i,k-1} - \frac{C_{j}}{\lambda \theta} \sum_{i=0}^{\infty} C_{i,k-1}.
\]

(6.54)

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References


