Valuation of contingent-claims characterising particular pension schemes

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Abstract

The benefits promised by pension schemes are often rather complex contingent-claims whose valuation requires the specification of the stochastic behaviour of several state-variables such as salaries, inflation rates, rates of return on investments, and so on. The object of this paper is, first of all, to present a valuation model suitable for their pricing, and then to apply this model to the valuation of very peculiar options embedded in the benefits offered by some “hybrid” pension plans. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The benefits promised by pension schemes, either of the defined-contribution type or of the defined-benefit one, are often rather complex contingent-claims, whose valuation requires the specification of the stochastic behaviour of several state-variables such as salaries, inflation rates, rates of return on investments, and so on. In particular, in some countries such as Australia, Canada and USA, these benefits are sometimes expressed as the greater between the values of a defined-benefit function and a defined-contribution one (“Greater Of Benefits”, GOB). This typical contingent-claim, the GOB, can then be regarded, along the lines of Stulz (1982) and Johnson (1987), as a call option with zero exercise price on the maximum of two risky assets: the defined-benefit value and the defined-contribution one.

The GOB was studied by Britt (1991), Bell and Sherris (1991), Sherris (1993, 1995), Cohen and Bilodeau (1996) that, under different assumptions and by various methodologies, approached the problem of its valuation. In particular, by applying the contingent-claims valuation approach, Sherris obtains a partial differential equation for the GOB value, and then solves it by numerical techniques.

Assume now that a pre-specified pension scheme, for instance of the defined-benefit type, offers its members the option of switching to a defined-contribution scheme, while maintaining the same contribution rate. This is an option offered by the Italian National Pension Plan to a limited class of workers with well-specified qualifications, which has been analysed by Bacinello (1997). Since any rational and non-satiated individual will certainly choose

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the scheme which grants the GOB value, this is just an option to receive the GOB (GOB option henceforth) instead of the initially specified benefit.

The object of this paper is to value, on an individual basis, such an option. It can also be regarded, along the lines of Margrabe (1978), as an option to exchange one asset, asset A, the defined-benefit (contribution) pension, for another, asset B, the defined-contribution (benefit) pension. To this end we assume an economic framework consistent with the two-factor model proposed by Moriconi (1994) to describe the stochastic evolution of real and nominal interest rates. In this framework we consider two additional state-variables: the individual salary, and the value of an index, representing the unit price of a reference portfolio where contributions are deemed to be invested. The return on this particular portfolio determines the interest rate credited to the accumulated contributions.

The individual under consideration is always entitled to receive benefits of the same type of those included in asset A (type A) until the moment in which he(she) exercises the GOB option. Hence it is obviously crucial, in the valuation problem, to specify: (a) whether this option is of European or of American style; (b) when it comes to maturity. If the GOB option is of European style, i.e., if it can be exercised only at maturity, all benefits due for contingencies occurred before this maturity are still of type A. Then the assets to be exchanged include only benefits due for contingencies occurred after maturity. In particular, if this maturity coincides with the retirement date, the assets to be exchanged consist only of the retirement annuity (usually reversible to survivors). On the contrary, if the GOB option, although being of European style, matures before the retirement date, the assets to be exchanged include also disability, withdrawal and death benefits due in case these events occur between maturity and the retirement date (besides retirement benefits). Even more so, if the option is of American style, i.e., if it can be exercised in any moment on or before its maturity, benefits for all kinds of events, no matter when they happen, are relevant in the valuation problem.

In the paper we assume neutrality with respect to all risks affecting life contingencies (mortality, disability, withdrawal, survivors, etc.), and independence of all these risks from those affecting financial markets. In this setting, by exploiting the martingale approach, we first obtain a pricing formula for a GOB option of European style maturing at the retirement date. Some comparative statics properties of this formula are then analysed by means of Monte Carlo methods. We consider the case in which the initial pension scheme (which delivers benefits of type A) is of the defined-benefit type as well as the one in which it is of the defined-contribution type instead.

The paper is structured as follows. In Section 2 we introduce our notation and assumptions concerning the economic framework. Section 3 is devoted to the description of our valuation model for a European-style GOB option with maturity at retirement, and Section 4 presents our simulation results. Section 5 concludes the paper.

2. The economic framework

This section is initially devoted to introduce our notation and assumptions concerning the economic framework. Then, we make some remarks about these assumptions, and finally we present a pricing formula for traded securities which constitutes the building block of our valuation model. We delay until the next section definitions and notation concerning life contingencies.

2.1. Notation and assumptions

As is usual in the financial literature, we assume a perfectly competitive and frictionless market, populated by rational and non-satiated agents, all sharing the same information revealed by a filtration. In this very theoretical

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1 See also Gerber and Shiu (1996, p. 202).
2 For details on this theory we refer to the fundamental papers of Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), Artzner and Delbaen (1989), or to that of Duffie (1996) that summarises the basic results.
framework we consider the following state-variables:

\( r(t) \) nominal instantaneous interest rate,
\( p(t) \) retail price index,
\( g(t) \) unit value of a portfolio in which contributions are deemed to be invested,
\( s(t) \) individual salary.

We assume that, under a risk-neutral measure \( Q \), the random behaviour of these variables is described by the following stochastic differential equations:

\[

dr(t) = \alpha_r [\theta_r - r(t)] \, dt + \sigma_r \sqrt{r(t)} \, dZ_r(t), \quad \alpha_r, \theta_r, \sigma_r > 0, \tag{2.1.1}
\]

\[
\frac{dp(t)}{p(t)} = \mu_p \, dt + \sigma_p \, dZ_p(t), \quad \sigma_p > 0, \tag{2.1.2}
\]

\[
\frac{dg(t)}{g(t)} = r(t) \, dt + \sigma_g \, dZ_g(t), \quad \sigma_g \geq 0, \tag{2.1.3}
\]

\[
\frac{ds(t)}{s(t)} = \mu_s \, dt + \sigma_s \, dZ_s(t), \quad \sigma_s > \sigma_p, \tag{2.1.4}
\]

where \( \alpha_r, \theta_r, \sigma_r, \mu_p, \sigma_p, \mu_g, \sigma_g, \sigma_s \) are real constants and \( Z_r, Z_p, Z_g, Z_s \) are correlated standard Brownian motions determining the sources of uncertainty in the economy. More precisely, we assume that

\[

\begin{align*}
\frac{dZ_r(t)}{dZ_p(t)} &= \rho_{rp} \, dt, \quad -1 \leq \rho_{rp} \leq 1, \\
\frac{dZ_r(t)}{dZ_g(t)} &= \rho_{rg} \, dt, \quad -1 \leq \rho_{rg} \leq 1, \\
\frac{dZ_p(t)}{dZ_g(t)} &= \rho_{pg} \, dt, \quad -1 \leq \rho_{pg} \leq 1, \\
\frac{dZ_p(t)}{dZ_s(t)} &= \frac{\sigma_p}{\sigma_s} \, dt,
\end{align*}
\]

while

\[

\begin{align*}
\frac{dZ_r(t)}{dZ_s(t)} = \frac{dZ_r(t)}{dZ_g(t)} = \frac{dZ_r(t)}{dZ_s(t)} = 0.
\end{align*}
\]

2.2. Remarks

Observe that relations (2.1.1) and (2.1.2) are compatible with those proposed by Moriconi (1994) under the physical measure (and nay, they have the same form but risk-adjusted parameters). In particular, the nominal interest rate follows the mean-reverting square root diffusion process proposed by Cox et al. (1979). The latter has the pleasant property of producing (almost surely) strictly positive interest rates, at least when \( 2 \alpha_r \theta_r \geq \sigma_r^2 \) (see Feller, 1951).

As for the instantaneous inflation rate, \( dp(t)/p(t) \, dt \), it can take also negative values, and its expected value, \( \mu_p \), is assumed to be deterministic and constant in time. An alternative specification for \( \mu_p \) suggested by Moriconi (1994) is a deterministic function of time, whereas in the multifactor model of Cox et al. (1985) it is an additional state-variable, following a mean-reverting diffusion process.\(^3\) However, the simplified version of the model with deterministic expected inflation rate has the appealing property of producing real interest rates that, consistent with the empirical evidence, are allowed to be negative, within a lower barrier equal to \( -\mu_p \). In fact, according to the celebrated Fisher equation, the real interest rate, which we denote by \( e(t) \), is given by

\[
e(t) = r(t) - \mu_p, \tag{2.2.1}
\]

\(^3\) See also Moriconi (1995).
so that, by exploiting relation (2.1.1),

\[ \frac{d e(t)}{e(t)} = \alpha_r [\theta_e - e(t)] \, dt + \sigma_r \sqrt{e(t)} + \mu_p \, dZ_r(t), \]

(2.2.2)

with \( \theta_e = \theta_r - \mu_p \). Finally, among the numerous models for inflation proposed in literature, we recall Chan (1998), who models the behaviour of the inflation rate directly by means of an Ornstein–Uhlenbeck diffusion process.

As far as the portfolio index \( g(t) \) is concerned, it follows a diffusion process when \( g > 0 \) (see relation (2.1.3)). In the very special case when it is risk-free (i.e., when \( g = 0 \)), \( g(t) = g(0) \exp\left[ \int_0^t r(u) \, du \right] \) is usually referred to as the deposit account (or money market account).

As for the modelling of the salary (or labour income), there are many debates in the actuarial literature (e.g., Carriere and Shand (1998) propose a deterministic trend for it). Undoubtedly, stochastic models are the most suitable, but the question is which ones. Salaries are generally characterised by a discontinuous behaviour since their paths are stepwise functions of time, so that jump processes seem to be realistic enough. However, especially in the financial literature, they are often modelled by geometric Brownian motions (see, e.g., Duffie and Zariphopoulou, 1993; Koo, 1998). Hence they are characterised by continuous paths. Here, as an approximation, we also make the geometric Brownian motion assumption (2.1.4), derived by the following considerations. Assume, for simplicity, that the expected variations in the individual income are due to seniority, according to a constant rate \( \alpha_s \), and to inflation. Assume also that the unexpected ones (due, for instance, to promotion, demotion, sickness, etc.) are generated by a standard Brownian motion uncorrelated with \( Z_r, Z_g \), and, in particular, with \( Z_p \). The behaviour of \( s(t) \) is then described by the stochastic differential equation

\[ \frac{d s(t)}{s(t)} = \alpha_s \, dt + \frac{\mu_p}{p(t)} \, dt + \sigma \, dZ(t), \quad \sigma > 0, \]

(2.2.3)

where \( \alpha_s \) and \( \sigma \) are real constants. Exploiting relation (2.1.2) we have also:

\[ \frac{d s(t)}{s(t)} = (\alpha_s + \mu_p) \, dt + \sigma \, dZ_p(t) + \sigma \, dZ(t). \]

(2.2.4)

with \( dZ_p(t) \, dZ(t) = 0 \). Then relation (2.1.4) is simply obtained by letting

\[ \mu_s = \alpha_s + \mu_p, \quad \sigma_s = \sqrt{\sigma^2 + \sigma_p^2}, \quad dZ_s(t) = \frac{\sigma_p}{\sigma_s} \, dZ_p(t) + \sqrt{1 - \left( \frac{\sigma_p}{\sigma_s} \right)^2} \, dZ(t). \]

2.3. The martingale representation for the price of traded securities

To conclude Section 2, we recall that, in our frictionless market framework, all traded securities can be priced according to a martingale representation. More precisely, consider a finite-variance contingent-claim that delivers, at a future date \( T \), a random payoff \( X_T \) which is functionally dependent on the values up to time \( T \) of the state-variables introduced in Section 2.1. Then its price at time \( t \), which we denote by \( \Pi_t(X_T) \), is given by the following relation:

\[ \Pi_t(X_T) = E_t \left[ X_T \exp \left\{ - \int_0^T r(u) \, du \right\} \right]. \]

(2.3.1)

where \( E_t \) denotes expectation taken with respect to the risk-neutral measure \( Q \) and conditional on information up to time \( t \).

This fundamental equation, expressible in closed-form only for very particular and simple expressions of the payoff \( X_T \), can be readily approached via Monte Carlo methods and will be the building-block in all our valuation framework.

We observe, however, that using this formula in the actual world can be problematic. Indeed, a very interesting topic of research is the study of valuation models that take into account the presence of friction in the market. Moreover,
in an incomplete market, there can be several risk-neutral measures and therefore the price of a non-redundant security can be not uniquely determined.

3. The valuation model for a European-style GOB option with maturity at retirement

In this section we describe our model for a European-style GOB option. To this purpose, we consider an active employee aged \( x \) at the present date, time \( t \), who has a spouse aged \( y \). The lifetime of the spouse is assumed to be independent of both the lifetime and the working time of the employee. We denote by \( t_0 \) the date at which this employee entered the pension scheme, and by \( T \) his/her retirement date, corresponding to the moment in which he/she attains a fixed age \( \xi \) known in advance. For simplicity, we suppose that retirement benefits can only be reverted to the surviving spouse and that there is no divorce or remarriage.

We assume neutrality with respect to all risks affecting the “demographic” aspects (death, disability, withdrawal, etc.), and independence of all the demographic variables from the economic ones introduced in the previous section. We define the following demographic functions, all depending on the age:

\[
\ell^a(\cdot) \quad \text{working life table of the employee},
\]

\[
\ell^e(\cdot) \quad \text{survival function of the employee},
\]

\[
\ell^s(\cdot) \quad \text{survival function of his/her spouse}.
\]

Suppose that the employee is entitled to a GOB option, of European style, exercisable at the retirement date \( T \). This means that, if he/she withdraws, dies or becomes invalid (permanently) before this date, he/she will receive benefits provided in the initially pre-specified scheme (of type A), but he/she loses the option right that, in this way, prematurely expires. On the contrary, if the employee is still active at the exercise date, he/she can opt for receiving benefits of type B instead of type A, i.e., he/she is entitled to receive the GOB. Then, the only benefits relevant in the valuation of this option are those supplied in case of retirement, while benefits at death, disability, withdrawal are quite irrelevant.

We assume that, both in the defined-benefit scheme and in the defined-contribution one, benefits at retirement consist of a continuous reversionary life annuity, with payments continuously adjusted according to the variations of the retail price index.

3.1. Benefits in the defined-benefit scheme

The initial payment in the defined-benefit scheme is proportional to an average of inflation-adjusted final salaries and to the total time of service. More precisely, denoting it by \( B(T, T) \), we assume:

\[
B(T, T) = \alpha \cdot \tilde{s}(T) \cdot (T - t_0),
\]

where \( \alpha \) is a positive constant and

\[
\tilde{s}(T) = \frac{1}{n} \int_{T-n}^{T} s(z) \frac{p(T)}{p(z)} \, dz, \quad \text{with } 0 < n \leq T - t_0.
\]

(3.1.2)

Note that, if \( n = T - t_0 \), then \( \tilde{s}(T) \) is an average of all adjusted salaries while, in the extreme case in which \( n \) approaches 0, \( \tilde{s}(T) \) tends to the last salary \( s(T) \). The instantaneous payment at time \( h > T \) is given by

\[
B(T, h) = B(T, T) \frac{p(h)}{p(T)}, \quad h > T,
\]

(3.1.3)

and its market value at time \( T \), the maturity of the GOB option, is obtained by applying the pricing formula (2.3.1):

\[
\Pi_T(B(T, h)) = E_T \left[ B(T, h) \exp \left\{ - \int_T^h r(u) \, du \right\} \right].
\]

(3.1.4)
The market value of the retirement annuity at time $T$ depends on life contingencies of the employee and his/her spouse at the same date. If the employee withdraws, becomes invalid or dies before this date, the retirement annuity is valueless. On the contrary, if the employee is still active at time $T$, but widowed, our no-remarriage assumption implies that its value is

$$B_w(T) = \int_T^{+\infty} \Pi_T(B(T, h)) \frac{\ell^g(x + h - t)}{\ell^s(\xi)} \, dh,$$

where $\ell^g(x + h - t)/\ell^s(\xi)$ represents the probability that the employee, alive (since active) at time $T$ and aged $\xi$, is still alive at time $h \geq T$ (aged $x + h - t$). Finally, in the case in which the employee is still active and married at time $T$, we assume that he/she will receive, at time $h > T$, the whole pension payment $B(T, h)$ if alive, else his/her widow(er) will receive only a portion, $w$, of the same payment. In this case the value of the reversionary annuity is given by

$$B_m(T) = \int_T^{+\infty} \Pi_T(B(T, h)) \left\{ \frac{\ell^g(x + h - t)}{\ell^s(\xi)} + w \left[ 1 - \frac{\ell^g(x + h - t)}{\ell^s(\xi)} \right] \frac{\ell^s(y + h - t)}{\ell^s(y + T - t)} \right\} \, dh,$$

where $\ell^s(y + h - t)/\ell^s(y + T - t)$ represents the probability that the spouse is still alive at time $h$ (aged $y + h - t$), being alive at time $T$ (aged $y + T - t$). Note that to obtain relations (3.1.5) and (3.1.6) we have exploited the assumptions of demographic-risk-neutrality and independence of this source of risk from the financial and economic uncertainty.

3.2. Benefits in the defined-contribution scheme

As far as the retirement benefits in the defined-contribution scheme are concerned, we assume first of all that contributions are proportional to salaries according to a fixed rate $\gamma$. We denote, moreover, by $C(t)$ the value at time $t$ of the accumulated contributions up to that time, and observe that, when $t$ coincides with the present date $T$, this value is given. Since the contributions are deemed to be invested in the reference portfolio with unit price $g$, we have in particular, at the retirement date $T$:

$$C(T) = \gamma \int_0^T s(z) \frac{g(T)}{g(z)} \, dz = C(t) \frac{g(T)}{g(t)} + \gamma \int_t^T s(z) \frac{g(T)}{g(z)} \, dz. \quad (3.2.1)$$

The value $C(T)$ is then assumed to be transformed into a life annuity, with inflation-adjusted payments, according to the market interest rate prevailing at the retirement date. Moreover, this annuity will be reversible to the widow(er) if the employee is still married at time $T$ and not reversible otherwise. Therefore its market value at time $T$ is equal to $C(T)$ in both cases.

3.3. The valuation formula for a GOB option

We have now all the elements to price a GOB option, once again by means of the fundamental equation (2.1.3). In particular, we recall that such an option vanishes if the employee is no longer active at time $T$. Moreover, we actually have two kinds of GOB options, according to the type of the initially pre-specified pension scheme (type A).

If indeed this is of the defined-benefit type, the maturity value of the option is given by $\max\{C(T) - B_w(T), 0\}$ in case of widowhood of the employee at time $T$, and by $\max\{C(T) - B_m(T), 0\}$ otherwise. Denoting by $\Pi_T(DB \rightarrow DC)$ its time $t$ value, i.e., the value at time $t$ of an option to exchange a defined-benefit retirement pension for a defined-contribution one, of European style and maturing at the retirement date $T$, it is

$$\Pi_T(DB \rightarrow DC) = \frac{\ell^g(\xi)}{\ell^s(\xi)} \left\{ 1 - \frac{\ell^s(y + T - t)}{\ell^x(y)} \right\} E_t \left[ \max\{C(T) - B_w(T), 0\} \exp \left\{ -\int_t^T r(u) \, du \right\} \right]$$

$$\quad + \frac{\ell^s(y + T - t)}{\ell^x(y)} E_t \left[ \max\{C(T) - B_m(T), 0\} \exp \left\{ -\int_t^T r(u) \, du \right\} \right], \quad (3.3.1)$$
where $\ell^a(\xi) / \ell^a(x)$ represents the probability that the employee, active at the present time $t$ (aged $x$), is still active at the retirement age $\xi$, while $\ell^a(t + T - t) / \ell^a(y)$ represents the probability that his/her spouse, alive at time $t$ (aged $y$), is still alive at time $T$.

In the same way, if the initially pre-specified pension scheme is of the defined-contribution type, the value at time $t$ of an option to exchange a defined-contribution retirement pension for a defined-benefit one, of European style and maturing at the retirement age $T$, is given by

$$
\Pi_t(\text{DC} \rightarrow \text{DB}) = \frac{\ell^a(\xi)}{\ell^a(x)} \left\{ \left[ 1 - \frac{\ell^a(y + T - t)}{\ell^a(y)} \right] E_t \left[ \max\{B_w(T) - C(T), 0\} \exp \left\{ -\int_t^T r(u) \, du \right\} \right] \right. \\
+ \frac{\ell^a(y + T - t)}{\ell^a(y)} E_t \left[ \max\{B_m(T) - C(T), 0\} \exp \left\{ -\int_t^T r(u) \, du \right\} \right] \right\}.
$$

By exploiting the linearity of the expectation operator and recalling relations (3.1.4)–(3.1.6), the numerical valuation of formulae (3.3.1) and (3.3.2) is reduced to the computation of two-stages iterated conditional expectations under the risk neutral measure $Q$: a first-stage expectation conditional on information up to the future time $T$, and a second-stage expectation conditional on information up to the present date $t$. In the next section we approach the numerical-valuation problem by means of the Monte Carlo simulation.

4. Numerical results and comparative statics analysis

In this section we present some numerical results, obtained via Monte Carlo simulation, for the prices of the GOB options defined by relations (3.3.1) and (3.3.2). The demographic functions $\ell^a$ and $\ell^s$ used in all the numerical experiments are extracted from the 1991 Italian Statistics about Mortality, taking into account the sex of the employee. Moreover, from the same statistics and from the experience of an Italian Pension Plan about disability and withdrawals, we built the multidecrement table $\ell^a$, separately for males and females.

In order to get a first insight into the numerical entity of the GOB options, we report in Table 1, by way of an example, some results obtained when all the parameters governing the evolution of the economic variables are maintained constant at the following levels: $\alpha_r = 0.3$, $\theta_r = 0.05$, $\sigma_r = 0.06$, $\mu_p = 0.02$, $\sigma_p = 0.03$, $\sigma_g = 0.2$, $\mu_s = 0.03$, $\sigma_s = 0.05$, $\rho_{p_p} = \rho_{p_g} = \rho_{p_g} = 0$. Moreover, the values at time $t$ of the state-variables are $r(t) = 0.03$, $p(t) = 1$, $g(t) = 1$, $s(t) = 4 \times 10^{-7}$, while the parameters characterising the pension schemes are $\alpha = 0.02$, $n = 10$, $w = 0.6$, $\gamma = 0.3$. Given the age $x$ of the employee, we assume that the age of his/her spouse, $y$, is equal to $x - 2$, if the employee is a male and to $x + 2$ otherwise, and denote by $x_0$ the entry age $x - t + t_0$. To get these results we

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simulate 1000 paths of all the state-variables between \( t \) and \( T \) and, for each of them, 1000 paths of \( r \) and \( p \) between \( T \) and the first time, say \( T^* \), such that \( \ell^x(x + T^* - t) = 0 \) and \( \ell^y(y + T^* - t) = 0 \) (in practice \( T^* \) represents the time when the youngest between the employee and his/her spouse becomes about 110 years old).

The results reported in Table 1 require just a few comments. Owing to our particular assumptions about the values of the “economic” and “contractual” parameters, we notice that the value of the GOB option turns out to be greater when the initial pension scheme is of the defined-benefit type. Furthermore, in this case it is nearly always greater for men than for women whilst it is lower when the scheme is of a defined-contribution type. This probably happens because women, on the average, live more than men do. This fact implies that, when all the other elements (in particular \( B(T, T) \) and \( C(T) \)) remain unchanged, the value of the defined-benefit annuity is greater for women than for men.

It is reasonable enough to expect the values of both the defined-benefit annuity and the defined-contribution annuity to decrease with the entry age \( x_0 \) when the other elements, except \( C(t) \), remain unchanged. Nonetheless, it is not clear enough which of the above two values is more sensitive, and therefore what the effect of changes in \( x_0 \) (and correspondingly in \( C(t) \)) is on the GOB options’ prices. Anyway, the results of most of our numerical experiments (not only the very few ones reported in Table 1) show that also the values of both the GOB options decrease with \( x_0 \).

The reported example should not lead one to believe that the prices of the GOB options are monotonic with respect to their time to maturity \( T - t = \xi - x \). Too many elements, including the demographic probabilities, are indeed affected by the parameters \( x \) and \( \xi \). However, while we did not observe any kind of regularity with respect to \( x \), there is a prevailing trend of the GOB options’ prices with respect to \( \xi \). More precisely, this trend is positive when the initial pension scheme is of the defined-benefit type and negative when the scheme is of the defined-contribution one. This is probably due to the fact that, while \( C(T) \) obviously increases with \( \xi \), \( B(T, T) \) increases as well but the length of the retirement annuity decreases, so that it is not clear which is the global effect on the value of the same annuity. Moreover, one should also take into account that \( \xi \) negatively affects the probabilities \( \ell^a(\xi)/\ell^a(x) \) and \( \ell^a(y + T - t)/\ell^a(y) = \ell^a(y + \xi - x)/\ell^a(y) \).

Let us now consider the comparative statics properties of our pricing formulae with respect to the “economic” and “contractual” parameters. It is quite evident that the parameters affecting only the time \( T \) value of the defined-benefit annuity in a specific way and not affecting the time \( T \) value of the defined-contribution one, or vice versa, have a clear effect on the prices of the GOB options. This is the case, for instance, of the rates \( \alpha \) and \( w \), with respect to which the defined-benefit value is an increasing function while the defined-contribution one is constant. Then an increase in these parameters raises the value of the GOB option when the initial pension scheme is of the defined-contribution type and lowers it when the initial pension scheme is of the defined-benefit one. The same effect is due to changes in the expected inflation rate \( \mu_p \), which determines the drift of the geometric Brownian motion describing the evolution of the retail price index. The relative variations in this index, obviously increasing with \( \mu_p \), positively affect both the initial payment, \( B(T, T) \), and the subsequent ones, \( B(T, h) \), \( h > T \), in the defined-benefit scheme,\(^4\) and do not affect at all the value at time \( T \) of the defined-contribution annuity, \( C(T) \). On the other hand, variations in the rate of contribution \( \gamma \) and in the time \( T \) value of the accumulated contributions \( C(t) \) have a positive effect on the defined-contribution value and do not affect the defined-benefit one: hence the value \( \Pi_t(DB \rightarrow DC) \) is increasing with them while \( \Pi_t(DC \rightarrow DB) \) is decreasing.

Also the parameter \( n \), used for computing the average of adjusted salaries, affects only the defined-benefit value and not the defined-contribution one. However, the effect of \( n \) on the defined-benefit value is somewhat unexpected, unless the growth rate of the individual salary is always greater than the inflation rate. This may happen if \( \mu_s > \mu_p \) and the volatility parameters \( \sigma_s \) and \( \sigma_p \) are low. In this case the initial payment in the defined-benefit scheme is decreasing with \( n \), so that \( \Pi_t(DC \rightarrow DB) \) is decreasing too, while \( \Pi_t(DB \rightarrow DC) \) is increasing.

\(^4\) See relations (3.1.1)–(3.1.3).
Furthermore, it is clear that the values at time \( t \) of the retail price index, \( p(t) \), and of the reference portfolio, \( g(t) \), do not affect any of the quantities under scrutiny. In fact, only their relative variations, \( dp/p \) and \( dg/g \), respectively, are relevant in the valuation framework, and these variations do not depend on the initial values.\(^5\)

As for the influence of the spot rate on the GOB options’ prices, it is clear that the time \( T \) value of the defined-benefit annuity is a decreasing function with respect to \( r \), while the corresponding value of the defined-contribution one, \( C(T) \), is increasing with the drift of the diffusion process followed by the portfolio \( g \), hence with \( r \). Moreover, \( r \) is also employed for discounting the GOB options’ payoffs between \( t \) and \( T \). Both these facts obviously determine a decreasing trend of the GOB option price with respect to the current spot rate \( r(t) \) when the initial scheme is of the defined-contribution type. When the initial scheme is of the defined-benefit type instead, the global effect of \( r(t) \) on the GOB option price has a positive sign.

Changes in the mean reversion coefficient \( \alpha_r \) when \( r(t) < \theta_r \), and changes in the long-term rate \( \theta_r \), affect the GOB options’ prices in the same way as changes in \( r(t) \) do, while changes in \( \alpha_r \) have an opposite effect when \( r(t) > \theta_r \).

Finally, an increase in the parameter \( \mu_s \) governing the evolution of the individual salary, as well as in the time \( t \) value of this state-variable, \( s(t) \), raises the time \( T \) values of both the defined-benefit annuity and the defined-contribution one. However, the defined-benefit value is more sensitive to variations in such parameters than the defined-contribution one when the effects on the paths of the individual salary between \( t \) and \( T \), induced by changes in \( \mu_s \) or \( s(t) \), are not supported by a corresponding variation in \( C(t) \). This implies that \( \Pi_f(DB \rightarrow DB) \) is always an increasing function with respect to these parameters, while \( \Pi_f(DB \rightarrow DC) \) can decrease with them when \( t - t_0 \) is not negligible or \( n \) is sufficiently smaller than \( T - t \). On the contrary, when a variation in \( s(t) \) is supported by a corresponding variation in \( C(t) \) and, if \( n > T - t \), in the average of adjusted salaries up to time \( t \),

\[
\tilde{s}(t) := \frac{1}{n - (T - t)} \int_{T-n}^t s(z) \frac{p(t)}{p(z)} \, dz,
\]

the effect on all the quantities here considered may be the same. The values of both the defined-benefit annuity and the defined-contribution annuity, as well as the values of both the GOB options, are indeed linear homogeneous functions in the pair \((C(t), s(t))\), when \( n \leq T - t \), and in the triplet \((C(t), s(t), \tilde{s}(t))\), when \( n > T - t \). The same happens with respect to the triplet \((C(t), \alpha_r, \gamma)\).

The behaviour of the time \( t \) prices of both the GOB options with respect to the volatility parameters and to the correlation coefficients is completely undefined. Then, to give at least a partial idea of it, in the following figures we plot the option prices when only one parameter changes while the others remain unchanged. To this end, we consider a male employee with age \( x = 50 \), a spouse aged \( y = 48 \), entry age \( x_0 = 25 \), time \( t \) value of accumulated contributions \( C(t) = 3 \times 10^6 \), and retirement age \( \xi = 60 \). We assume that the initial payment in the defined-benefit scheme is proportional only to the last salary \( s(T) \), and all (but one) of the remaining parameters are fixed to the same values employed for getting the results reported in Table 1. The prices plotted in Figs. 1 and 2 are obtained by simulating 100 paths of all the state-variables between \( t \) and \( T \) and, for each of them, 100 paths of \( r \) and \( g \) between \( T \) and \( T^o \). In all these figures the value \( \Pi_f(DB \rightarrow DC) \) is represented by a straight line, while the value \( \Pi_f(DB \rightarrow DB) \) is represented by a dotted line.

More in detail, in Fig. 1a we plot the time \( t \) prices of both the GOB options when the volatility parameter \( \sigma_s \) varies between 0.005 and 0.155 with step 0.005; Fig. 1b displays the same prices when \( \sigma_p \) varies between 0.004 and 0.049 with step 0.0015; in Fig. 1c we plot the results obtained when \( \sigma_g \) varies between 0 and 0.9 with step 0.03; Fig. 1d shows these results when \( \sigma_s \) varies between 0.035 and 0.185 with step 0.005. Finally, in Fig. 2 the prices of the GOB options are plotted against the correlation coefficients \( \rho_{rp}, \rho_{rg}, \rho_{pg} \), that vary between \(-0.75 \) and 0.75 with step 0.05.

\(^5\) See relations (2.1.2) and (2.1.3).
Some comments about our findings are in order. First of all, we point out that, as already observed in comments to Table 1, $\Pi_r(DB \rightarrow DC)$ is almost always greater than $\Pi_r(DC \rightarrow DB)$, although the latter hardly ever turns out to be negligible.

Secondly, we remark that the prices of both the GOB options are not very sensitive to variations in the volatility parameter $\sigma_r$ (governing the evolution of the spot rate) and in the correlation coefficient $\rho_{pg}$ (see Figs. 1a and 2c). Yet, they seem to be hardly, if at all, sensitive to variations in the volatility parameter $\sigma_p$ (governing the evolution of the retail price index) and in the correlation coefficient $\rho_{rp}$ (see Figs. 1b and 2a). Instead, these prices are very sensitive to variations in the volatility parameter $\sigma_g$ (governing the evolution of the portfolio $g$ (see Fig. 1c). More precisely, both of them, and especially $\Pi_r(\text{DC} \rightarrow \text{DB})$, increase very rapidly with this parameter. In particular, $\Pi_r(\text{DC} \rightarrow \text{DB})$ is very low when $\sigma_g = 0$, crosses $\Pi_r(DB \rightarrow DC)$ when $\sigma_g$ is between 5400 and 5700 bp, and reaches the value of $713 \times 10^6$, which is about the double of $\Pi_r(DB \rightarrow DC)$, when $\sigma_g = 0.9$.  

Fig. 1. Values of the GOB options $\Pi_r(DB \rightarrow DC)$ (---) and $\Pi_r(DC \rightarrow DB)$ (- - -) as functions of the volatility parameter $\sigma_r$ (a), $\sigma_p$ (b), $\sigma_g$ (c), $\sigma_g$ (d).
Moreover, the prices of both the GOB options are slowly increasing with the volatility parameter $\sigma_s$ governing the evolution of the individual salary; in particular, from Fig. 1d we notice that the difference between $\Pi_1(\text{DB} \rightarrow \text{DC})$ and $\Pi_1(\text{DC} \rightarrow \text{DB})$ seems to be nearly constant.

Finally, the prices of the GOB options, although having a different trend, approach each other as the correlation coefficient $\rho_{rg}$ becomes greater (see Fig. 2b).

5. Summary and conclusions

In this paper we have presented a valuation model suitable for pricing contingent-claims intervening in pension schemes. This model has been applied to a very peculiar option (GOB option), offered by some "hybrid" pension plans. A GOB option is used to exchange benefits computed on the grounds of a defined-benefit formula with benefits computed according to a defined-contribution one, or vice versa. As for the economic framework, we have assumed an arbitrage-free market. Directly under an equivalent martingale measure and consistently with the two-factor model of Moriconi (1994), we have modelled the following state-variables: nominal interest rates, retail price index, unit value of a portfolio in which contributions are deemed to be invested, individual salary. As expected, given the complexity of the contingent-claims under scrutiny, the valuation formulae obtained are not expressed in closed form, so that a numerical approach is called for. In particular, we have presented some numerical results obtained by means of the Monte Carlo simulation. Furthermore, the comparative statics properties of our pricing formula for a European-style GOB option with maturity coinciding with the retirement date have been discussed.

The paper does not supply an empirical validation of the economic model assumed and does not present consistent estimates for all its relevant parameters. This is undoubtedly a gap to be filled. Moreover, an interesting topic of research consists, on the one hand, in considering alternative assumptions about the economic framework and, in particular, in analysing the valuation problem in the presence of frictions in the market. On the other hand, the pricing problem should be framed into an incomplete market setting. In fact, while it can be realistic enough to assume that financial instruments (such as index-linked and nominal bonds) allowing to hedge the risk connected to the first three state-variables are traded, contingent-claims depending on the individual salary could be hardly attainable. Finally, in our valuation model we have assumed the rate of contribution to be given exogenously; it would be interesting to study how an equilibrium premium could be determined endogenously within the same model.
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