Pricing catastrophe insurance products based on actually reported claims

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Abstract
This paper deals with the problem of pricing a financial product relying on an index of reported claims from catastrophe insurance. The problem of pricing such products is that, at a fixed time in the trading period, the total claim amount from the catastrophes occurred is not known. Therefore, one has to price these products solely from knowing the aggregate amount of the reported claims at the fixed time point. This paper will propose a way to handle this problem, and will thereby extend the existing pricing models for products of this kind. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction
Modelling claims from catastrophe actuaries use heavy-tailed distributions, such as the Pareto distribution. This means that the aggregate claim basically is determined by the largest claim; see Embrechts et al. (1997) or Rolski et al. (1999). This effect became clearly visible in the early 1990s, when the insurance industry had to cover huge aggregate claims incurred from catastrophes. Because certain catastrophic events like earthquakes, hurricanes or flooding are typical for some areas, a properly calculated annual premium would be nearly as high as the loss insured. From an actuarial point of view, such events are not insurable. But people living in such areas need protection. One possibility would be the government (tax payer) to take over the risk, as it is the case for flooding in the Netherlands. Another possibility are futures or options based on a loss index. Here, the risk is transferred to private investors. A description of these products can be found for example in Albrecht et al. (1994) or Schradin and Timpel (1996).

In 1992, the Chicago Board of Trade (CBoT) introduced the CAT-futures. This future is based on the ISO-index, which measures the amount of claims occurred in a certain period and reported to a participating insurance company until a certain time. The product never became popular among private investors. The reasons were that the index was announced only once before the settlement date, there was information asymmetry between insurers and investors, and that there was a lack of realistic models. In 1995, the CAT-future was replaced by the PCS-option. This option is based on a loss index, the PCS-index, estimated by an independent authority. The latter index is announced daily. In this paper, we study a model for indices like the ISO-index or the PCS-index. In the case of the CAT-future

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where the information stream is generated by a delayed reporting of claims from the catastrophes, in the case of the PCS-option by more and more refined estimates. For simplicity we will formulate the model as a model for the ISO-index. More specifically, we study the case where the number of claims from a single catastrophe has a fixed distribution (Section 3.1) and thereafter the case where the number of claims depend on an unobserved “severity” of the catastrophe (Section 3.2).

The recently introduced PCS-options (see Christensen, 1997) do not directly depend on reported claims. But there is a strong correlation between actually reported claims and the PCS-index. Because these options serve as a sort of reinsurance instrument, an insurance company exposed to catastrophic risk would have to estimate the PCS-index and its price in order to determine their hedging strategy. It therefore seems natural to use the information on the claims reported to this company. Therefore, our (ISO-index) model may also be of interest for a company investing into the new catastrophe options.

The main purpose of this paper is to introduce a model taking reporting lags into account. As an illustration as to how calculations can be done in this model, we will approximate the CAT-future price, even though this product is not traded anymore. For pricing the catastrophe insurance futures and options we use the exponential utility approach of Aase (1994), Bühlmann (1980) or Embrechts and Meister (1997). This approach will work only for aggregate claims with an exponentially decreasing tail. But data give evidence that the distribution tail of the aggregate claims is heavy-tailed. In our model, a heavy-tail can be obtained by a heavy-tailed distribution for the number of individual claims of a single catastrophe. For pricing we approximate the claim number distribution by a negative binomial distribution; more precisely, by a mixed Poisson distribution with a $\Gamma(\gamma, \eta)$-mixing distribution. Choosing $\gamma$ and $\eta$ small a heavy-tail behaviour can be approximated. The reader should note that the value of the security is based on a capped index and therefore has an upper bound. This justifies the light-tail approximation.

In the remaining parts of this introduction we describe the CAT-future. In Section 2, we introduce the model. In contrast to the existing literature, the reporting lags are explicitly taken into account. In Sections 3.1 and 3.2, we calculate approximations to the prices. Finally, in Section 4 we study the approximation error.

1.1. Description of the CAT-futures

CAT-futures are traded on a quarterly cycle, with contract months March, June, September, and December. A contract for a calendar quarter (called the event quarter) is based on losses occurring in the listed quarter and being reported to the participating companies by the end of the following quarter. A contract also specifies an area and the type of claim to be taken into account. The additional 3 months following the reporting period is attributable to data processing lags. Six months period following the start of the event quarter is called reporting period. The three reporting months following the event quarter are to allow for settlement lags that are usual in insurance. The contracts expire on the fifth day of the fourth month following the end of the reporting period. We will use arbitrary times $T_1 < T_2$ for the end of the event quarter and the end of the reporting period, respectively. This will allow for redesigning the futures. As a matter of fact a longer reporting period would be much more suitable for the need of the insurance world.

The settlement value of the contract is determined by a loss index, the ISO-index. Let us now consider the index. Each quarter approximately 100 American insurance companies report property loss data to the ISO (Insurance Service Office, a well known statistical agent). ISO then selects a pool of at least 10 of these companies on the basis of size, diversity of business, and quality of reported data. The ISO-index is calculated as the loss-ratio of this pool:

$$\text{ISO-index} = \frac{\text{reported incurred losses}}{\text{earned premiums}}.$$

The list of companies which are included in the pool is announced by the CBoT prior to the beginning of the trading period for that contract. The CBoT also announces the premium volume for companies participating in the pool prior to the start of the trading period. Thus the premium in the pool is a known constant throughout the trading period, and price changes are attributable solely to changes in the market’s expectation of loss liabilities.
The settlement value for the CA T-futures is
\[ F_{T_2} = 25000 \times \min(I_{T_2}, 2), \]
where \( I_{T_2} \) is the ISO-index at the end of the reporting period, i.e. the ratio between the losses incurred during the event quarter and reported up till 3 months later and the premium volume for the companies participating in the pool.

**Example 1.1.** The June contract covers losses from events occurring in April, May and June and are reported to the participating companies by the end of September. The June contract expires on 5 January, the following year. The contract is illustrated in Fig. 1.

### 1.2. The CAT-future pricing problem

Cummins and Geman (1993) were the first to price the insurance futures. Their approach was quite different from the approach used in this paper. As model they used integrated geometric Brownian motion. This allowed them to apply techniques arising from pricing Asian options. The model, however, seems to be far from reality. At times where a catastrophe occurs or shortly thereafter, one would expect a strong increase of the loss index. It therefore is preferable to use a marked point process as it is popular in actuarial mathematics.

The price to pay for the more realistic model is “non-uniqueness” of the market; see Aase (1994), Embrechts and Meister (1997) for further details. In fact, the index \((I_t)\) is not a traded asset. Thus markets cannot be complete. Moreover, as it is the case for term structure models, any equivalent measure may be used for no-arbitrage pricing. However, the preferences of the agents in the market will determine which martingale measure applies.

In this paper, we follow the approach of Embrechts and Meister (1997). There the general equilibrium approach is used, where all the agent’s utility functions are of exponential type. More precisely, let \( F_t \) be the price of the future, \( L_t \) the value of the losses occurred in the event quarter and reported till time \( t \), \( \mathcal{F}_t \) the information at time \( t \), \( \Pi \) premiums earned and let \( c = 25000/\Pi \). Then the price at time \( t \), is (see Embrechts and Meister, 1997, p. 19)

\[ F_t = c \frac{E_P[\exp(\alpha L_{\infty}) (L_{T_2} \wedge 2\Pi) | \mathcal{F}_t]}{E_P[\exp(\alpha L_{\infty}) | \mathcal{F}_t]} \quad (1.1) \]

In particular, \( E_P[\exp(\alpha L_{\infty})] \) has to exist. The market will determine the risk aversion coefficient \( \alpha \).

The term \( \exp(\alpha L_{\infty}) / E_P[\exp(\alpha L_{\infty})] \) is strictly positive and integrates to one. Thus it is the Radon–Nikodym derivative \( dQ/dP \) of an equivalent measure. In the specific model we will consider, the process \( (L_t) \) follows under to the measure \( Q \) the same model (but with different parameters) as under \( P \). We will use this fact to calculate the price of the CAT-future and the PCS-option. This change of measure is similar to the Esscher method described in Gerber and Shiu (1994) and Schrardin and Timpel (1996). If we assume that proportional reinsurance is possible, the premiums are split fairly between insurer and reinsurer, and that the proportion held in the portfolio can be changed at any time, then the index \( (L_{\infty}(T) - \Pi(T)) \) would become a traded asset, where \( L_{\infty}(T) \) are all the claims
occurred till time $T$ and $\Pi(T)$ are the premiums earned to cover the claims occurring till time $T$. In our model $\Pi(T)$ would be a linear function. This would imply that the process $(L_\infty(T) - \Pi(T) : T \geq 0)$ is a martingale under the pricing measure. This condition will determine the risk aversion coefficient $\alpha$; see for instance, Sondermann (1988).

To proceed further in the calculation of the future price, one has to choose a model for $(L_t)$. Aase (1994) used a compound Poisson model. This can be seen as catastrophes occurring at certain times and claims are reported immediately. In such a model there would not be a need for the prolonged reporting period. In Embrechts and Meister (1997), a doubly stochastic Poisson model is introduced. Here, a high intensity level will occur shortly after a catastrophe, where more claims are expected to be reported. In Klüppelberg and Mikosch (1997, Example 5.3), the asymptotic expected value and asymptotic variance for a general compound process are obtained.

The aim of this paper is to model the claims reported to the companies as individual claims with a reporting lag. This is done by modelling the aggregate claim from a single catastrophe as a compound (mixed) Poisson model. We thereby obtain the possibility to separate the individual claims and to model the reporting times of the claims. In Section 3.1, we calculate the future price using a compound Poisson model, whereas in Section 3.2, the results are extended by using a compound negative binomial model, represented as a mixed compound Poisson model. We thereby can estimate the mixing parameter from the reporting flow.

2. The model and assumptions

Let $T_1$ denote the end of the event period and $T_2 > T_1$ the end of the reporting period. We work on a complete probability space $(\Omega, \mathcal{F}, P)$ containing the following random variables and stochastic processes:

- $L_t$ the aggregate amount of reported claims till time $t$
- $N_t$ the number of catastrophes occurred in the interval $[0, t]$
- $M_t$ the number of claims from the $i$th catastrophe
- $M_{ij}(t)$ the number of claims from catastrophe $i$ reported until $t$
- $Y_{ij}$ the claim size for the $j$th claim from the $i$th catastrophe
- $D_{ij}$ the reporting lag for the $j$th claim from the $i$th catastrophe
- $\tau_i$ the occurrence time of the $i$th catastrophe

We assume the following:

- $(\mathcal{F}_t)$ is the smallest right continuous complete filtration, such that the aggregate amount of reported losses $L_t$ at time $t$ is $(\mathcal{F}_t)$-adapted.
- $(N_t)$ is a Poisson process with rate $\Lambda \in (0, \infty)$.
- $(M_i : i \in \mathbb{N})$, $(N_t : 0 \leq t \leq T_1)$, $(D_{ij} : i, j \in \mathbb{N})$, $(Y_{ij} : i, j \in \mathbb{N})$ are independent.
- $M_t$ is the mixed Poisson distributed with mixing distribution $F_\lambda$. That is, there are random variables $(\lambda^i)$ with distribution $F_\lambda$ such that, given $\lambda^i$, $M_t$ is conditionally Poisson distributed with parameter $\lambda^i$. If the distribution $F_\lambda$ is degenerated $\lambda^i = \lambda$ for some constant $\lambda$ the (unconditional) distribution of $M_t$ is Poisson with parameter $\lambda$.
- $(\lambda^i : i \in \mathbb{N})$ are i.i.d. and independent of $(N_t)$, $(D_{ij})$, $(Y_{ij})$.
- $D_{ij} \sim F_D, Y_{ij} \sim F_Y$. We denote by $Y$ ($D$, respectively) a generic variable for $Y_{ij}(D_{ij})$, and by $m_Y(r) = E[e^{rY}]$ the moment generating function of the claim sizes.
- the $j$th claim $Y_{ij}$ from the $i$th catastrophe is reported at time $\tau_i + D_{ij}$.

We have $N_{T_2} - N_{T_1} \sim \text{Poi}(\Lambda(T_2 - T_1))$ and $(\tau_{N_{T_2}}, \ldots, \tau_{N_T} | N_{T} - N_{T_1} = n)$ has the same distribution as $(U_{(1)}, \ldots, U_{(n)})$ where the $(U_i)$ are i.i.d. uniformly distributed on the interval $[T, T]$ and $(U_{(i)})$ denotes the order statistics; see for instance, Rolski et al. (1999, Theorem 5.2.1).

Moreover, it can be shown, which may seem a little bit surprising, that the number of claims $M_i(T_2) - M_i(t)$ from catastrophe $i$ reported in the period $[t, T_2]$, given $\lambda^i$, is conditionally independent of the number of claims $M_i(t)$ reported in the period $[\tau_i, t]$.
Moreover, for $1 \leq i \leq N_{T_1}$, given $(\tau_i)$, $\lambda^i$, we have

$$i \leq N_t : \quad M_i(t) | \lambda^i, \tau_i \sim \text{Poi}(\lambda^i (F_D(t - \tau_i))), \tag{2.1}$$

and

$$i > N_t : \quad M_i(T_2) - M_i(t) | \lambda^i, \tau_i \sim \text{Poi}(\lambda^i (F_D(T_2 - \tau_i) - F_D(t - \tau_i))).$$

In our model the claims $Y_{ij}$ from the $i$th catastrophe are randomly ordered. This simplifies the modelling of the reporting lags $D_{ij}$. Let $(D_{i;j})_{1 \leq j \leq M_i}$ be the order statistics of the $(D_{ij})_{1 \leq j \leq M_i}$, and $Y_{i;j}$ be the claim corresponding to $D_{i;j}$. Then the claims occurred before $T_1$ and reported till $t \leq T_2$ amount to

$$L_t = \sum_{i=1}^{N_{i\cap T_1}} \sum_{j=1}^{M_i(t)} Y_{i;j}.$$ 

In particular, the final aggregate amount $L_{T_2}$ can be represented as

$$L_{T_2} = L_t + \sum_{i=1}^{N_{i\cap T_1}} \sum_{j=M_i(t)+1}^{M_i(T_2)} Y_{i;j} + \sum_{i=N_{i\cap T_1}+1}^{N_{T_1}} \sum_{j=1}^{M_i(T_2)} Y_{i;j}.$$ 

For the rest of this section we work with the measure $P$ conditioned on $\mathcal{F}_t$. Let

$$S_i = \sum_{j=M_i(t)+1}^{M_i(T_2)} Y_{i;j}.$$ 

For $i \leq N_t$, given $\lambda^i$, $S_i$ is then compound Poisson distributed with intensity parameter $\lambda^i (F_D(T_2 - \tau_i) - F_D(t - \tau_i))$. At time $t$, $N_t$ is known, so $S_1 + \cdots + S_{N_t}$ conditioned on $\lambda^1, \ldots, \lambda^{N_t}$ is again compound Poisson distributed with parameter $\lambda^1 (F_D(T_2 - \tau_1) - F_D(t - \tau_1)) + \cdots + \lambda^{N_t} (F_D(T_2 - \tau_{N_t}) - F_D(t - \tau_{N_t}))$. The latter is known from risk theory; see for instance Gerber (1979, p. 13) or Rolski et al. (1999, Theorem 4.2.2).

For $N_t < i \leq N_{T_1}$, given $\lambda^i$ and $\tau_i$, $S_i$ is then compound Poisson distributed with intensity parameter $\lambda^i (F_D(T_2 - \tau_i))$. We again have that $S_{N_t + 1} + \cdots + S_{N_{T_1}}$ conditioned on $N_{T_1}, \lambda^{N_{T_1} + 1}, \ldots, \lambda^{N_{T_1}}$ and $\tau_{N_t + 1}, \ldots, \tau_{N_{T_1}}$ is compound Poisson distributed with intensity parameter $\sum_{i=N_{T_1}+1}^{N_{T_1}} \lambda^i (F_D(T_2 - \tau_i))$. So all in all we get that $L_{T_2} - L_t = S_1 + \cdots + S_{N_t} + S_{N_t + 1} + \cdots + S_{N_{T_1}}$ is again compound Poisson distributed with intensity parameter

$$\sum_{i=1}^{N_t} \lambda^i (F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \sum_{i=N_{T_1}+1}^{N_{T_1}} \lambda^i (F_D(T_2 - \tau_i))$$

$$\overset{d}{=} \sum_{i=1}^{N_t} \lambda^i (F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \sum_{i=N_{T_1}+1}^{N_{T_1}} \lambda^i (F_D(T_2 - \tau_i)) \tag{2.2}$$

where $\bar{\tau}_i$ are i.i.d. uniformly distributed on $(t, T_1)$ and independent of $\mathcal{F}_t$. Here $\overset{d}{=}$ means equality in distribution. Thus, for $t$ fixed $L_{T_2} - L_t$ becomes a mixed compound Poisson model.
3. Calculation of the CAT-future price

3.1. Deterministic \( \lambda^i \)

In this section, we will derive the future price (1.1) when \( \lambda^i = \lambda \) is deterministic. We first need the value \( E_P[\exp(\alpha L_{\infty})] \). We remark that \( \sum_{j=1}^{M_i} Y_{ij} \) has a compound Poisson distribution with moment generating function \( \exp(\lambda(m_Y(r) - 1)) \). This yields

\[
E_P[\exp(\alpha L_{\infty})] = \exp\{\Lambda(\exp(\lambda(m_Y(r) - 1)) - 1)\}.
\]

Let us consider now the process \( (L_t) \) under the measure \( Q \). For an introduction to change of measure methods we refer to Rolski et al. (1999). A simple calculation yields that under \( Q \) the process \( (L_t) \) is of the same type, only with different parameters. \( (N_t) \) is a Poisson process with rate

\[
\tilde{\lambda} = \Lambda E_P \left[ \exp \left\{ \sum_{j=1}^{M_I} Y_{ij} \right\} \right] = \Lambda \exp(\lambda(m_Y(\alpha) - 1)).
\]

The number of claims of catastrophe \( i \) is Poisson distributed with parameter \( \tilde{\lambda} = \lambda(m_Y(\alpha)) \) and the individual claims have the distribution function \( \tilde{F}_Y(s) = \int_0^s e^{\phi(t)} dF_Y(y)/m_Y(\alpha) \). The lags \( D_{ij} \) have the same distribution as under \( P \).

The price of a CAT-future is therefore \( cE_Q[L_{T_2} \wedge 2\Pi] \). Denoting the distribution function of \( L_{T_2} - L_t \) under \( Q \) conditioned on \( F_t \) by \( \tilde{F}_L(\cdot; t) \) we can express the price as

\[
c(L_t + E_Q[(L_{T_2} - L_t) - ((L_{T_2} - L_t) - (2\Pi - L_t))]^+ | F_t) \]
\[
= c \left( L_t + E_Q[(L_{T_2} - L_t) | F_t] - \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x; t)) \text{d}x \right).
\]

But the problem with the above expression is that we have to find the \( n \)-fold convolutions of \( F_D[T_2 - \bar{\tau}] \), in order to calculate the last term. To find an explicit expression seems to be hard.

Historical data show that, so far, the cap 2 in the definition of the CAT-future has not been reached. The largest loss-ratio was hurricane Andrew with \( L_\infty = 1.79\Pi \). Under the measure \( P \) we have that \( (L_{T_2} > 2\Pi) \) is a rare event. Because we are dealing with catastrophe insurance, the market risk aversion coefficient \( \alpha \) cannot be large. Otherwise, catastrophe insurance would not be possible. We therefore assume that \( (L_{T_2} > 2\Pi) \) is also a rare event with respect to the measure \( Q \); see also Embrechts and Meister (1997). The light-tail approximation to our model then assures that the tail of \( \tilde{F}_L(\cdot; t) \) is exponentially decreasing. That is \( \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x; t)) \text{d}x \) will be small as long as \( Q(L_{T_2} > 2\Pi) \) is small, see also the discussion in Section 4. The latter depends of course on the risk aversion coefficient \( \alpha \), which has to be small in order to be able to neglect the last term. As in Embrechts and Meister (1997) we therefore propose the approximation \( c(L_t + E_Q[(L_{T_2} - L_t) | F_t]) \) to the price of the CAT-future, and we then make the following definition.

**Definition 3.1.** Let \( p_t \) be the price of the CAT-future at time \( t \). The upper bound \( p_t^{\text{approx}} \) of \( p_t \) defined as

\[
p_t^{\text{approx}} = p_t + \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x; t)) \text{d}x = c(L_t + E_Q[(L_{T_2} - L_t) | F_t])
\]

is used as an approximation to the future price \( p_t \).

**Theorem 3.1.** Let the assumptions be as in Section 2 with a fixed risk aversion coefficient \( \alpha \). Assume further that
\[ \lambda_i = \lambda \text{ is deterministic. Then } p_t^{\text{approx}} \text{ is given by} \]
\[
\frac{25000}{T} \left( L_t + \left( \sum_{i=1}^{N_t} \left( F_D(T_2 - \tau_i) - F_D(t - \tau_i) \right) + \tilde{A}(t - T_1) E_Q[F_D(T_2 - \bar{\tau})] \right) \tilde{\lambda} E_Q[Y] \right)
\]
for \( t \in [0, T_1] \) and
\[
\frac{25000}{T} \left( L_t + \sum_{i=1}^{N_t} (F_D(T_2 - \tau_i) - F_D(t - \tau_i)) \tilde{\lambda} E_Q[Y] \right)
\]
for \( t \in [T_1, T_2] \).

**Proof.** From the considerations in Section 2 we know that for \( t < T_1 \)
\[
E_Q[(L_{T_2} - L_t) | \mathcal{F}_t] = \left( E_Q \left[ \sum_{i=1}^{N_t} \tilde{\lambda}(F_D(T_2 - \tau_i) - F_D(t - \tau_i)) | N_t, \tau_1, \ldots, \tau_N \right] + E_Q \left[ \sum_{i=1}^{N_t} \tilde{\lambda} F_D(T_2 - \bar{\tau}_i) \right] \right) E_Q[Y_t]
\]
\[
= \left( \sum_{i=1}^{N_t} (F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \tilde{A}(t - T_1) E_Q[F_D(T_2 - \bar{\tau})] \right) \tilde{\lambda} E_Q[Y]. \tag{3.1}
\]
Note that
\[
E_Q[F_D(T_2 - \bar{\tau})] = E_P[F_D(T_2 - \bar{\tau})] = \frac{1}{T_1 - t} \int_{T_2 - T_1}^{T_2 - t} F_D(s) ds, \tag{3.2}
\]
provided \( t < T_1 \), and \( E_Q[Y] = m_Y(\alpha)/\gamma_Y(\alpha) \). If \( T_1 \leq t \leq T_2 \), we find
\[
E_Q[(L_{T_2} - L_t) | \mathcal{F}_t] = \sum_{i=1}^{N_t} (F_D(T_2 - \tau_i) - F_D(t - \tau_i)) \tilde{\lambda} E_Q[Y].
\]

The approximation error will be discussed in Section 4.

### 3.2. Stochastic \( \lambda \)

We now assume that the \( \lambda_i \)'s are stochastic and independent. This can be seen as a measure of the severity of the catastrophe. For simplicity of the model, we assume that \( \lambda_i \) can be observed via reported claims only. Of course, in reality other information as TV-pictures or reports from the affected area will be available. Then for claims occurring before \( t \) we have some information on the intensity parameter \( \lambda_i \). We therefore have to work with the posterior distribution of \( \lambda_i \) given \( \mathcal{F}_t \). It would be desirable if the prior and the posterior distribution would belong to the same class; see the discussion in Cox and Hinkley (1974, Chapter 10). We therefore choose \( \lambda_i \) to be \( \Gamma \) distributed. Let \( \tilde{\lambda} \sim \Gamma(\gamma, \eta) \).

We find
\[
E_P[\exp(\alpha L_\infty)] = \exp \left\{ A \left( \frac{\eta}{\eta - m_Y(\alpha) + 1} \right)^\gamma - 1 \right\}.
\]
It again turns out that under the measure \( Q \) the process \( (L_t) \) is of the same type with different parameters. \( (N_t) \) is a Poisson process with rate \( \tilde{\lambda} = \Lambda \eta^\gamma (\eta - m_Y(\alpha) + 1)^{-\gamma} \), \( \lambda_i \) is \( \Gamma(\gamma, \eta - m_Y(\alpha) + 1) \) distributed, \( M_t \)
given $\lambda^i$ is conditionally Poisson distributed with parameter $\tilde{\lambda}^i = \lambda^i m_Y(\alpha)$ and $Y$ has distribution $F_Y(x) = \int_0^x e^{\alpha y} dF_Y(y)/m_Y(\alpha)$. As before the lags $(D_y)$ have the same distribution under $Q$ as under $P$. Thus $M_i$ has a mixed Poisson distribution where the mixing variable $\tilde{\lambda}^i$ is $\Gamma(\eta, (\eta - m_Y(\alpha) + 1)/m_Y(\alpha))$ distributed. Let $\tilde{\gamma} = \gamma$ and $\tilde{\eta} = (\eta - m_Y(\alpha) + 1)/m_Y(\alpha)$.

We now fix the time $t$ at which we want to find the CAT-future price. For $i \leq N_t$, the posterior distribution of $\tilde{\lambda}^i$ at time $t$ is then

$$\tilde{\lambda}^i | F_t \sim \Gamma(\gamma + M_i(t), F_D(t - \tau_i) + \tilde{\eta}).$$

(3.3)

**Theorem 3.2.** Let the assumptions be as in Section 2 with a fixed risk aversion coefficient $\alpha$. Assume further that $\lambda^i \sim \Gamma(\gamma, \eta)$. Then $P_t^\text{approx}$ is given by

$$\frac{25000}{\Pi} \left(L_t + \sum_{i=1}^{N_t} E_Q[\tilde{\lambda}^i | F_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i)) + \frac{\gamma m_Y(\alpha) \tilde{\Lambda}(T_1 - t)}{\eta - m_Y(\alpha) + 1} E_Q[F_D(T_2 - \tau_i)] E_Q[Y] \right)$$

for $t \in [0, T_1]$ and

$$\frac{25000}{\Pi} \left(L_t + \sum_{i=1}^{N_{T_1}} E_Q[\tilde{\lambda}^i | F_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y] \right).$$

for $t \in [T_1, T_2]$.

**Proof.** For the calculation of $E_Q[L_{T_2} - L_t | F_t]$ we again split $L_{T_2} - L_t$ into the terms occurring from catastrophes occurred before and catastrophes that will occur in the future. Consider first the case $t < T_1$. The first terms have expectation

$$\sum_{i=1}^{N_t} E_Q[\tilde{\lambda}^i | F_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y].$$

Note that $E_Q[\tilde{\lambda}^i | F_t] = (\gamma + M_i(t))/(F_D(t - \tau_i) + \tilde{\eta})$.

For the expectation of the second terms we obtain

$$\tilde{\Lambda}(T_1 - t)E_Q[\tilde{\lambda}^i F_D(T_2 - \tau_i) | Y] = \frac{\gamma m_Y(\alpha) \tilde{\Lambda}(T_1 - t)}{\eta - m_Y(\alpha) + 1} E_Q[F_D(T_2 - \tau_i)] E_Q[Y].$$

The expectation was already calculated in (3.2). This yields the desired expressions. If $T_1 \leq t \leq T_2$, we find

$$E_Q[(L_{T_2} - L_t) | F_t] = \sum_{i=1}^{N_{T_1}} E_Q[\tilde{\lambda}^i | F_t](F_D(T_2 - \tau_i) - F_D(t - \tau_i))E_Q[Y].$$

Note that with the exception of (3.2) the upper bound can be found explicitly.

**4. The approximation error**

The results in Theorems 3.1 and 3.2 are both approximations, so it is relevant to ask how good these approximations are. In this section, we will investigate this question. We only consider the case where $\lambda^i = \lambda$ is constant. For the mixed Poisson case the results are similar.
4.1. The approximation of the approximation error

From Section 3.1 we know that the approximation error (AE) is given by the following expression:

$$c \left( \int_{2\Pi - L_t}^{\infty} (1 - \tilde{F}_L(x; t)) \, dx \right),$$

where $\tilde{F}_L(\cdot; t)$ denotes the distribution function of $L_{T_2} - L_t$ under $Q$ conditioned on $\mathcal{F}_t$. The reason for omitting this term was that it is hard to calculate $\tilde{F}_L(\cdot; t)$. In order to find an approximation to the expression above we will now try to use some of the approximations to $L_{T_2} - L_t$ known from actuarial mathematics. Namely the translated gamma approximation and the Edgeworth approximation.

The idea behind the translated gamma approximation is to approximate the distribution function by $k + Z$, where $k$ is a constant and $Z$ is $\Gamma(g, h)$ distributed, such that the first three moments of $L_{T_2} - L_t$ and $k + Z$ coincide. We already have calculated the mean value $\mu_L$ of $L_{T_2} - L_t$ in (3.1). Standard calculations yield also the (conditional) variance $\sigma_L^2$ and the (conditional) coefficient of skewness $s_L$. From this the parameters of the translated gamma distribution are found to be

$$g = \frac{4}{s_L^2}, \quad h = \frac{2}{s_L \sigma_L}, \quad k = \mu_L - \frac{2\sigma_L}{s_L}.$$

The approximation error therefore is approximated by

$$\text{AP}(G) = c \left( \int_{2\Pi - L_t}^{\infty} \int_{-\infty}^{\infty} \frac{\frac{h^g}{\Gamma(g)} y^g e^{-hy} \, dy \, dx}{(2\Pi - L_t - k)^{g/2}} \right),$$

where

$$Z = \frac{L_{T_2} - L_t - E_Q[L_{T_2} - L_t]}{\sqrt{\text{Var}_Q[L_{T_2} - L_t]}}.$$

The Taylor expansion of log $M_Z(r)$ around $r = 0$ has the form

$$\log M_Z(r) = a_0 + a_1 r + a_2 \frac{1}{2} r^2 + a_3 \frac{1}{6} r^3 + a_4 \frac{1}{24} r^4 + \cdots,$$

where

$$a_k = \left. \frac{d^k \log M_Z(r)}{dr^k} \right|_{r=0}.$$

Simple calculations show that $a_0 = 0, a_1 = E[Z] = 0, a_2 = \text{Var}[Z] = 1, a_3 = s_L$ and $a_4 = (E[(L_{T_2} - L_t - E[L_{T_2} - L_t])^4]/\text{Var}[L_{T_2} - L_t]) - 3$. In our case $a_3$ and $a_4$ are calculated under $Q$ and conditioned on $\mathcal{F}_t$, and both the values can be found by standard calculations. We truncate the Taylor series after the term involving $r^4$. The moment generating function of $Z$ can be written as

$$M_Z(r) \approx \exp\left(\frac{1}{2}r^2\right) \exp\left(\frac{1}{6}a_3 r^3 + \frac{1}{24}a_4 r^4\right) \approx \exp\left(\frac{1}{2}r^2\right) \left(1 + a_3 \frac{1}{6} r^3 + a_4 \frac{1}{24} r^4 + a_3^2 \frac{1}{72} r^6\right).$$

The inverse of $\exp\left(\frac{1}{2}r^2\right)$ is easily found to be the normal distribution function $\Phi(x)$. For the other terms we derive

$$r^n e^{r^2/2} = \int_{-\infty}^{\infty} (e^{\Phi(x)})^{(n)} \phi(x) \, dx = (-1)^n \int_{-\infty}^{\infty} e^{\Phi(x)}^{(n+1)}(x) \, dx.$$
Thus, the inverse of $x^n e^{x^2/2}$ is $(-1)^n$ times the $n$th derivative of $\Phi$. The approximation yields

$$P[L_{T_2} - L_1 \leq x] = P[Z \leq z] \approx \Phi(z) - \frac{1}{6}a_3 \Phi^{(3)}(z) + \frac{1}{24}a_4 \Phi^{(4)}(z) + \frac{1}{72}a_5^2 \Phi^{(6)}(z),$$

where $z = (x - E[L_{T_2} - L_1])/(\text{var}[L_{T_2} - L_1])^{1/2}$. The approximation error therefore is approximated by

$$\text{AP}(E) = c \left( \sqrt{\text{Var}[L_{T_2} - L_1]} \int_{z_0}^{\infty} \Phi(z) - \frac{1}{6}a_3 \Phi^{(3)}(z) + \frac{1}{24}a_4 \Phi^{(4)}(z) + \frac{1}{72}a_5^2 \Phi^{(6)}(z) \, dz \right),$$

where $z_0 = ((2\Pi - L_1) - E[L_{T_2} - L_1])/\sqrt{\text{Var}[L_{T_2} - L_1]}$.

We now have constructed two ways of approximating the AE. The question is then whether we obtain a better price if we correct the uncapped future price with these approximations, or we are better off just using the uncapped future price directly? We will now look at an example in order to answer this question.

**Example 4.1.** The capped future price ($F_t^C$) is calculated according to Eq. (1.1),

$$F_t^C = c \frac{E_p[\exp(aL_\infty)(L_{T_2} \wedge 2\Pi)|F_t]}{E_p[\exp(aL_\infty)|F_t]},$$

where a reliable value of the expression is obtained by Monte-Carlo (MC) simulations. In order to use MC we make the following assumptions:

- The claim sizes are exponentially distributed with parameter $\mu$.
- The reporting lags are exponentially distributed with parameter $\beta$.

The uncapped future price ($F_t^U$) is calculated according to Theorem 3.1.

We will keep all the parameters fixed in the example, except from the premium $\Pi$ and the risk aversion coefficient $\alpha$ in order to see how the approximations depend on these two parameters. We use the following parameters:

$$T_1 = 1, \quad T_2 = 2, \quad t = 0.5, \quad A = 6, \quad N_t = 3, \quad \mu = 0.0005, \quad \tau_1 = 0.1,$$

$$\tau_2 = 0.25, \quad \tau_3 = 0.4, \quad M_1(t) = 698, \quad M_2(t) = 528, \quad M_3(t) = 259, \quad \beta = 3,$$

$$\lambda = 1000, \quad L_t = E[L_t] = 2.97 \times 10^6, \quad \Pi = (1 + \theta)12 \times 10^6.$$ 

$\Pi$ is calculated by the expected value principle with safety loading $\theta$ under the assumption that all the claims will be reported.

The parameters are chosen such that $P(L_{T_2} > 2\Pi)$ is consistent with the few data that we had. None of the approximately 80 available settlement values exceeded the level $2\Pi$. For dates before 1992, the ISO-index had to be estimated from the final aggregate loss value ($L_\infty$). The largest values of the ratio $L_{T_2}/\Pi$ there have been seen so far is 1.7893 (the Eastern Loss Ratio from Hurricane Andrew, September 1992) and 1.0508 (the Western Loss Ratio from Northridge Earthquake, 2 March 1994). In our example with $\theta = 0.05$ we have that $P(L_{T_2}/\Pi > 1.79) \approx 0.01$.

For different values of $\alpha$ and $\theta$, Table 1 shows the values of $F_t^C$, $F_t^U$, the approximation error $AE = F_t^U - F_t^C$ by using the uncapped future price, the approximation error $AE(G) = (F_t^U - \text{AP}(G)) - F_t^C$ if we correct $F_t^U$ by the gamma approximation to (4.1), and finally the approximation error $AE(E) = (F_t^U - \text{AP}(E)) - F_t^C$ if we correct $F_t^U$ by the Edgeworth approximation (4.1).

From Table 1 we see, that for all the chosen parameters the uncapped future price seems to approximate the capped future price fairly well, and best when the risk aversion coefficient is small or the safety loading is large. But are the chosen parameters reasonable?

Let us first discuss the $\theta$ parameter. In insurance the safety loading is always positive, and looking at real data the safety loading seems to be “large” when we are considering catastrophe insurance. By “large” we mean that the event $\{L_{T_2} > 2\Pi\}$ never has occurred.

The $\alpha$ parameter is the risk aversion coefficient for the single insurance company when pricing in a utility maximization framework (see Embrechts and Meister (1997) for further details), or the markets risk aversion when
pricing in a general equilibrium model (see Embrechts and Meister (1997) for further details). The first thing to note on the \( \alpha \) parameter is that the parameter is price defined, i.e. it depends on the way we price the losses. Here, the values of the losses are “large” and therefore the \( \alpha \) parameter becomes “small”. The parameters are then chosen in such a way that different prices are represented, for \( \theta = 0.15 \) and \( \alpha = 1 \times 10^{-8} \) the capped future price is 21608.8 and for \( \theta = 0.05 \) and \( \alpha = 3 \times 10^{-7} \) the capped future price is 32812.7. An indication that the single insurance company or the market should have a low risk aversion coefficient, the market conditions: The CAT-future pays a high profit with a small probability and a low profit with a high probability. After these remarks on the parameters we now turn to the figures.

From Table 1, we see that there is some variance on the figures from the MC simulations. This is observed in the column AE, where the AE should be decreasing when the \( \theta \)’s are increasing. But apart from this it is clear that both approximations give reasonable values for the approximation error, i.e. if the uncapped future price is corrected with one of the approximations we in general obtain a more accurate price. From the values it seems like the AE(E) underestimates the AE, but even though it is the case in this example this does not hold in general. Based on this example the gamma approximation gives the best approximations for nearly all the values. The only exception is for \( \alpha = 1 \times 10^{-8} \), and \( \theta = 0.05 \), but this is probably caused by the variance in the MC. So based on this example the gamma approximation is the best one to use.

Finally, we conclude that in our model under the above assumption the uncapped future price is a good approximation. But as mentioned above we obtain a more accurate price if we correct with one of the approximations, and in this example the gamma approximation is the best one to use.

5. Conclusion

This paper develops a model for insurance future pricing, which only relies on the information available. The products are priced solely from observing the reporting stream. Contrary to the existing literature we model the reporting times explicitly. We thereby obtain a more realistic model.

The results of this paper rely on an approximation to the exact future price. One therefore has to be careful applying the results derived, because the results will be inaccurate if the cap-probability \( (P(L_{T_2} > 2I)) \) or the risk aversion coefficient is “too large”.

This paper suggest two ways to approximate the approximation error, the gamma approximation and the Edgeworth approximation. It is shown that they both are useful in the determination of the error level even though the gamma approximation seems to be the best.

The results are derived specially for the CAT-futures, even though an improved financial catastrophe insurance product, the PCS-option, was introduced in 1995. For a description of the PCS-option and an explanation of why the
CAT-future was improved (see Christensen, 1997). The results from this paper cannot directly be used for pricing PCS-options because they have another structure. But, because of a strong correlation between claims reported and the PCS-index some of the ideas may be used. This is a topic for further research.

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