A no arbitrage approach to Thiele’s differential equation

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Abstract

The multi-state life insurance contract is reconsidered in a framework of securitization where insurance claims may be priced by the principle of no arbitrage. This way a generalized version of Thiele’s differential equation (TDE) is obtained for insurance contracts linked to indices, possibly marketed securities. The equation is exemplified by a traditional policy, a simple unit-linked policy and a path-dependent unit-linked policy. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The reserve on an insurance contract is traditionally defined as the expected present value of future contractual payments and is provided by the insurance company to cover these payments. The reserve thus defined can be calculated under various conditions depending e.g. on the choice of discount factor used for calculation of the present value. We shall take a different approach and define the reserve as the market price of future payments. This redefinition of the reserve inspires a reconsideration of the premium calculation principle. Financial mathematics suggests the principle of no arbitrage, and our purpose is to derive the structure of the reserve imposed by this principle. Fortunately, this structure specializes to well-known results in actuarial mathematics like Thiele’s differential equation (TDE), introduced by Thiele in 1875, and since then generalized in various directions. Thus, the traditionally defined reserve coincides with the price under certain market conditions.

The key to market prices is the notion of securitization of insurance contracts. Securitization of insurance contracts is making progress in various respects these years. On the stock exchanges all over the world attempts are made at securitizing insurance risk as an alternative to traditional exchange of risk by reinsurance contracts. This development on the exchanges has caused a discipline of modelling and pricing a variety of new products (see e.g. Cummins and Geman, 1995; Embrechts and Meister, 1995). Parallel to this development securitization has become a discipline in the unification of actuarial mathematics and mathematical finance since it plays an important role in stating actuarial problems in the framework of mathematical finance and vice versa (see e.g. Delbaen and Haezendock, 1989; Sondermann, 1991). Our approach to TDE belongs to the latter.

Financial theory applies to markets where there exist assets correlated with the claim subject to pricing, and finance is thus particularly apt to analysis of insurance contracts if such a market exists. An obvious example is

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unit-linked life insurance, at least if the unit is traded, and this subject of actuarial mathematics has been an aim of financial theory since Brennan and Schwartz (1976) recognized the option structure of a unit-linked life insurance with a guarantee. For an overview of the existing literature, see Aase and Persson (1994). Aase and Persson (1994) attained a generalized version of TDE for unit-linked insurance contracts. Our model framework covers their set-up, and we show how the securitization leads to further generalization of TDE by means of purely financial arguments. The fundamental connection between the celebrated TDE and the Black–Scholes differential equation (just as celebrated but in a different forum) is indicated by Aase and Persson (1994). Our derivation brings to the surface more directly this connection by treatment of financial risk and insurance risk in equal terms.

The target group of the paper is two-sided. On one hand, we approach an actuarial problem of evaluating an insurance payment process. The tools are imported from financial mathematics, and the reader with a background in traditional actuarial mathematics will benefit from knowledge of the concept of arbitrage as well as its connection to martingale measures (see e.g. Harrison and Pliska, 1981; Delbaen and Schachermayer, 1994). On the other hand, the paper may also form an introduction to life insurance mathematics for financial mathematicians. A statistical model frequently used in life insurance mathematics is presented, an insurance contract is constructed, and our main result is specialized to a cornerstone in insurance mathematics, Thiele’s Differential Equation. However, the statistical model and the construction of an insurance contract is hardly motivated, and the reader with a background only in financial mathematics may consult a textbook on basic life insurance mathematics e.g. Gerber (1990), for such a motivation.

In Section 2, we present the basic stochastic model. In Section 3, we define an index and a market based on this model and in Section 4, we introduce a payment process and an insurance contract based on the index. In Section 5, the price process of an insurance contract is derived, whereas the differential equation imposed by a no arbitrage condition on the market forming this price process is derived in Section 6. Section 7 contains three examples of which one is the traditional actuarial set-up, whereas the other two treat unit-linked insurance in a simple and a path-dependent set-up, respectively.

2. The basic stochastic environment

We take as given a probability space \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P)\). We let \((X_t)_{t \geq 0}\) be a cadlag (i.e. its sample paths are almost surely right continuous with left limits) jump process with finite state space \(J = \{1, \ldots, J\}\) and associate a marked point process \((T_n, \Phi_n)\), where \(T_n\) denotes the time of the \(n\)th jump of \(X_t\), and \(\Phi_n\) is the state entered at time \(T_n\), i.e. \(X_{T_n} = \Phi_n\). We introduce the counting processes

\[
N^j_t = \sum_{n=1}^{\infty} I(T_n \leq t, X_{T_n} = j), \quad j \in J,
\]

and the \(J\)-dimensional vector

\[
N_t = \begin{bmatrix}
N^1_t \\
\vdots \\
N^J_t
\end{bmatrix}.
\]

The process \(X\) generates information formalized by the filtration \(\mathbb{F}^X = \{\mathcal{F}^X_t\}_{t \geq 0}\), where

\[
\mathcal{F}^X_t = \sigma(X_s, 0 < s \leq t) = \sigma(N^j_s, j \in J, 0 < s \leq t).
\]

We let \((W_t)_{t \geq 0}\) be a standard \(K\)-dimensional Brownian motion such that \(W^k_t\) and \(W^l_t\) are independent for \(k \neq l\). The information generated by \(W\) is formalized by the filtration \(\mathbb{F}^W = \{\mathcal{F}^W_t\}_{t \geq 0}\), where

\[
\mathcal{F}^W_t = \sigma(W_s, 0 < s \leq t).
\]
Finally, we let the filtration $F = \{F_t\}_{t \geq 0}$ formalize the flow of information generated by both $X$ and $W$, i.e. $F_t$ is the smallest sigma-field containing $F^X_t$ and $F^W_t$.

$$F_t = F^X_t \vee F^W_t.$$ 

For a matrix $A$, we let $A^T$ denote the transpose of $A$ and $A^{i\cdot}$ and $A^{\cdot j}$ the $i$th row and the $j$th column of $A$, respectively. For a vector $a$, we let $\text{diag}(a)$ denote the diagonal matrix with the components of $a$ in the principal diagonal and 0 elsewhere. We shall write $\delta^{i \times j}$ and $\delta^{j \times 1}$ instead of $(\delta, \ldots, \delta)$ and $(\delta, \ldots, \delta)^T$, respectively.

For derivatives, we shall use the notation $\partial_x = \partial / \partial x$ and $\partial_{xy}^2 = \partial^2 / \partial x \partial y$. For a vector $a$, we let $\int a$ and $da$ mean componentwise integration and componentwise differentiation, respectively.

### 3. The index and the market

In the subsequent sections, we shall define and study an insurance contract. Instead of letting the payments in the insurance contract be directly driven by the stochastic basis we shall work with an index which is driven by the stochastic basis and which will form the basis for the payments.

We introduce an index $S$, an $(I + 1)$-dimensional vector of processes, the dynamics of which is given by

$$dS_t = \alpha_t^i \, dt + \beta_t^{ij} \, dN_t + \sigma_t^i \, dW_t, \quad S_0 = s_0,$$

where $\alpha^S \in \mathbb{R}^{I+1}$, $\beta^S \in \mathbb{R}^{(I+1) \times J}$, and $\sigma^S \in \mathbb{R}^{(I+1) \times K}$ are functions of $(t, S_t)$ and $s_0 \in \mathbb{R}^{I+1}$ is $F_0$-measurable. We denote by $S_t^i$, $\alpha_t^i$, $\beta_t^{ij}$, and $\sigma_t^{jk}$ the $i$th entry of $S_t$, the $i$th entry of $\alpha_t^S$, the $(i, j)$th entry of $\beta_t^S$, and $(i, k)$th entry of $\sigma_t^S$, respectively. The information generated by $S$ is formalized by the filtration $F^S = \{F^S_t\}_{t \geq 0}$, where

$$F^S_t = \sigma(S_t, 0 < t \leq t) \subseteq F_t.$$ 

The index will form the basis for the payments to be defined in Section 4. As examples in Section 4, we define $S^1 = X$, $S^2$ is the duration of the sojourn in the current state of $X$, $S^3$ the total number of jumps and $S^4$ the price of a risky asset, and the reader may benefit from having these examples in mind already at this stage.

We assume that $S$ is a Markov process and that there exist deterministic piecewise continuous functions $\mu^j(t, s)$, $j \in \mathcal{J}$, $s \in \mathbb{R}^{I+1}$ such that $N^j_t$ admits the $F^S_t$-intensity process $\mu^j_t = \mu^j(t, S_t)$, informally given by

$$\mu^j_t \, dt = E(dN^j_t \mid F^S_{t-}) + o(dt) = E(dN^j_t \mid S_{t-}) + o(dt),$$ 

where $o(h)/h \to 0$ as $h \to 0$. We know that the compensated counting processes

$$M^j_t = N^j_t - \int_0^t \mu^j_s \, ds, \quad j \in \mathcal{J}$$

are $F^S_t$-martingales and, more generally, that $\int_0^t H(s)(dN^j_t - \mu^j_t \, ds)$ is an $F^S_t$-martingale for every $j \in \mathcal{J}$, where $H$ is an $F^S_t$-predictable process such that $E \int_0^t H(s) \mu^j_s \, ds < \infty$. Introduce the $J$-dimensional vectors containing the intensity processes and martingales associated with $N$,

$$\mu_t = \begin{bmatrix} \mu^1_t \\ \vdots \\ \mu^J_t \end{bmatrix}, \quad M_t = \begin{bmatrix} M^1_t \\ \vdots \\ M^J_t \end{bmatrix}.$$ 

If e.g. $X$ is included in the index $S$, $\mu(t, X_t)$ candidates to the intensity process corresponding to the classical situation (see e.g. Hoem, 1969). However, in general, the intensity process $\mu$ may differ from the intensity process w.r.t. the natural filtration of $N$. 
We introduce a market \( Z \), an \((n+1)\)-dimensional vector \((n \leq I)\) of price processes assumed to be positive, and denote by \( Z_i \) the \( i \)th entry of \( Z \). The market \( Z \) consists of exactly those entries of \( S \) that are marketed, i.e. traded on a given market. We assume that there exists a short rate of interest such that the market contains a price process \( Z^0 \) with the dynamics given by

\[
\text{d}Z^0_t = r_t Z^0_t \, \text{d}t, \quad Z^0_0 = 1.
\]

This price process can be considered as the value process of a unit deposited on a bank account at time 0, and we shall call this entry for the risk-free asset even though \( r_t \) is allowed to depend on \((t, S_t)\). Furthermore, we assume that the set of martingale measures, \( Q \), i.e. the set of probability measures \( Q \) equivalent to \( P \) such that \( Z_i/Z^0 \) is a \( Q \)-martingale for all \( i \), is non-empty. From the fundamental theory of asset pricing this assumption is known to be essentially equivalent to the assumption that no arbitrage possibilities exist on the market \( Z \).

The entries of an index \( S \) will also be called indices, and the indices appearing in \( Z \) will then be called marketed indices or assets. With this formulation, the set of marketed indices is a subset of the set of indices and it contains at least one entry, namely \( Z^0 \).

A Markov trading strategy is a sufficiently integrable predictable process \( \theta \in \mathbb{R}^{n+1} \) that can be written in the form \( \theta_t = \theta(t-, S_{t-}) \), and the value process \( U \) corresponding to \( \theta \) is defined by

\[
U_t = \theta_t \cdot Z_t = \sum_{i=0}^n \theta^i_t Z^i_t.
\]

The Markov trading strategy \( \theta \) is said to be self-financing if

\[
\text{d}U_t = \theta_t \, \text{d}Z_t.
\]

It is easily shown that if \( \theta \) is self-financing and \( Q \in \mathcal{Q} \), then \( U/Z^0 \) is a \( Q \)-martingale.

We say that a trading strategy is admissible if it is a Markov trading strategy such that \( U_t > 0 \) for all \( t \) and complying with whatever institutional requirements there may be. We denote by \( \Theta \) the set of admissible strategies.

Throughout this paper one can think of \( \Theta \) as the strategy corresponding to a constant relative portfolio, i.e. a strategy such that for a constant \( c \in \mathbb{R} \)

\[
\theta^i_t Z^i_t = c^i U_{t-}, \quad i = 0, \ldots, n.
\]

This strategy reflects an investment profile possibly restricted by the supervising authorities e.g. such that \( \theta^i \) is non-negative for all \( i \) if short-selling is not allowed. Hereby we point out that \( \theta \), in general, is not a strategy aiming at hedging some contingent claim. As it will be seen, it is rather a part of the contingent claim itself.

4. The payment process and the insurance contract

The insurance company will henceforth be referred to as the agent. Fixing some time horizon \( T \), we define an insurance contract to be a triplet \((S, Z, B)\), where \( B \) is an \( \mathcal{F}^S_T \)-adapted, cadlag process of finite variation starting at \( B_0 \) at time 0 and with dynamics given by

\[
\text{d}B_t = b^c_t \, \text{d}t - b^d_t \, \text{d}N_t - \Delta B_T \, \text{d}1_{(t \geq T)}, \quad t \in (0, T],
\]

where \( b^c \in \mathbb{R} \) and \( b^d \in \mathbb{R}^J \) are functions of \((t, S_t)\) and \( \Delta B_T \) is a function of \( S_{T-} \). We denote by \( b^j \) the \( j \)th entry of \( b^d \). Note that the \( \mathcal{F}^S_T \)-adaptedness of \( B \) makes demands on the connection between the coefficients of \( B \) and the coefficients of \( S \).

Whereas \( S \) is called the index (process) of the insurance contract, \( B \) is called the payment process since \( B_t \) denotes the cumulative payments from the contract holder to the agent over \([0, t] \). Both continuous payments and lump sum payments are thus allowed to depend on the present state of the process \((t, S)\). The minus sign in front of \( b^d \) and \( \Delta B_T \) in \( \text{d}B_t \) agrees with actuarial tradition, and insurance payment functions are often constructed such that \( b^c, b^d \),
and $\Delta B_T$ in that case are positive corresponding to the case where $b^i_t$ are premiums and $b^d_t$ and $\Delta B_T$ are benefits. To simplify notation, lump sum payments at deterministic times are restricted to time 0 and time $T$.

Thus, an insurance contract is given by an index on which contractual payments depend, a market on which contractual payments are invested in a way specified below, and the payment process itself. It should be noted that the inclusion of $Z$ in $S$ opens for insurance contracts with payments depending on marketed indices. On the other hand, payments do not in general depend on all entries of $Z$, and they will typically not, since the number of assets on the market in realistic situations is larger than the number of indices on which payments depend.

We now consider some specific cases to show that the general insurance contract specializes to a number of well-known simple and advanced life insurance policies. We will focus on the process $S$ and take as given that this process is Markov and that the $F^S_t$-intensity process exists in all cases. Given $S$, interesting insurance contracts are easily devised. The reader will recognize some elements from this illustration in the examples in Section 7.

In classical life insurance mathematics the state of the policy is described by a process of the same type as $X$. Payments are allowed to depend on the state of the policy, i.e. $X$, and are assumed to be invested at a deterministic rate of interest. Reserving the first entry of $S$, $S_0$, for the process $Z_0$, we can put $S^1 = X$ by defining

$$\alpha^1_t = 0, \quad \beta^1_t = j - S^1_t, \quad \sigma^1_t = 0, \quad s^1_0 = X_0.$$  

This contract can be extended in various directions. We can e.g. allow for payments and intensities depending on the duration of the sojourn in the current state by letting $S^2$ be defined by

$$\alpha^2_t = 1, \quad \beta^2_t = -S^2_t, \quad \sigma^2_t = 0, \quad s^2_0 = 0,$$

and payments and intensities depending on the total number of jumps by letting $S^3$ be defined by

$$\alpha^3_t = 0, \quad \beta^3_t = 1, \quad \sigma^3_t = 0, \quad s^3_0 = 0.$$  

In Møller (1996) and Norberg (1996), generalized versions of TDE have been studied where payments depend on the duration of the sojourn in the current state.

Now we extend the contract by letting payments depend on the present state of a marketed index e.g. a geometric Brownian motion. The geometric Brownian motion and the process $Z_0$ constitute the Black–Scholes model and is obtained by defining

$$\alpha^4_t = S^4_t \alpha^4, \quad \beta^4_t = 0, \quad \sigma^4_t = S^4_t \sigma^4.$$  

A Markovian multi-dimensional diffusion model is obtained by just adding further processes similar to $S^4$ to the market. The Black–Scholes case and the multi-dimensional diffusion case have been studied previously in connection with unit-linked insurance in Aase and Persson (1994) and Ekern and Persson (1996), respectively, with a particular construction of $X$, namely a classical life–death model.

The pure diffusion price processes are continuous. Price processes involving jumps involve a development of the traditional actuarial idea of the process $X$, where $X$ describes the state of life of an individual or a group of individuals only. In a jump model for prices, the process $X$ may also partly describe the financial state. This development of the traditional actuarial idea has been studied previously in e.g. Norberg (1995). We mention that e.g. a price process modelled by a geometric compound Poisson process, where the jump distribution is discrete and finite, is included in the model.

As we, through addition of the state variables $S^2$ and $S^3$, introduced path-dependence on the non-marketed index $X$, we can also have dependence on the path of the marketed index $S^4$ e.g. by defining

$$\alpha^5_t = g(t, S^4_t), \quad \beta^5_t = 0, \quad \sigma^5_t = 0,$$

where $g$ is some specified function. Introduction of path-dependence of marketed indices through addition of state variables is well known in the theory of Asian derivatives and opens for quite exotic unit-linked products as will
be exemplified in Section 7.3. Previously, path-dependent unit-linked insurance has been studied in Bacinello and Ortu (1993) and Nielsen and Sandmann (1995) in set-ups quite different from ours.

In Bacinello and Ortu (1993), Aase and Persson (1994), Nielsen and Sandmann (1995) and Ekern and Persson (1996), standard arbitrage pricing theory is applied to (complete market) financial risk in unit-linked insurance. Working in a framework of securitization, we apply, however, (possibly incomplete market) financial mathematics to both financial risk and insurance risk, and we obtain thereby a generalized version of TDE and a corresponding pricing formula, where financial risk and insurance risk are treated in equal terms.

5. The derived price process

The insurance contract forms the basis for the introduction of two price processes, $F$ and $V$:

- $F_t$: the price at time $t$ of the contractual payments to the agent over $[0,T]$, i.e. premiums less benefits,
- $V_t$: the price at time $t$ of the contractual payments from the agent over $(t,T]$, i.e. benefits less premiums.

We make some preliminary comments on these processes, preparing for a detailed study and motivating our focus.

By the price at time $t$ of contractual payments we mean the amount against which the payments stipulated by a contract are taken over by one agent from another. Thus, buying and selling means ‘taking over’ and ‘handing over’, respectively, the contractual payments over some specified period of time. This consideration of contractual payments as dynamically marketed objects is called ‘securitization’ of insurance contracts and plays an important role in the adaptation and application of financial theory to insurance problems. Important contributions are from Delbaen and Haegendorn (1989) and Sondermann (1991).

By the securitization of contractual payments, we have implicitly taken as given the existence of a market, on which these contractual payments are allowed to be traded and, furthermore, that these contractual payments actually are bought and sold by the agents on the market. We shall assume that the market $Z$ constitutes such a market.

In many countries government regulations appear to prohibit a securitization of insurance contracts. One of the reasons may be that the supervisory authorities are not at all prepared for a free exchange of the kind of financial interests appearing on the insurance market. On the other hand, reinsurance contracts represent one allowable way of forwarding risk to a third party and financial markets all over the world are, generally speaking, being deregulated in these years. The agents on the insurance market, i.e. the customers, the direct insurance companies, and the reinsurance companies are, of course, the primary investors, but also other parties may consider the contractual payments of insurance contracts as possible investment objects. This statement is substantiated by the fact that $F$ can be interpreted as the surplus of the company stemming from the insurance contract. This surplus is reflected in the equity, which is definitely a relevant investment object for all investors.

We have introduced two price processes, one covering all contractual payments and one covering future payments only. Even though we may be interested in the price of the future payments only, we shall work with the process $F$ since this process forms the asset, in a financial sense, arising from the securitization of the insurance contract. One could consider the introduction of the process $F$ as a preliminary step leading to the definition of the process $V$.

Thus, when we henceforth refer to trading and marketing of $F$, this should be considered as equivalent to trading and marketing of $V$.

In actuarial terminology, the outstanding liabilities are called the reserve, and these liabilities can be calculated under various assumptions. Since $F$ and $V$ are price processes arising on a market, it seems natural to call $V_t$ the market reserve at time $t$. We will, however, suppress the word ‘market’, and simply speak of $V_t$ as the reserve at time $t$. One should carefully note that, whereas the reserve is traditionally defined as the expected present value of future payments, we take the reserve to be the market price of future payments.

We have assumed that a non-empty set of martingale measures exists on the market $Z$ prior to the marketing of $F$, essentially equivalent to a no arbitrage condition on $Z$. Our approach to $F$ will be the requirement that also on the market $(Z, F)$ posterior to the marketing of $F$ there exists a non-empty set of martingale measures, essentially
equivalent to a no arbitrage condition on \((Z, F)\). We shall see how this requirement lays down structure on the price process \(F\) of an insurance contract. If the no arbitrage condition is fulfilled for \((Z, F)\) we shall speak of \((S, Z, B)\) as an arbitrage-free insurance contract and about \(V\) as the corresponding arbitrage-free reserve.

Already at this stage we will argue for side conditions on the price process \(V\). These side conditions are due to the no arbitrage condition on the market \((Z, F)\) and the structure of the payment process \(B\). Here it is important to make clear the problems we actually want to solve and, at least to financial mathematicians, it may be beneficial to relate these problems to problems in traditional contingent claims analysis. Given \((S, Z)\) we wish to determine a payment process \(B\) such that no arbitrage possibilities arise from marketing the insurance contract. Afterwards, given the payment process \(B\) we wish to determine arbitrage-free prices of the insurance contract.

When determining the payment process, this process is to be considered as a balancing tool and is as such comparable with the delivery price of a future or the price of an option. However, the payment process contains a continuum of balancing elements (premiums and benefits during various periods and at various events) and in practice all but one of these elements are predetermined by the customer and the last one acts as the balancing tool of the agent. Which elements are predetermined and which element is the balancing tool depends on the type of insurance contract (defined benefits, defined contributions, etc.). Since the contract specifies the payment of \(B_0\) at time 0, \(B\) should be balanced such that the equivalence relation

\[
F_0 = B_0 - V_0 = 0
\]

is fulfilled in order to prevent the obvious arbitrage possibility that arises if the customer can buy the insurance contract on a lump sum payment of \(B_0\) at time 0 and immediately sell the future payments of the same contract on the market at a price different from \(B_0\). Hereby, the insurance contract is somewhat similar to the future contract.

The side condition at time \(T\) is also given by a no arbitrage argument. Since the price at time \(T\) of a payment of \(\Delta B_T\) at time \(T\) in an arbitrage-free market must be \(\Delta B_T\), we have

\[
V_T = \Delta B_T.
\]

So, the side conditions (1) and (2), imposed by the no arbitrage condition, should be included in the basis for balancing the payment process \(B\). Given \(B\), this payment process is to be considered as an, indeed unusual, contingent claim and achieves as such in an arbitrage-free market at least one arbitrage-free price at any time \(t\). Here again, the insurance contract is somewhat similar to the future contracts, it has a price, positive or negative, at any time during the term of the contract.

The agent receives payments according to the insurance contract \((S, Z, B)\). These payments are assumed to be invested in a value process corresponding to some admissible strategy \(\theta\). Consequently, the present value at time \(t\) of the contractual payments over \([0, T]\) becomes

\[
U_t \int_0^T \frac{1}{U_t} \, dB_t,
\]

where \(U\) is the value process corresponding to the chosen trading strategy \(\theta\). This present value is composed of an \(\mathcal{F}^S_t\)-measurable part

\[
L_t = U_t \int_0^t \frac{1}{U_t} \, dB_t,
\]

and a part which is not \(\mathcal{F}^S_t\)-measurable \(U_t \int_t^T \frac{1}{U_t} \, dB_t\). If the price operator, denoted by \(\pi_t\), is assumed to be additive, pricing the contractual payments over \([0, T]\) amounts to replacing \(U_t \int_t^T \frac{1}{U_t} \, dB_t\) by some \(\mathcal{F}_t\)-measurable process, the price process \(-V_t\). Thus,

\[
F_t = \pi_t \left( U_t \int_0^T \frac{1}{U_t} \, dB_t \right) = L_t - V_t.
\]

We restrict ourselves to price operators allowing \(V_t\) to be written in the form \(V(t, S_t)\). This restriction seems reasonable since \(S\) is Markov, but it should be noted that this is actually an assumption on the structure of the price operator. In Section 6 we consider, correspondingly, only a certain class of martingale measures imposed by the representation \(V_t = V(t, S_t)\).
If \( X_t \) jumps to state \( j \) at time \( t \), \( S_t \) will jump to \( S_{t-} + \beta_{t-}^{S,j} \), and thus \( V_t \) jumps to \( V_t^j = V(t, S_{t-} + \beta_{t-}^{S,j}) \). Each \( V_t^j \) is \( \mathcal{F}_t \)-predictable, and we can introduce the \( J \)-dimensional \( \mathcal{F}_t \)-predictable row vector

\[
V_t^J = [V_t^1, \ldots, V_t^J].
\]

Assume that the partial derivatives \( \partial_t V, \partial_s V, \) and \( \partial_{ss} V \) exist and are continuous, and abbreviate \( \psi_t = \frac{1}{2} \text{tr}(\sigma_t^S)^T \partial_{ss} V \). Then Ito’s lemma applied to the process \( V \) gives the differential form

\[
dV_t = \left( \partial_t V_t + (\partial_s V_t)^T \alpha_t^S + \psi_t \right) dt + \left( V_t^J - V_t^{1 \times J} \right) dN_t + (\partial_s V_t)^T \sigma_t^S dW_t.
\]

Ito’s lemma also gives the differential form of the process \( L_t \),

\[
dL_t = b_t^c dt - b_t^d dN_t + \frac{L_{t-}}{U_{t-}} dU_t = b_t^c dt - b_t^d dN_t + L_{t-} r_t dt - L_t r_t dt + \frac{L_{t-}}{U_{t-}} dU_t
\]

\[
= b_t^c dt - b_t^d dN_t + L_t r_t dt + \frac{L_{t-} - Z_t^0}{U_{t-}} d\left( \frac{U_t}{Z_t^0} \right).
\]

It should be noted that we can also write \( dL_t = dB_t + (\partial_t L_{t-} / U_{t-}) dZ_t \) and consider the process \( L \) as a value process corresponding to a (non-self-financing) trading strategy given by \( \nu_t = \partial_t L_{t-} / U_{t-} \).

Now, collecting terms give the differential form of the process \( F_t \),

\[
dF_t = dL_t - dV_t = r_t F_t dt + (b_t^c + r_t V_t - \partial_t V_t - (\partial_s V_t)^T \alpha_t^S - \psi_t - (b_t^d + V_t^J - V_t^{1 \times J}) \mu_t) dt
\]

\[-(\partial_s V_t)^T \sigma_t^S dW_t - (b_t^d + V_t^J - V_t^{1 \times J}) dM_t + \frac{L_{t-} - Z_t^0}{U_{t-}} d\left( \frac{U_t}{Z_t^0} \right).
\]

Upon introducing

\[
R_t = b_t^c + V_t^J - V_t^{1 \times J}, \quad \text{TD}(\alpha_t^S, \mu_t) = b_t^c + r_t V_t - \partial_t V_t - (\partial_s V_t)^T \alpha_t^S - R_t \mu_t - \psi_t,
\]

and abbreviating

\[
\alpha_t^F = r_t F_t + \text{TD}(\alpha_t^S, \mu_t), \quad \beta_t^F = -R_t, \quad \sigma_t^F = -(\partial_s V_t)^T \sigma_t^S, \quad \rho_t^F = \frac{L_{t-} - Z_t^0}{U_{t-}},
\]

we arrive at the simple form

\[
dF_t = \alpha_t^F dt + \beta_t^F dM_t + \sigma_t^F dW_t + \rho_t^F d\left( \frac{U_t}{Z_t^0} \right).
\]

The abbreviations \( R \) and TD are motivated by the terms sum at Risk and Thiele’s Differential, respectively. In Section 6, we shall see that TD, taken in a point different from \((\alpha_t^S, \mu_t)\), set equal to 0 constitutes a generalized version of TDE. A differential equation for the reserve of a life insurance contract was derived by Thiele in 1875 but we shall refer to Hoem (1969) for a classical version presented in probabilistic terms.

Note that (3) is not the semimartingale form under \( P \), since \( U / Z^0 \) is not in general a \( P \)-martingale. This is, however, a convenient form as Section 6 will show.

6. The set of martingale measures and TDE

In this section, we study the consequences of the no arbitrage condition on the markets \( Z \) and \((Z, F)\) by studying the conditions for existence of an equivalent martingale measure on these markets.
For construction of a new measure $Q$, we shall define a likelihood process $\Lambda$ by

$$d\Lambda_t = \Lambda_{t-} \left( \sum_j g^j_{t-} dM^j_t + \sum_k h^k_t dW^k_t \right) = \Lambda_{t-} (g^T_{t-} dM_t + h^T_t dW_t), \quad \Lambda_0 = 1,$$

where we have introduced

$$g^j_t = g^j(t, S_t), \quad h^k_t = h^k(t, S_t),$$

(4)

Assume that $g_t$ and $h_t$ are chosen such that the conditions

$$E^P [\Lambda_T] = 1, \quad g^j(t, s) > -1, \quad j \in J$$

are fulfilled. Then we can change measure from $P$ to $Q$ on $(\Omega, \mathcal{F}_T)$ by the definition

$$\Lambda_T = \frac{dQ}{dP},$$

and it follows from Girsanov’s theorems that $W^k_t$ under $Q$ has the local drift $h^k_t$ and that $N^j_t$ under $Q$ admits the $\mathcal{F}^S_t$-intensity process $\sum_{j=1}^J (1 + g^j_t) \mu^j_t$. In vector notation, $W_t$ has the local drift $h_t$ and $N_t$ admits the $\mathcal{F}^S_t$-intensity process $\sum_{j=1}^J (1 + g^j_t) \mu^j_t$ under $Q$. Note that by (4) we consider only the part of possible measure transformations that allow $g$ and $h$ to be stochastic processes on a particular form. This restriction on the measure transformation is imposed by the restriction on the price operator leading to $V_t = V(t, S_t)$. This will be argued at the end of this section. Defining the $Q$-martingales

$$M^Q_t = N_t - \int_0^t \text{diag}(1 \times 1 + g_t) \mu_t \, dt, \quad W^Q_t = W_t - \int_0^t h_t \, dt,$$

we can write the dynamics of $(Z, F)$ as

$$dZ_t = \alpha^Z_t \, dt + \beta^Z_t \, dM^Q_t + \sigma^Z_t \, dW^Q_t, \quad dF_t = \alpha^F_t \, dt + \beta^F_t \, dM^Q_t + \sigma^F_t \, dW^Q_t + \rho^F_t \, d\left( \frac{U_t}{Z^0_t} \right),$$

(6)

where

$$\alpha^Z_t = \alpha^Z_t + \sigma^Z_t h_t + \beta^Z_t \text{diag}(1 \times 1 + g_t) \mu_t, \quad \alpha^F_t = r_t F_t + \text{TD}(\alpha^S_t + \sigma^S_t h_t, \text{diag}(1 \times 1 + g_t) \mu_t).$$

We define the market prices of diffusion and jump risk, respectively, by

$$\eta_t = -h_t, \quad \gamma_t = -\text{diag}(g_t) \mu_t,$$

and say that the agent is risk-neutral w.r.t. diffusion risk $k$ or jump risk $j$ if $\eta^k_t = 0$ or $\gamma^j_t = 0$, respectively.

Now we determine the set of martingale measures $Q$ in the market $Z$ by requiring $Z/Z^0$ to be a martingale under $Q$. We see that $g_t$ and $h_t$ should be chosen such that

$$\alpha^Z_t + \sigma^Z_t h_t + \beta^Z_t \text{diag}(1 \times 1 + g_t) \mu_t - r_t Z_t = 0.$$

We have that also $U/Z^0$ is a martingale under $Q$ and (6) is seen to be written on semimartingale form under $Q$. Thus, requiring that also $F/Z^0$ is a martingale under $Q$ gives the equation

$$\text{TD}(\alpha^S_t + \sigma^S_t h_t, \text{diag}(1 \times 1 + g_t) \mu_t) = 0.$$
which constitutes a generalized version of TDE. In Section 7, we recognize the classical version from Hoem (1969).

Adding to TDE the side conditions \( V_0 = B_0 \) and \( V_{T-} = \Delta B_T \), we formulate our result as a theorem.

**Theorem 1.** Assume that the partial derivatives \( \partial_t V, \partial_s V, \) and \( \partial_{ss} V \) exist and are continuous. Assume that \((g, h)\) can be chosen such that

\[
\alpha^Z_t + \sigma^Z_t h_t + \beta^Z_t \text{diag}(1^J \times 1 + g_t) \mu_t - r_t Z_t = 0. \tag{7}
\]

Then, if the arbitrage-free reserve on an insurance contract \((S, Z, B)\) can be written in the form \( V(t, S_t), V(t, s) \) solves for some \((g, h)\) subject to (7) the differential equation

\[
\partial_t V_t = b^Q_t + r_t V_t - (\partial_s V_t)^T(\alpha^Z_t + \sigma^Z_t h_t) - R_t \text{diag}(1^J \times 1 + g_t) \mu_t - \psi_t, \quad V_{T-} = \Delta B_T.
\]

The payments \( B \) of an arbitrage-free insurance contract fulfills the equivalence relation

\[
V_0 = B_0. \tag{8}
\]

Although the semimartingale form of \( F \) under \( P \) was not needed in our derivation of TDE, it may be interesting for other reasons. After some straightforward calculations one gets

\[
dZ_t = (r_t Z_t + \beta^Z_t \gamma_t + \sigma^Z_t \zeta_t) \, dt + \beta^Z_t \, dM_t + \sigma^Z_t \, dW_t,
\]

\[
dF_t = \left( r_t F_t + \left( \beta^F_t L_t + \frac{\partial_t L_t}{U_t} \beta^F_t \right) \gamma_t + \left( \sigma^F_t L_t + \frac{\partial_t L_t}{U_t} \sigma^F_t \right) \zeta_t \right) \, dt + \left( \beta^F_t L_t - \beta^F_t \frac{\partial_t L_t}{U_t} \right) \, dM_t + \left( \sigma^F_t L_t + \frac{\partial_t L_t}{U_t} \sigma^F_t \right) \, dW_t. \tag{9}
\]

This representation of \((Z, F)\) motivates the term market price of risk and shows how the expected return on the marketed indices, now including \( F \), is increased compared to the return on the asset \( Z_0 \). From Theorem 1, it is seen that the price process \( V \) does not depend on \( \theta \), but (9) shows that the price process \( F \) indeed does. This implies that when it comes to laying down the payment process \( B \), the only marketed indices of importance are the ones that actually appear as indices in \( B \). Only if we consider the price process \( F \), the remaining entries of \( Z \), i.e., those entries of \( S \) that play the role of investment possibilities but do not appear as indices in \( B \), are important. In Section 7, we consider examples of insurance contracts. Since we focus on the process \( V \), we limit the market to contain assets on which payments depend. The representation in (9) may be an appropriate starting point for choice of an admissible strategy \( \theta \), taking into account e.g. the preferences of (the owners of) the agent.

Here we finish the general study of the process \((V, F)\) by putting down its stochastic representation formula. The traditionally educated life insurance actuary may rejoice at recognizing the reserve as an expected value. We have postponed the representation of the reserve as an expected value in order to emphasize that this is rather a fortunate consequence of the no arbitrage condition than a (measure-adjusted) consequence of traditional actuarial reasoning. In order to prevent arbitrage possibilities we have constructed \( Q \) such that \((F/Z^0, U/Z^0)\) under \( Q \) is a martingale, and then it follows that

\[
\frac{F_t}{Z^0_t} = E^Q \left( \frac{F_T}{Z^0_T} \left| \mathcal{F}^S_t \right. \right) = E^Q \left( \frac{1}{Z^0_T} \int_{t}^{T} U_T dB_T \left| \mathcal{F}^S_t \right. \right) + E^Q \left( \frac{1}{Z^0_T} \int_{t}^{T} U_t dB_t \left| \mathcal{F}^S_t \right. \right)
\]

\[
= E^Q \left( \frac{U_T}{Z^0_T} \left| \mathcal{F}^S_t \right. \right) \int_{0}^{t} \frac{1}{U_t} \, dB_t + \int_{t}^{T} E^Q \left( \frac{1}{U_t} \, dB_t \left| \mathcal{F}^S_t \right. \right) \left| \mathcal{F}^S_t \right. \right)
\]

\[
= \frac{U_t}{Z^0_t} \int_{0}^{t} \frac{1}{U_t} \, dB_t + \int_{t}^{T} E^Q \left( \frac{1}{U_t} \, dB_t \left| \mathcal{F}^S_t \right. \right) = \frac{L_t}{Z^0_t} + E^Q \left( \int_{t}^{T} \frac{1}{Z^0_t} \, dB_t \left| \mathcal{F}^S_t \right. \right).
\]
Thus,

$$V_t = E^Q \left( Z_t^0 \int_t^T \frac{1}{Z_{t'}^0} d(-B_{t'}) \bigg| \mathcal{F}_t^S \right),$$

and the equivalence relation $V_0 = B_0$ can be written as

$$E^Q \left( \int_0^T \frac{1}{Z_{t'}^0} d B_{t'} \right) = -B_0.$$

A special case of this constraint is known in actuarial mathematics as the equivalence principle, namely the case of risk-neutrality. In general, market prices of risk are not zero, and here we have taken into account the existence of a market $Z$, that may contain information of these market prices of risk.

A calculation similar to the one leading to (10) shows that the equivalence relation $V_0 = B_0$ corresponds to the relation

$$E^Q \left( \frac{L_T}{Z_T^0} \right) = 0. \tag{11}$$

The prices $L$ and $V$ correspond to the (individual) retrospective and prospective reserve, respectively, as defined in Norberg (1991). Corresponding to the involved quantities we shall, therefore, call (8) and (11) for the prospective equivalence relation and the retrospective equivalence relation, respectively.

The representation in (10) explains why the restricted class of martingale transformations corresponds to reserves in the form $V(t, S_t)$. The structure of $\Lambda$ determined by (4) is necessary and sufficient for the following relation to hold,

$$V_t = E^Q \left( Z_t^0 \int_t^T \frac{1}{Z_{t'}^0} d(-B_{t'}) \bigg| \mathcal{F}_t^S \right) = E^P \left( \frac{\Lambda_T}{\Lambda_t} Z_t^0 \int_t^T \frac{1}{Z_{t'}^0} d(-B_{t'}) \bigg| \mathcal{F}_t^S \right)$$

$$= E^P \left( \frac{\Lambda_T}{\Lambda_t} Z_t^0 \int_t^T \frac{1}{Z_{t'}^0} d(-B_{t'}) \bigg| S_t \right) = V(t, S_t).$$

### 7. Examples

#### 7.1. A classical policy

In this section, we consider a model where payments depend on the present state of $X$. Hoem (1969) obtained in this model a version of TDE which has taken a central position in life insurance mathematics and is widely used by practitioners.

Let $r$ be deterministic, put $K = 0$ and define $S$ by

$$\alpha_t^S = \begin{bmatrix} r_t S_0^0 \\ 0 \end{bmatrix}, \quad \beta_t^S = \begin{bmatrix} 0^{1 \times J} \\ 1 - S_1^1 \cdots J - S_1^1 \end{bmatrix}, \quad s_0 = \begin{bmatrix} 1 \\ X_0 \end{bmatrix},$$

where $Z = S^0$. Thus, the market $Z$ consists of the risk-free paper only and contains thereby no information on market prices of risk. We fix a martingale measure by assuming that the agent is risk-neutral w.r.t. risk due to the policy state, which is the only risk present in this model i.e.

$$g_t = 0.$$

Then the reserve function solves the classical TDE

$$\partial_t V_t = b_t^S + r_t V_t - (b_t^d + V_t^S - V_t 1^{1 \times J}) \mu_t, \quad V_T = \Delta B_T.$$
and the payment process of the insurance contract should be based on the equivalence relation

\[ V_0 = B_0. \]

### 7.2. A simple unit-linked policy

In this section, we consider a model where payments depend on the present state of \( X \) and the present state of a marketed index given by a geometric Brownian motion. Aase and Persson (1994) obtained a version of TDE in a similar model. We use the word simple since the payments depend only on the present state of the marketed index. Modelling the marketed index by a geometric Brownian motion, we now work with the Black–Scholes model.

Let \( r \) be deterministic, put \( K \) and define \( S \) by

\[
\begin{align*}
\alpha_t^S &= \begin{bmatrix} r_t & 0 \\ 0 & \sigma_t^2 \end{bmatrix}, \\
\beta_t^{S'} &= \begin{bmatrix} 1 - S_t^1 \cdots J - S_t^J \\ 0^1 \times J \\ 0^1 \times J \end{bmatrix}, \\
\sigma_t^S &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
s_0 &= \begin{bmatrix} 1 \\ X_0 \\ S_0^0 \end{bmatrix},
\end{align*}
\]

where

\[ Z = \begin{bmatrix} S^0 \\ S^2 \end{bmatrix}. \]

Theorem 1 states that \((g, h)\) should be chosen subject to

\[ \alpha_t^3 S_t^2 + \sigma_t^2 h_t - r_t S_t^2 = 0, \]

implying that

\[ h_t = \frac{r_t - \alpha}{\sigma}. \]

The market contains no information on the price of jump risk, and as in Section 7.1, we fix a martingale measure by assuming that the agent is risk-neutral w.r.t. risk due to the state of \( X \), i.e.

\[ g_t = 0. \]

Then the reserve function solves the TDE

\[ \partial_t V_t = b_t V_t - \partial_z V_t r_t s_t^2 - (b_t^d + V_t^d \mu_t - \frac{1}{2} \sigma^2 \partial_x s_t^2 V_t \sigma s_t^2), \quad V_T = \Delta B_T, \]

and the payment process should be based on the equivalence relation

\[ V_0 = B_0. \]

### 7.3. A path-dependent unit-linked policy

In this section, we consider a model where payments depend on the present state of \( X \) and the present state of two accounts. One account is defined as the value of as-if investments in a marketed index and the other is defined as the value of as-if investments in a non-marketed index. We call it a path-dependent policy since the payments depend on the path of the marketed index through the account defined by the value of as-if investments in this index. Modelling the marketed index by a geometric Brownian motion, the market \( Z \) constitutes the Black–Scholes model, and as in Section 7.2 the geometric Brownian motion will be described by \( S^2 \).

Now we construct the process \( S \). First we introduce a pseudo-bank account \( \hat{S}^0 \) given by

\[ d\hat{S}_t^0 = \hat{r}_t \hat{S}_t^0 \, dt, \quad \hat{S}_0^0 = 1, \]

where \( \hat{r} \) is a pseudo-short rate of interest, which may differ from the short rate of interest \( r \).
The insurance contract specifies two pseudo-payment processes, \( B^{S_0} \) and \( B^{S_2} \), defined in the same way as \( B \). The notation \( b_t^{S_0}c, b_t^{S_0}d, b_t^{S_2}c, b_t^{S_2}d \) for continuous pseudo-payments respectively lump sum pseudo-payments is natural. Now we make up two pseudo-accounts \( A^{S_0} \) and \( A^{S_2} \) by pretending to invest payments from \( B^{S_0} \) and \( B^{S_2} \) in \( S_0 \) and \( S_2 \), respectively. Consequently, the two accounts can be written

\[
A_t^{S_0} = S_0^t \int_0^t \frac{dB_t^{S_0}}{S_0^t}, \quad A_t^{S_2} = S_2^t \int_0^t \frac{dB_t^{S_2}}{S_2^t}.
\]

The payment process \( B \) is specified to depend on these two accounts. It is easily seen that if we put \( K = 1 \) and define \( S \) by

\[
\alpha_t^S = \begin{bmatrix}
    r_t S_0^t \\
    0 \\
    \alpha S_2^t \\
    b_t^{S_2}c \\
    b_t^{S_2}d + \hat{r}_t S_4^t
\end{bmatrix}, \quad \beta_t^S = \begin{bmatrix}
    0^I \
    1 - S_1^t \cdots - J - S_1^t \\
    0^I \
    b_t^{S_2}d_1 \
    b_t^{S_2}d + \hat{S}_2^t d_1
\end{bmatrix}, \quad \sigma_t^S = \begin{bmatrix}
    0 \\
    0 \\
    \sigma_{S_2}^t \\
    0 \\
    0
\end{bmatrix}, \quad s_0 = \begin{bmatrix}
    1 \\
    X_0 \\
    S_2^0 \\
    0 \\
    0
\end{bmatrix},
\]

where

\[
Z = \begin{bmatrix}
    S_0^t \\
    S_2^t
\end{bmatrix}.
\]

we actually have that

\[
A_t^{S_0} = S_4^t, \quad A_t^{S_2} = S_2^t S_4^t.
\]

As an example, one could think of a policy specifying that e.g. continuous premiums \( b_t^c \) are invested in the pseudo-bank account, the payment process \( B_t^{S_2} = t \) is invested in the index \( S_2 \), and the benefit payment \( b_t^d \) is the maximum of the two accounts made up hereby. In that case one could interpret \( \hat{r} \) as the guaranteed rate of interest and the pseudo-bank account thus represents one way of introducing an interest rate guarantee. By this example, we point out that elements of the payment process \( B \) are allowed to appear in the processes \( B^{S_2} \) and \( B^{S_0} \).

Bacinello and Ortu (1993) and Nielsen and Sandmann (1995) have studied a variant of this set-up and the occurrence of elements of \( B \) in \( B^{S_0} \) is exactly what the endogeneity in the title of Bacinello and Ortu (1993) refers to. Bacinello and Ortu (1993) indicate that the resulting insurance product is an Asian-like derivative, and comparing our construction of state variables with the one known from the theory of Asian options shows in which way the derivative is Asian-like. Bacinello and Ortu (1993) end up with a delicate fix point problem and discuss conditions for existence of a payment process satisfying the equivalence relation \( V_0 = 0 \) (no lump sum payment at time 0). These conditions constrain the two accounts and the payments depending on them. Apart from the mathematical conditions, one could deal with conditions of the payments being practically feasible. However, we shall not enter into any of these discussions, but rather allow of general contributions to the two accounts as described above.

Theorem 1 states that \((g, h)\) should be chosen subject to

\[
\alpha S_2^t + \sigma S_2^t h_t - r_t S_2^t = 0,
\]

implying that

\[
h_t = \frac{r_t - \alpha}{\sigma}.
\]
The market contains no information on jump risk, and as in Section 7.1 we fix a martingale measure by assuming that the agent is risk-neutral w.r.t. risk due to the state of $X$, i.e.

$$g_t = 0.$$ 

TDE can now easily be found by Theorem 1.

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References