An investigation into parametric models for mortality projections, with applications to immediate annuitants’
and life office pensioners’ data

Terry Z. Sithole, Steven Haberman, Richard J. Verrall*

Department of Actuarial Science and Statistics, City University, Northampton Square, London EC1V 0HB, UK

Received 1 December 1999; received in revised form 1 May 2000; accepted 30 June 2000

Abstract

This paper investigates the use of parametric models for projecting mortality rates. The basic framework used is that of
geneneralised linear and non-linear models and can be considered as an extension of the Gompertz–Makeham models [Forfar
et al., J. Inst. Actuaries 115 (1988) 1; Trans. Faculty Actuaries 41 (1988) 97] to include calendar period. The data considered
are the CMI ultimate experience for immediate annuitants (male and female) over the period 1958–1994, and for life office
pensioners (male and female) over the period 1983–1996. The modelling structure suggested by Renshaw et al. [British
Actuarial J. 2 (II) (1996) 449] is used to investigate the data sets pertaining to the ultimate experiences, and to determine
a range of suitable models, analysing the data by age and calendar period. The properties of these models are investigated
and recommendations are made on which models are appropriate for use in projections. The select experience for immediate
annuitants’ is modelled using the structure suggested by Renshaw and Haberman [Insurance: Math. Econ. 19 (2) (1997) 105].
Projected forces of mortality using the recommended model are given for each experience. These are compared with the CMI
projected mortality rates. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Mortality projections; Immediate annuitants; Life office pensioners

1. Introduction

Mortality has shown a gradual decline over time, with the rates of decline not being necessarily uniform across the
age range. For pensioners and annuitants, it is important to be able to accurately measure changes in mortality over
time since the policyholders’ benefits depend on survival. If the standard mortality table used to calculate annuity
rates and reserves predicts higher mortality rates than actually experienced by the policyholders, the policyholders
will have been undercharged, and the company will incur losses. Also reserves will be understated.

In UK, the Continuous Mortality Investigations (CMI) committee of the Institute and Faculty of Actuaries makes
projections of future improvements in the mortality of pensioners and annuitants. The procedure essentially involves
two stages. Firstly, for a given investigation period, the data are graduated and mortality tables produced. In the
second stage, projected mortality tables are produced by applying reduction factors derived from a consideration of
past improvements and likely future improvements in the mortality rates.

* Corresponding author.
E-mail address: r.j.verrall@city.ac.uk (R.J. Verrall).

0167-6687/00 $ – see front matter © 2000 Elsevier Science B.V. All rights reserved.
PII: S0167-6687(00)00054-8
Renshaw et al. (1996) suggested a modelling structure in the framework of generalised linear models, which incorporates both the age variation in mortality and the underlying trends in the mortality rates. In this paper, we use this modelling structure to investigate mortality trends for immediate annuitants and life office pensioners. We focus on the projected forces of mortality and mortality improvement factors derived from the appropriate model for each experience.

Section 2 covers the current practice of the CMI in projecting mortality rates. In Section 3, the modelling structure suggested by Renshaw et al. (1996) is given. The application of the structure in modelling immediate annuitants’ and pensioners’ mortality experiences is discussed in Section 4, while in Section 5 we give a brief outline of a procedure for modelling select mortality and how this can be applied to the annuitants’ select data.

2. Current CMI practice in projecting mortality

The current practice of the CMI is to graduate the force of mortality at age $x$, $\mu_x$, by fitting the “Gompertz–Makeham” class of formulae:

$$\mu_x = GM(r, s) = \sum_{i=1}^{r-1} \alpha_i x^i + \exp \left( \sum_{j=0}^{s-1} \beta_j x^j \right), \quad (2.1)$$

with the convention that when $r = 0$, the polynomial term is absent, and when $s = 0$, the exponential term is absent (Forfar et al., 1988).

Tables resulting from the graduation, referred to as base tables, are then projected by applying time reduction factors, $RF(x, t)$, for an ultimate life attaining exact age $x$ at time $t$, where $t$ is measured in years from an appropriate origin. The projected mortality rate at time $t$ is

$$q_{x,t} = q_{x,0} RF(x, t), \quad (2.2)$$

where $q_{x,0}$ is the rate of mortality from the new base table for the appropriate experience. Section 4 of CMI Report 10 contains a full description of this method of projecting mortality rates.

The CMI have recently proposed a new mortality improvement model for pensioners and annuitants (CMI Report 17), to be used with mortality tables based on the 1991–1994 mortality experiences. The form of the model assumes that at each age, the limiting rate of mortality is non-zero and that the rate of mortality decreases to its limiting value by exponential decay. There is a further assumption that a given percentage of the total future decrease in mortality will occur in the first 20 years, with the percentage varying by age.

The mortality improvement model adopted by the CMI is

$$RF(x, t) = \alpha(x) + \left[ 1 - \alpha(x) \right] \left[ 1 - f(x) \right]^{t/20}, \quad (2.3)$$

where

$$\alpha(x) = \left\{ \begin{array}{ll}
& c, \quad x < 60, \\
& 1 + (1 - c)(x - 110) / 50, \quad 60 \leq x \leq 110, \\
& 1, \quad x > 110,
\end{array} \right.
$$

$$f(x) = \left\{ \begin{array}{ll}
& p, \quad x < 60, \\
& \frac{[(110 - x)p + (x - 60)q]}{50}, \quad 60 \leq x \leq 110, \\
& q, \quad x > 110,
\end{array} \right.$$

with $c = 0.13$, $p = 0.55$ and $q = 0.29$. 
The new model is such that the rate of improvement in mortality is assumed to depend on both age and time for lives aged between 60 and 110 years only. At ages below 60 years, the rate of improvement is assumed to depend only on time, while no improvement is assumed for lives aged 110 years and above. The same factors apply for all experiences, male and female, for data based on lives and amounts.

3. Modelling with respect to age and time

In this section, details of the modelling structure proposed by Renshaw et al. (1996) are given. The same procedures have been applied in modelling the immediate annuitants’ and pensioners’ mortality experiences.

For each experience and the specific duration \( d \), the force of mortality, \( \mu_{xt} \), at age \( x \), in calendar year \( t \), is modelled using formulae of the type

\[
\mu_{xt} = \exp \left[ \beta_0 + \sum_{j=1}^{s} \beta_j L_j(x') + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{s} \sum_{j=1}^{r} \gamma_{ij} L_j(x') t'^i \right],
\]

subject to the convention that some of the \( \gamma_{ij} \) terms may be pre-set to 0.

\( x_0 \) and \( t_0 \) are the transformed ages and transformed calendar years, respectively, such that both \( x_0 \) and \( t_0 \) are mapped onto the interval \([-1, 1]\). \( L_j(x) \) are Legendre polynomials generated by

\[
L_0(x) = 1, \quad L_1(x) = x, \quad (n + 1) L_{n+1}(x) = (2n + 1)x L_n(x) - n L_{n-1}(x),
\]

where \( n \) is an integer and \( n \geq 1 \).

Rewriting the equation for \( \mu_{xt} \) as

\[
\mu_{xt} = \exp \left\{ \sum_{j=0}^{s} \beta_j L_j(x') \right\} \exp \left\{ \sum_{i=1}^{r} \left( \alpha_i + \sum_{j=1}^{s} \gamma_{ij} L_j(x') \right) t'^i \right\},
\]

it is noted that the first of the two multiplicative terms is the equivalent of a Gompertz–Makeham graduation term, \( \text{GM}(0, s + 1) \) as defined by Eq. (2.1). The second multiplicative term may be interpreted as an age-specific trend adjustment term, provided at least one of the \( \gamma_{ij} \) terms is not pre-set to zero.

To estimate the unknown parameters \( \alpha_i, \beta_j \) and \( \gamma_{ij} \), the actual number of deaths \( A_{xt} \) are modelled as independent Poisson response variables \( A_{xt} \) of a generalised linear model with mean and variance given by

\[
E[A_{xt}] = m_{xt} = R_{xt}^c \mu_{xt}, \quad \text{var}(A_{xt}) = \phi m_{xt}.
\]

\( R_{xt}^c \) is the central exposed-to-risk, and \( \phi \) a scale parameter to take account of the fact that as the data are based on policy numbers rather than head counts, there may be duplicate policies issued on the same lives, resulting in over-dispersion of the Poisson random variable.

The unknown parameters are linked to the mean through the log function

\[
\log m_{xt} = \log R_{xt}^c + \beta_0 + \sum_{j=1}^{s} \beta_j L_j(x') + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{s} \sum_{j=1}^{r} \gamma_{ij} L_j(x') t'^i.
\]

so that

\[
\log m_{xt} = \log R_{xt}^c + \beta_0 + \sum_{j=1}^{s} \beta_j L_j(x') + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{s} \sum_{j=1}^{r} \gamma_{ij} L_j(x') t'^i.
\]
Estimation of the parameters is carried out using the quasi-log-likelihood approach, which, in this case, involves maximising the expression

\[
\frac{1}{\phi} \sum_{x,t} \left( -m_{xt} + a_{xt} \log m_{xt} \right).
\]

To determine the optimum values of \( r \) and \( s \), the improvement in the scaled deviance for successive increases in the values of \( r \) and \( s \) is compared with a \( \chi^2 \) random variable with 1 d.f. as an approximation. The optimum values chosen are the minimum values of \( r \) and \( s \) beyond which improvement in the deviance is not statistically significant.

The unscaled deviance corresponding to the predicted rates, \( \hat{\mu}_{xt} \), is

\[
D(e, f) = 2 \sum_{xt} \left\{ a_{xt} \log \left( \frac{\hat{a}_{xt}}{\hat{m}_{xt}} \right) - (a_{xt} - \hat{m}_{xt}) \right\},
\]

where \( \hat{m}_{xt} = R^c_{xt} \hat{\mu}_{xt} \), i.e., \( \hat{m}_{xt} \) is the number of deaths predicted by the model. The corresponding scaled deviance is defined as the unscaled deviance divided by the scale parameter \( \phi \), with \( \phi \) estimated from dividing the optimum unscaled deviance by the number of degrees of freedom \( v \).

Diagnostic tests on the model formula are carried out using the deviance residuals and the Pearson residuals. In addition, the parameter estimates are checked for statistical significance.

For this study, implementation was carried out using S-PLUS and details of statistical modelling in S-PLUS can be found in Chambers and Hastie (1993) and Venables and Ripley (1994).

4. Trend analysis of the immediate annuitants’ and pensioners’ mortality experiences

In general, for each experience analysed, determining a model that provides the best fit to the data did not present many problems. The difficulty was in identifying a model that not only provided a good fit for the data, but also had a good shape for the purpose of making projections. The appropriate model for each experience was therefore determined in stages.

Firstly, a model that provided the best fit to the data was determined. Projections based on the model over a 20-year period were then considered. Finally, using the information obtained by fitting the model, and the projected forces of mortality, the model was revised.

In Section 4.1, we describe the data analysed. A discussion of the results obtained from analysing the immediate annuitants’ experience is presented in Sections 4.2–4.4 and the pensioners’ experience is discussed in Sections 4.5 and 4.6. A comparison of the mortality improvement factors derived from the projected forces of mortality for pensioners and immediate annuitants is presented in Section 4.7.

4.1. The data

The data considered are the CMI immediate annuitants’ experience and the life office pensioners’ experience based on ‘lives’, or more accurately, based on policy counts.

The immediate annuitants’ ultimate experience at duration \( d = 1 \) year and over (1+ years) has been modelled for individual calendar years \( t = 1958–1994 \) for males and females separately. However, data pertaining to calendar years 1968, 1971 and 1975 were not available. The annuitants’ mortality experience for the period 1947–1957 was available and was modelled, but the results are not included in this paper. The calendar year 1958 was chosen as the starting point for our study because there was a well-documented change in the class of lives taking out immediate annuity contracts as a result of the Finance Act 1956. The female immediate annuitants’ experience was modelled for individual ages \( x \) ranging from 65 to 100 years, giving a total of 1224 data cells, while the males’ experience was modelled for individual ages 65–95 years, excluding age 94 in calendar year 1970, giving 1053 data cells.
An initial analysis of the males’ data revealed some inconsistencies notably at age 94 in 1970, where there were 96 recorded deaths corresponding to a central exposed-to-risk of 52.5. Hence this data cell was excluded from subsequent analyses. For the immediate annuitants’ experience modelled, the females exposed-to-risk constitutes about 74% of the total exposed-to-risk.

In line with current CMI practice, the life office pensioners’ experience was modelled for all durations combined, for individual calendar years \( t = 1983–1996 \). Data for the period prior to 1983 were not available for study. The experience was analysed for the age ranges \( x = 60–100 \) years for males, and \( x = 60–95 \) years for females, giving a total of 574 and 504 data cells, respectively. In terms of the exposed-to-risk, the male pensioners’ experience is just over 75% of the total pensioners’ experience analysed.

It should be noted that a subset of the male pensioners’ data analysed in this paper for ages 60–95 years and observation period 1983–1990 was modelled by Renshaw and Hatzoupoulos (1996), using the same modelling structure as suggested by Renshaw et al. (1996).

4.2. Trend analysis of the female immediate annuitants’ ultimate experience, duration 1+ years

In analysing the female annuitants’ experience at duration 1+ years, an examination of the improvement in deviance as a result of increasing terms in \( r \) and \( s \) in the first instance, and then introducing \( \gamma_{ij} \) terms leads to the adoption of a 7-parameter formula:

\[
\mu_{xt} = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j (x') + \left\{ \alpha_1 + \sum_{j=1}^{2} \gamma_{ij} L_j (x') \right\} t' \right].
\] (4.1)

Details of the parameter estimates are given in Table 1. The fitted model (4.1) consists of the GM(0,4) term in age effects \( x \), and a trend adjustment term which is linear in time \( t \) on the log scale. The coefficient of the trend adjustment term is itself a quadratic in \( x \).

Although the data are supportive of the model, some of the predicted forces of mortality outside the range of ages modelled are in fact increasing with time rather than decreasing. Fig. 1 shows the predicted forces of mortality to calendar year \( t = 2014 \) plotted against \( t \) on the log scale. At both ends of the age range, the predicted forces of mortality are such that there is a crossing over of forces of mortality. As an example, the predicted force of mortality for a life aged 55 in 2005 is higher than the predicted force of mortality for a life aged 66 years in the same year. Hence it appears that this model would only be useful for predictions of future forces of mortality provided the predictions are made within the range of ages over which the model has been fitted.

In fitting the model (4.1), it was observed that when the parameter \( \gamma_{12} \) is excluded from the formula, all the remaining six parameters in the model are statistically significant. Therefore the 6-parameter model which excludes the quadratic coefficient in age effects from the trend adjustment term was next fitted to the data. The revised

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>S.E.</th>
<th>( t )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_0 )</td>
<td>-2.6195</td>
<td>0.0070</td>
<td>-376.8458</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>1.8238</td>
<td>0.0158</td>
<td>115.4447</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>-0.0601</td>
<td>0.0165</td>
<td>-3.6557</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>-0.1174</td>
<td>0.0171</td>
<td>-6.8481</td>
</tr>
<tr>
<td>( \alpha_1 )</td>
<td>-0.1865</td>
<td>0.0121</td>
<td>-15.4538</td>
</tr>
<tr>
<td>( \gamma_{11} )</td>
<td>0.0226</td>
<td>0.0230</td>
<td>0.9838</td>
</tr>
<tr>
<td>( \gamma_{12} )</td>
<td>0.1163</td>
<td>0.0279</td>
<td>4.1752</td>
</tr>
</tbody>
</table>
Fig. 1. Female immediate annuitants, duration 1+ years, predicted forces of mortality based on the 7-parameter log-link model (4.1).
model was
\[
\mu_{x,t} = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + (\alpha_1 + \gamma_{11} L_1(x')) t' \right].
\] (4.2)

The predicted rates based on each of the two models (4.1) and (4.2), together with the crude mortality rates, are plotted against time \(t\) on the log scale at 5-year age intervals as shown in Fig. 2. There is little difference between the predicted rates from the two models for the age range 75–95 years. However, outside this range of ages, the forces of mortality predicted from the two models follow differing patterns.

Fig. 3 is a plot of the predicted forces of mortality based on the 6-parameter model given by Eq. (4.2), plotted at 5-year age intervals from age 55 to 110 years. The predictions have been made to calendar year 2014.

It is observed that the revised model provides reasonable predictions within and outside the range of ages over which the model was fitted. All the predicted forces of mortality progress smoothly with respect to both age and time, and it can be seen that the model naturally predicts a reduction in the rate of improvement in mortality at the older ages. Therefore the model described by (4.2) would seem to be appropriate for making predictions of future forces of mortality for female immediate annuitants, although the model defined by (4.1) provides the better fit to the data. It would therefore appear that in searching for a model that has a good shape for the purpose of making predictions, there has to be a trade-off between goodness-of-fit and predictive shape. Details of the revised fit are given in Table 2.

Using the modelling structure proposed by Renshaw et al. (1996), the advantage that we are able to make is the predictions of future forces of mortality directly from the model formula. In addition, a mortality improvement model can be derived also from the model formula for use with a given set of mortality tables for a given base year.

In deriving a mortality improvement model, we focus on the force of mortality and use the same format as that used by the CMI, except that we define the reduction factor to be a ratio of forces of mortality (rather than of mortality rates). Thus
\[
\mu_{x,t} = \mu_{x,0} \text{RF}(x, t),
\]

where \(\mu_{x,0}\) is the force of mortality for a life attaining exact age \(x\) in the base calendar year (taken as year 0), i.e., the base rate from the mortality table for the appropriate experience; \(\mu_{x,t}\) the force of mortality for a life attaining exact age \(x\) in calendar year: base year + \(t\).

The formula for the reduction factor for the 6-parameter model (4.2) is then given by
\[
\text{RF}(x, t) = \exp \left[ \frac{t}{w_t} (\alpha_1 + \gamma_{11} x') \right],
\] (4.3)
where \(w_t\) denotes half of the calendar year range for the investigation period. For the immediate annuitants’ experience analysed, \(w_t = 18\) for both males and females.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>S.E.</th>
<th>(t)-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta_0)</td>
<td>-2.6274</td>
<td>0.0068</td>
<td>-384.9264</td>
</tr>
<tr>
<td>(\beta_1)</td>
<td>1.8506</td>
<td>0.0148</td>
<td>125.3246</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>-0.0833</td>
<td>0.0158</td>
<td>-5.2843</td>
</tr>
<tr>
<td>(\beta_3)</td>
<td>-0.0991</td>
<td>0.0167</td>
<td>-5.9264</td>
</tr>
<tr>
<td>(\alpha_1)</td>
<td>-0.2206</td>
<td>0.0090</td>
<td>-24.4358</td>
</tr>
<tr>
<td>(\gamma_{11})</td>
<td>0.0616</td>
<td>0.0213</td>
<td>2.8877</td>
</tr>
</tbody>
</table>

Table 2
Female immediate annuitants, duration 1+ years, 6-parameter log-link model (deviance = 2292.8 on 1218 d.f.; \(\phi = 1.8846\))
Fig. 2. Female immediate annuities, duration 1+ years, crude mortality rates and predicted forces of mortality based on the 7-parameter (4.1) and the 6-parameter (4.2) log-link models.
Fig. 3. Female immediate annuitants, duration 1+ years, predicted forces of mortality based on the 6-parameter log-link model (4.2).
In particular, based on the parameter estimates given in Table 2 and the data-set for female immediate annuitants, the formula for $RF(x, t)$ is

$$RF(x, t) = \exp \left[ \frac{t}{18} \left(-0.2206 + 0.0616 \left(\frac{x - 82.5}{17.5}\right)\right) \right].$$

or

$$RF(x, t) = \exp\left[(-0.028389 + 0.000196x)t\right].$$

\section*{4.3. Trend analysis of the male immediate annuitants’ ultimate experience, duration 1+ years}

For the male immediate annuitants’ ultimate experience at duration 1+ years, a simple 3-parameter model was found to fit the data adequately. In this model, the trend adjustment term does not involve any age-specific terms (i.e., all the $\gamma_{ij}$ terms are zero), so that the changes in the forces of mortality over time are assumed to be independent of age. The parameter estimates, which are all statistically significant, are shown in Table 3.

The model formula is

$$\mu_{xt} = \exp[\beta_0 + \beta_1 L_1(x') + \alpha_1 t'],$$

which may be written as

$$\log \mu_{xt} = \beta_0 + \beta_1 x' + \alpha_1 t'.$$

The formula for the reduction factors derived from Eq. (4.5) is

$$RF(x, t) = \exp\left[\alpha_1 \frac{t}{w_t}\right].$$

For the particular male annuitants data-set modelled and the parameter estimates given in Table 3, the reduction factor is

$$RF(x, t) = \exp[-0.010811t].$$

In applying this model to predict future forces of mortality, there is an underlying assumption that mortality trends over time only depend on the time factor and not on the age of the individual. This assumption does not seem reasonable since the rate of change in mortality over time would be expected to vary with age. It is possible that because the male immediate annuitants’ investigation is small, the underlying pattern might not be fully captured by the model derived from analysing this experience over a period of 34 years. A longer investigation period could be considered. However, with the changes in mortality occurring over recent years, a longer period might also result in a model that does not represent the more recent trends adequately. It was therefore considered appropriate to model the males’ data using the 6-parameter model (4.2) determined for the larger female immediate annuitants’ experience.

Increasing the number of parameters can only improve the goodness-of-fit for the model, although the additional parameters introduced would not be expected to be statistically significant. The particular form of the 6-parameter model adopted results in rates that progress smoothly over both age and time, so that the smoothness criteria would
Table 4  
Male immediate annuitants, duration 1+ years, 6-parameter log-link model (deviance = 1891.48 on 1047 d.f.; $\phi = 1.8314$)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>S.E.</th>
<th>$t$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>$-2.4735$</td>
<td>$0.0086$</td>
<td>$-287.86$</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>$1.3616$</td>
<td>$0.0178$</td>
<td>$76.36$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$-0.0248$</td>
<td>$0.0210$</td>
<td>$-1.18$</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>$0.0102$</td>
<td>$0.0228$</td>
<td>$0.45$</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>$-0.1995$</td>
<td>$0.0136$</td>
<td>$-14.69$</td>
</tr>
<tr>
<td>$\gamma_{11}$</td>
<td>$0.0293$</td>
<td>$0.0292$</td>
<td>$1.00$</td>
</tr>
</tbody>
</table>

also be satisfied. Table 4 gives the parameter estimates derived from again fitting the model:

$$
\mu_{x,t} = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + (\alpha_1 + \gamma_{11} L_1(x'))t' \right].
$$

Fig. 4 is a comparative plot of the predicted forces of mortality at 5-year age intervals from age $x = 55$ to 110 years, for male immediate annuitants at duration 1+ years. The predicted rates are based on the two models given by Eqs. (4.5) and (4.2), with parameter estimates given in Tables 3 and 4, respectively. As for the females experience, predictions have been made to calendar year $t = 2014$.

There is little difference in the predicted rates from the two models for the period and the age range over which the data were analysed. As expected, the predicted forces of mortality based on each model progress smoothly with respect to age and time. However, the preferred model would be the 6-parameter model (4.2), which has been identified from modelling a larger amount of data. This model exhibits the desired features for making predictions of future forces of mortality. In addition, except for ages above 100 years, forces of mortality predicted on the basis of this model are generally lower for the projection period.

It should be borne in mind that the male immediate annuitants’ experience is small, so that at each individual age $x$ in each calendar year $t$, the observed experience is small and this can present problems in modelling the data.

Based on the 6-parameter model, the appropriate formula for the reduction factor for male immediate annuitants at duration 1+ years is again given by Eq. (4.3), i.e.

$$
RF(x, t) = \exp \left[ \frac{t}{w_t} (\alpha_1 + \gamma_{11} x') \right],
$$

where $x' = (x - 80)/15$ for male annuitants.

Based on the parameter estimates given in Table 4, the reduction factor is

$$
RF(x, t) = \exp \{(-0.019765 + 0.000109x)t\}.
$$

4.4. Comparison of predicted forces of mortality and mortality improvement factors for immediate annuitants, duration 1+ years

Based on the 6-parameter log-link model with a trend adjustment term that is linear in time $t$ on the log scale, predicted forces of mortality for male and female immediate annuitants are compared in Section 4.4.1. The forces of mortality are projected over a 20-year period from 1995 to 2014. Section 4.4.2 deals with comparisons of reduction factors derived from the projected forces of mortality.

4.4.1. Comparison of predicted forces of mortality for immediate annuitants

Fig. 5 is a comparative plot of predicted forces of mortality for male and female immediate annuitants. The parameter estimates are as given in Table 2 for females and Table 4 for males.
Fig. 4. Male immediate annuities duration 1–years: predicted forces of mortality based on the 3-parameter (4, 5) and the 6-parameter (4, 2) log-link models.
We observe that the predicted forces of mortality for males are consistently higher than the predicted forces of mortality for females at all ages and in each of the 20 calendar years for which the rates have been projected. The highest differences occur between ages 100 and 110 years, where there is a rapid increase with age in the predicted forces of mortality for males. For example, in calendar year 1995, the force of mortality for a male aged 100 years is just under 0.5, while the corresponding predicted rate at age 110 is above 1. The predicted forces of mortality for males at these older ages seem rather high. It would appear that the marked improvements in mortality which have occurred in the male population in the more recent years are not adequately reflected in the predicted forces of mortality at the older ages. In order to give more weight to the mortality experience in the more recent years, it might be necessary to determine the parameter estimates from an observation period which excludes the earlier years, while still fitting the same model.

4.4.2. Comparison of mortality improvement factors for immediate annuitants, 1993–2014

Fig. 6 shows comparative plots of the new CMI reduction factors (Eq. (2.3)) and reduction factors derived from the projected forces of mortality based on the 6-parameter log-link model for male and female immediate annuitants. For consistency with the CMI improvement model, 1992 has been taken as the base year.

Given a base table of mortality rates, the CMI reduction factors predict lower mortality rates at the younger ages, and higher mortality rates at the older ages. The reduction factors based on the log-link model are closest to the CMI factors at ages between 75 and 85 years. At ages above 90 years, there is a large difference between the CMI and the log-link reduction factors. By the age of 110 years, the CMI mortality improvement model assumes that there is no longer improvement in mortality over time.

It would therefore appear that the CMI reduction factors have a shape with respect to age and time that is different from that which emerges from a model-based analysis of the historic mortality experience. We have doubts about the suitability of the CMI reduction factors for making predictions of future mortality rates for immediate annuitants.

4.5. Trend analysis of the male life office pensioners’ experience

From an analysis of deviance and the statistical significance of additional parameters introduced to the model formula, the best fitting model was determined to be a 7-parameter model that is quadratic in time on the log scale. Thus the model was

$$
\mu_{xt} = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + ((\alpha_1 + \gamma_{11}x')t' + \alpha_2 t^2) \right].
$$

(4.7)

In a recent study, Renshaw and Hatzoupoulos (1996) have modelled the male pensioners’ mortality experience by targeting the probability of death, $q_{xt}$. The number of pension policies was modelled as a binomial response variable of a generalised linear model with possible over-dispersion, in conjunction with the complementary log-log link function:

$$
\log(-\log(1 - q_{xt})) = \eta_{xt}.
$$

The formula adopted in that study was

$$
\log(-\log(1 - q_{xt})) = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + ((\alpha_1 + \gamma_{11}x')t' + \alpha_2 t^2 + \alpha_3 t^3) \right].
$$

(4.8)

We note that the difference between Eqs. (4.7) and (4.8) is that in (4.8), there is one additional term involving $t^3$, i.e., the trend adjustment term in (4.8) is a cubic in time $t$ on the log scale. In this study, no attempt has been made to fit the same 8-parameter model as it turns out that higher order terms in $t$ result in unreasonable predictions.
Fig. 6. Comparison of time reduction factors CMI model and suggested model for immediate annuitants.
Fig. 7 shows the predicted forces of mortality and mortality improvement factors for male pensioners for $t = 1992–2016$ based on the 7-parameter model given by Eq. (4.7). It is observed that although the forces of mortality predicted progress gradually with respect to age, they appear to progress rapidly with time, a feature that would not be desirable.

In view of these results and the conclusions drawn in Sections 4.2 and 4.3 in the analyses of immediate annuitants’ experiences, a 6-parameter model excluding the term involving $t^2$ was then fitted to the data. Thus the revised model was again

$$
\mu_{x+t} = \exp \left[ \beta_0 + \sum_{j=1}^{3} \beta_j L_j(x') + (\alpha_1 + \gamma_{11}x')t' \right].
$$

Fig. 8 shows the crude mortality rates and the predicted forces of mortality for male pensioners based on the 7-parameter (4.7) and the 6-parameter (4.2) models plotted against calendar year $t = 1983–1996$ on the log scale. The rates are shown at 5-year age intervals from $x = 60$ to 100 years. Although the 7-parameter model has a trend adjustment term that is quadratic in time $t$ on the log scale, while the 6-parameter model is linear in $t$ also on the log scale, it is difficult to discern any differences between the predicted rates from the two models based on a visual inspection of Fig. 8. Both models fit the data adequately.

A detailed examination of the predicted forces of mortality reveals that in general, the rates based on the 7-parameter model are lower than the rates based on the 6-parameter model. However, the differences are very small.

The projected forces of mortality and mortality improvement factors for male life office pensioners, based on the 6-parameter log-link model, are shown in Fig. 9. The projections have been made over the 20-year period from 1997 to 2016. It is observed that the projected rates have a declining trend at all ages below 106 years. The model was therefore adopted since it has a reasonable shape for predictions for most ages other than extreme old age. Table 5 gives details of the fit of the 6-parameter model given by Eq. (4.2) to the male pensioners’ mortality experience.

The mortality improvement model for male life office pensioners is again

$$
RF(x, t) = \exp \left[ \frac{t}{u_t} (\alpha_1 + \gamma_{11}x') \right],
$$

with $u_t = 6.5$ for both male and female pensioners, and $x' = (x - 80)/20$ for male pensioners. For the particular parameter estimates given in Table 5, the reduction factor is

$$
RF(x, t) = \exp\{(-0.078846 + 0.000744x)t\}.
$$

### 4.6. Trend analysis of the female life office pensioners’ experience and comparison with male life office pensioners’ experience

In modelling the female life office pensioners’ experience, an analysis of the deviance suggested choosing $s = 1$ or 3, $r = 2$ and the age-specific trend adjustment term involving $x$ and $t$, i.e., $\gamma_{11}$. The value of $s$ was then chosen to be

| Table 5 |
|---|---|---|
| Parameter | Estimate | S.E. | $t$-value |
| $\beta_0$ | -2.5356 | 0.0050 | -510.5671 |
| $\beta_1$ | 1.8232 | 0.0142 | 128.3969 |
| $\beta_2$ | -0.2649 | 0.0124 | -21.3053 |
| $\beta_3$ | -0.0475 | 0.0155 | -3.0559 |
| $\alpha_1$ | -0.1257 | 0.0045 | -28.2352 |
| $\gamma_{11}$ | 0.0967 | 0.0129 | 7.5275 |
Fig. 7. Male life office pensioners, predicted forces of mortality and mortality improvement factors 1992–2016; 7-parameter log-link model (4.7).
Fig. 8. Male life office pensioners, crude mortality rates and predicted forces of mortality based on the 7-parameter (4.7) and the 6-parameter (4.2) log-link models.
Fig. 9. Male life office pensioners, predicted forces of mortality and mortality improvement factors 1992–2016; 6-parameter log-link model (4.2).
as this was the optimal value of $s$ determined from fitting the larger male life office pensioners’ experience. Given the analyses of the immediate annuitants’ experience in Sections 4.2 and 4.3, and the male life office pensioners’ experience in Section 4.5, the value of $r$ was chosen to be 1. Therefore the same 6-parameter model given by Eq. (4.2) was also adopted for female pensioners.

As for the male life office pensioners’ experience, it was observed that the 6-parameter model provided a good fit to the female life office pensioners’ data. Fig. 10 is a comparative plot of male and female life office pensioners’ predicted forces of mortality up to calendar year 2016. The predicted forces of mortality show a declining trend at all ages below 104 years for females (and 106 years for males). At the younger ages, the predicted rates for males appear to be improving at a faster rate than the females’ predicted rates. For ages above 95 years, the predicted rates for male pensioners are lower than the predicted rates for females.

It should be noted that although the life office pensioners’ mortality experience has been graduated over a period of 13 years only, the model derived still predicts reasonable future forces of mortality at most ages and for the 20-year period over which forces of mortality have been forecast. Table 6 gives parameter estimates for the adopted female life office pensioners’ model.

For female life office pensioners, the mortality improvement model is given by

$$RF(x, t) = \exp \left\{ \frac{t}{6.5} \left[ \alpha_1 + \gamma_{11} \left( \frac{x - 77.5}{17.5} \right) \right] \right\},$$

i.e., $x' = \frac{x - 77.5}{17.5}$.

Based on the parameter estimates given in Table 6, the formula for the reduction factor is

$$RF(x, t) = \exp \{ (-0.050651 + 0.000489x) t \}.$$

Fig. 11 is a comparison of CMI mortality improvement factors with the improvement factors derived from the log-link model for male and female pensioners. The factors are shown at 5-year age intervals from age 55 to 100 years. It is observed that in general, the reduction factors derived for life office pensioners exhibit a similar trend to CMI reduction factors. Given a base table of mortality rates for male pensioners, the reduction factors based on the log-link model would predict lower mortality rates than predicted by applying the CMI factors at all the ages. However, for female pensioners, the reduction factors based on the log-link model predict less improvement in mortality than the CMI reduction factors.

### 4.7. Comparison of mortality improvement factors for life office pensioners and immediate annuitants

Mortality improvement factors derived from the projected forces of mortality based on the log-link model, together with the CMI mortality improvement factors, are shown in Fig. 12 for male immediate annuitants with policy duration 1+ years and male life office pensioners. In deriving the mortality improvement factors, 1992 has been taken as the base year. The reduction factors are again shown at 5-year age intervals from age 55 to 110 years.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>S.E.</th>
<th>$t$-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>-3.1771</td>
<td>0.0069</td>
<td>-462.4352</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>1.8380</td>
<td>0.0143</td>
<td>128.5235</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-0.0259</td>
<td>0.0170</td>
<td>-1.5197</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>-0.0364</td>
<td>0.0189</td>
<td>-1.9252</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>-0.0830</td>
<td>0.0098</td>
<td>-8.4430</td>
</tr>
<tr>
<td>$\gamma_{11}$</td>
<td>0.0556</td>
<td>0.0215</td>
<td>2.5910</td>
</tr>
</tbody>
</table>
Fig. 10 Life office pensions: predicted forces of mortality 1983–2016 based on the 6-parameter log-link model (4.2).
Fig. 11. Comparison of time reduction factors CMI model and suggested model for life office pensioners.
Fig. 12. Comparison of log-link reduction factors and CMI reduction factors. Male immediate annuitants duration 1+ years and male life office pensioners.
Fig. 13. Comparison of English reduction factors and CMI reduction factors. Female immediate annuities duration 1 year and female life office pensions.
Table 7
Reduction factors predicted to 2014

<table>
<thead>
<tr>
<th>Age</th>
<th>Male annuitants</th>
<th>Female annuitants</th>
<th>Male pensioners</th>
<th>Female pensioners</th>
<th>CMI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>duration 1+ years</td>
<td>duration 1+ years</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>55</td>
<td>0.7382</td>
<td>0.6785</td>
<td>0.4340</td>
<td>0.5927</td>
<td>0.4915</td>
</tr>
<tr>
<td>60</td>
<td>0.7470</td>
<td>0.6932</td>
<td>0.4710</td>
<td>0.6255</td>
<td>0.4915</td>
</tr>
<tr>
<td>65</td>
<td>0.7560</td>
<td>0.7083</td>
<td>0.5112</td>
<td>0.6600</td>
<td>0.5630</td>
</tr>
<tr>
<td>70</td>
<td>0.7651</td>
<td>0.7237</td>
<td>0.5548</td>
<td>0.6965</td>
<td>0.6301</td>
</tr>
<tr>
<td>75</td>
<td>0.7743</td>
<td>0.7394</td>
<td>0.6021</td>
<td>0.7350</td>
<td>0.6927</td>
</tr>
<tr>
<td>80</td>
<td>0.7836</td>
<td>0.7555</td>
<td>0.6535</td>
<td>0.7756</td>
<td>0.7506</td>
</tr>
<tr>
<td>85</td>
<td>0.7930</td>
<td>0.7719</td>
<td>0.7092</td>
<td>0.8184</td>
<td>0.8039</td>
</tr>
<tr>
<td>90</td>
<td>0.8026</td>
<td>0.7887</td>
<td>0.7697</td>
<td>0.8636</td>
<td>0.8526</td>
</tr>
<tr>
<td>95</td>
<td>0.8122</td>
<td>0.8059</td>
<td>0.8354</td>
<td>0.9113</td>
<td>0.8966</td>
</tr>
<tr>
<td>100</td>
<td>0.8220</td>
<td>0.8234</td>
<td>0.9067</td>
<td>0.9617</td>
<td>0.9358</td>
</tr>
<tr>
<td>105</td>
<td>0.8319</td>
<td>0.8413</td>
<td>0.9840</td>
<td>1.0148</td>
<td>0.9703</td>
</tr>
<tr>
<td>110</td>
<td>0.8419</td>
<td>0.8596</td>
<td>1.0680</td>
<td>1.0709</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

However, the reduction factors applicable to male life office pensioners are only shown for ages up to 105 years, the ages at which the projected forces of mortality are declining with time. It should be noted that the CMI reduction factors at ages below 60 are given by the reduction factor at age 60 years. We observe that the CMI reduction factors and the reduction factors derived from the log-link model for male pensioners have a different shape from the reduction factors derived for male immediate annuitants.

Fig. 13 shows reduction factors based on the log-link model for female life office pensioners and female immediate annuitants with policy duration 1+ years, together with CMI reduction factors for comparison. As for male immediate annuitants, the reduction factors for female immediate annuitants are shown at 5-year age intervals from 55 to 110 years. Reduction factors in respect of female life office pensioners are shown up to age 100 years. As for the male immediate annuitants, the reduction factors for female immediate annuitants exhibit a pattern that is different from the pattern in the pensioners’ and the CMI reduction factors.

Table 7 shows the reduction factors which would apply in 2014 based on the log-link models adopted in this study. The CMI reduction factors are also shown for comparison purposes. In practice, where the model-based reduction factor exceeds 1, we would set the value to 1.

5. Modelling of immediate annuitants’ select experience

In general, immediate annuitants exercise a strong degree of self-selection, resulting in lower mortality rates during the initial years following selection. Consequently, in addition to modelling the ultimate experience, it is necessary to model the select experience for immediate annuitants.

Renshaw and Haberman (1997) suggested a modelling structure which involves the modelling of the ultimate experience by any suitable method, and then modelling the log crude mortality ratios for the select experience relative to the ultimate experience. We give a brief description of the modelling procedure in Section 5.1 and an example of its application in Section 5.2.

5.1. Modelling select mortality: methodology

For a given observation period, let $d\alpha_{x,t}$ be the observed number of deaths at age $x$, time $t$ and duration $d$ and $dR^c_{x,t}$ is the matching central exposed-to-risk at age $x$, time $t$ and duration $d$. 
Fig. 14. Female immediate annuitants, 1974–1994 combined experience.
For an individual select duration $d$ relative to the corresponding ultimate duration denoted $d_+$ at age $x$ and time $t$, define the statistic

$$dz_{x,t} = \log\left(\frac{d\mu_{x,t}}{d+\mu_{x,t}}\right) - \log\left(\frac{d+c\mu_{x,t}}{d+c+\mu_{x,t}}\right).$$

The underlying patterns are modelled by targeting

$$E[dz_{x,t}] \approx \log\left(\frac{d\mu_{x+1/2,t}}{d+\mu_{x+1/2,t}}\right) = d\eta_{x,t},$$

with weights

$$d\omega_{x,t} = \frac{d\mu_{x,t}}{d\mu_{x,t} + d+c\mu_{x,t}},$$

where $d\mu_{x,t}$ and $d+c\mu_{x,t}$ denote the force of mortality at age $x$, time $t$, for select duration $d$ and ultimate duration $d_+$, respectively.

The resulting forces of mortality for the individual select durations are given by

$$d\mu_{x+1/2,t} = d+c\mu_{x+1/2,t} \exp(d\eta_{x,t}).$$

The force of mortality at age $x$, in each time period $t$ and select duration $d$, can therefore be considered as a proportion of the corresponding force of mortality at age $x$ and ultimate duration $d_+$, where the adjustment factor is given by $\exp(d\eta_{x,t}).$


Assuming that the changes in mortality due to time are the same for select and ultimate lives, and that these are adequately represented in the modelling of the ultimate experience, the suffix $t$ can be dropped from $d\eta_{x,t}$ and the linear predictor denoted as $d\eta_x$. The responses $dz_x$ are then modelled from the combined experience over the observation period for each select duration $d = 0, 1, \ldots$ and the ultimate duration $d_+$. This is particularly useful when modelling the select experience for immediate annuitants since the data are scanty.

As an example, the select mortality experience for female immediate annuitants for the period 1974–1994 has been modelled at select durations 0 and 1–4 combined, relative to the ultimate experience at duration $d_+ = 5+$. The $dz_x$ response plots against age $x$ together with the fitted values are shown in Fig. 14. The linear predictor structure fitted is of the form

$$d\eta_x = \theta_d,$$

i.e., a model representing a horizontal straight line structure for each duration $d$. Plots of the linear predictor and the adjustment factor against age are also shown. Table 8 is a reproduction of the linear predictors and adjustment factors for each select duration $d$.

<table>
<thead>
<tr>
<th>Duration, $d$</th>
<th>Linear predictor, $\theta_d$</th>
<th>Adjustment factor $\exp(\theta_d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-0.397989$</td>
<td>0.671669</td>
</tr>
<tr>
<td>1–4</td>
<td>$-0.160523$</td>
<td>0.851698</td>
</tr>
</tbody>
</table>

Table 8
Female immediate annuitants, 1974–1994 experience: linear predictors and adjustment factors based on ultimate rates at duration 5+ years
The forces of mortality for each select duration $d$ are obtained by applying the relevant adjustment factor to the predicted forces of mortality at ultimate duration $5+\text{years}$ based on modelling the 1974–1994 mortality experience with respect to both age and time as detailed in Section 4. For a given select duration $d$, the form of the linear predictor (5.1) is such that the adjustment factor is the same at all ages $x$.

The constraints $0 \leq d \eta_x \leq 0$ for $d = 1–4$ combined, are satisfied, ensuring the ordering of forces of mortality with respect to duration $d$ at each age $x$.

6. Conclusion

We have shown that the modelling structure suggested by Renshaw et al. (1996) can in fact be used to predict future forces of mortality provided the trend adjustment term is linear in time on the log scale and the coefficient of this term is itself linear in age effects. Including $y_{ij}$ terms involving higher terms in $x$ resulted in forces of mortality that did not progress smoothly at ages outside the range of ages modelled. On the other hand, introducing higher order terms in time $t$ resulted in an unrealistically rapid improvement in mortality.

Although data pertaining to life office pensioners have been analysed over a period of 13 years only, predicted forces of mortality for both male and female pensioners appear to be reasonable at most ages, particularly ages below 100 years. In addition, the mortality improvement factors derived are consistent with CMI mortality improvement factors which are based on the pensioners’ mortality experience. However, data over a longer period might still need to be analysed in order to derive parameter estimates that give reasonable predictions at all ages.

It is also seen that although the CMI mortality improvement factors are close to the improvement factors derived from modelling the life office pensioners’ experience, the improvement factors derived from modelling the immediate annuitants’ experience do not have a similar pattern. This could be because the underlying mortality trends in the two experiences are different. For example, because the annuitants exercise some degree of self-selection, there is less variation in mortality over time than there is for pensioners. As a result, improvement factors appropriate for the pensioners’ experience might not be appropriate for the immediate annuitants’ experience.

Consideration of alternative approaches to the modelling of time trends in annuitants’ and pensioners’ mortality is the object of current research, e.g. using the methodology of Lee and Carter (1992).

Acknowledgements

The authors wish to thank the CMI Bureau for providing the data and ERC Frankona for financial support for this study.

References

Continuous Mortality Investigation Bureau, 1990. Continuous Mortality Investigation Reports CMIR 10. The Institute of Actuaries and the Faculty of Actuaries, UK.
Continuous Mortality Investigation Bureau, 1999. Continuous Mortality Investigation Reports CMIR 17. The Institute of Actuaries and the Faculty of Actuaries, UK.