An equilibrium asset pricing model based on Lévy processes: relations to stochastic volatility, and the survival hypothesis

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Abstract

This paper presents some security market pricing results in the setting of a security market equilibrium in continuous time. The model consists in relaxing the distributional assumptions of asset returns to a situation where the underlying random processes modeling the spot prices of assets are exponentials of Lévy processes, the latter having normal inverse Gaussian marginals, and where the aggregate consumption is inverse Gaussian. Normal inverse Gaussian distributions have proved to fit stock returns remarkably well in empirical investigations. Within this framework we demonstrate that contingent claims can be priced in a preference-free manner, a concept defined in the paper. Our results can be compared to those emerging from stochastic volatility models, although these two approaches are very different. Equilibrium equity premiums are derived, and calibrated to the data in the Mehra and Prescott [J. Monetary Econ. 15 (1985) 145] study. The model gives a possible resolution of the equity premium puzzle. The “survival” hypothesis of Brown et al. [J. Finance L 3 (1995) 853] is also investigated within this model, giving a very low crash probability of the market. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper investigates an asset pricing model, where the asset returns are distributed according to a normal inverse Gaussian (NiG) distribution. The underlying stochastic processes are assumed to be Lévy processes, an assumption that will be discussed below. One reason for studying asset returns with NiG marginals is that these have been found promising in fitting real asset returns, and to outperform other distributions in this regard, such as the normal and the stable Pareto distributions.

We attempt to derive NiG marginals of asset returns in a framework with one consumption good, assumed to be inverse Gaussian at each point in time, where the asset returns are assumed to be conditionally normally distributed, given this consumption good. We then proceed to construct stochastic processes for the returns, having the NiG marginal distributions, and otherwise being discontinuous Lévy processes, where numerous jumps of small sizes dominate the sample path behavior.

In this situation the arbitrage pricing theory does not provide us with a unique equivalent martingale measure, so here we find the general equilibrium setup particularly advantageous. We attempt to price derivative securities within the setting of one consumption good and a representative agent, and demonstrate how “preference-free”...
evaluations of contingent claims result, where only the short-term interest rate turns out to directly depend on preferences.

Given that a new asset pricing model is proposed, it seems natural to investigate how well it performs, and in particular if it is able to explain existing puzzles in the theory of asset pricing. This we attempt to do in Section 5, where we derive a consumption-based capital asset pricing model (CCAPM) based on the NiG model, and calibrate it to the data in the Mehra and Prescott study of 1985. Although we do not perform any type of empirical investigation, it turns out that the equity premiums may be explained for moderate values of the risk aversion and impatience rate in the market, yielding a possible explanation of the equity premium puzzle. Moreover, the survival hypothesis of Brown et al. from 1995 can be tested in this framework, and a market crash probability may be endogenously derived, calibrated to the above data. Based on the given 90 years history of market data, a rather low market crash probability was estimated.

The way we construct our model, it turns out to be related to stochastic volatility models of arbitrage pricing theory, the connection being that the aggregate consumption process in our framework plays the role of the exogenously imposed stochastic volatility process in the arbitrage pricing theory. The apparent advantage with the equilibrium theory is that the randomness in the volatility parameter now comes as a result of more basic assumptions, not being proposed as an extra exogenous source of uncertainty.

An early reference to a related model in finance is McCulloch (1978), who considered a return process with Cauchy-distributed marginals. Time series of daily returns usually do not exhibit significant autocorrelation, so it may therefore seem natural to model the returns in non-overlapping periods of time to be independent and identically distributed, hence the Lévy process seems reasonable. This does not, however, imply a zero correlation structure of each associated asset price. Since a Lévy process is also a semimartingale, we have to our disposal the associated stochastic calculus for this latter category of processes, a fact we make use of.

The first to introduce the NiG distribution to finance was Barndorff-Nielsen (1994), based on a paper of Eberlein and Keller (1994). The latter demonstrated that asset returns were well fitted by hyperbolic distributions. This class of distributions turns out to outperform other distributions as models of returns of stocks in the German stock market. The hyperbolic distribution lacks the desirable feature of being closed under convolutions, and based on this observation a stochastic process was constructed, built on a similar class of distributions, having the convolution property and termed the NiG distribution. Later work by, e.g., Rydberg (1996) has indicated that the NiG distribution provides on its own a promising fit to returns on stocks.

This paper is organized as follows. The economic model is explained in Section 2.1, and the probability model of asset returns is given in Sections 2.2–2.4. In this part we model returns as cumulative logarithmic returns. Preference-free evaluation is discussed in Section 3, and a special option pricing formula is derived in Section 4, where its relation to stochastic volatility models is briefly discussed. A CCAPM is derived in Section 5. In Section 6, we view returns as cumulative, simple returns by the use of the semimartingale exponential, in which case we can develop an approximation for the ruin intensity of a risky asset. In Section 7, the results of Section 6 are used to discuss the survival hypothesis. In Section 8, we consider several risky assets, and Section 9 concludes.

2. Equilibrium pricing of derivative securities with Lévy process driven uncertainty

2.1. Introduction: the economic model

In this section, we present the economic model. It consists of $N + 1$ primitive securities, where $N$ of them are risky with return processes $R(t) = (R_1(t), \ldots, R_N(t))$, paying no dividends and with associated price processes $S(t) = (S_1(t), \ldots, S_N(t))$, where $t$ is the time variable, $0 \leq t \leq T$, and one is a risk-less security with real spot-price unity, and real return rate equal to the short-term interest rate $r(t)$. There are $m$ agents in this market each one being characterized by a nonzero consumption/endowment process $e_i$, and by strictly increasing utility functions $U_i(e_i)$ which are time-additive and of the von Neumann–Morgenstern type, i.e., $U_i(e_i(t)) = E\{\int_0^T u_i(e_i(t), r(t)) \, dr\}$. Special to our model is that $c_i(t)$ is the total amount of consumption by time $t$ for agent $i$. From this starting point we know
the conditions under which there exists a capital market equilibrium with a representative agent felicity index $u$, in which all the agents are maximizing their utility functions subject to budget constrains such that markets clear. The main analytic issue is then the determination of equilibrium price behavior.

In the model to be specified below, we will find it convenient to let the felicity index $u$ depend on accumulated consumption $c_t$ during the time period $[0, t]$. This may seem unusual to some readers. There is, however, a recent literature on deviations from the standard separable, additive utility representations. Following, e.g., Constantinides (1990), let us consider

$$z_t = c_0 e^{-at} + \int_0^t e^{-a(t-s)} d_c s,$$  

(1)

where $a$ and $d$ are constants, $a \geq 0$ and $d > 0$, i.e., $z_t$ is a process of weighted past consumption derived from the consumption process $c_t$, the latter denoting the total amount of consumption by time $t$, aggregated over all the consumers. Clearly past consumption matters more, the smaller the parameter $a$ is. Let $y$ be a process defined analogous to (1) but with a different set of parameters. Then the felicity index $u$ considered would be a function $u(z, y, t)$ of two variables and time, say, where $\partial u/\partial z > 0$, and $\partial u/\partial y < 0$. The utility function of the market is then given by $U(c) = E[\int_0^T u(z_s, y_s, t) \, dt]$. In such a case we would have a situation with both habit formation and durability. Habit formation because of the process $y$ and the negative partial derivative of the felicity index with respect to its second argument, durability because of the process $z$ and the positive partial derivative of $u$ with respect to its first argument (see, e.g., Hindy et al., 1997).

Returning to our model, we have a situation with only durability, where $d = 1$ and $a = 0$. This is a strong form of durability, where past consumption clearly matters. Along a possible optimal consumption path $c$, a large value of the total consumption for some $t$, would typically imply a smaller value at time $t+$ with high probability.

Prices are then determined as follows. If $(q, D)$ is any of the given securities with real price process $q$ and accumulated dividend process $D$, the real market value $q$ at each time $t$ satisfies the following:

$$q(t) = \frac{1}{u'(c_t, t)} E \left\{ \int_t^T (u'(c_v, v) \, dD(v) + d[D, u'(c)](v))|\mathcal{F}_t \right\},$$  

(2)

where $u'$ signifies the marginal utility of the representative agent, and $[D, u'(c)](t)$ the square covariance process between accumulated dividends $D(t)$ and the marginal utility process $u'(c_t)$ (see Aase, 1997). Here $c(t) = \sum_{i=1}^{m} c_i(t) = \sum_{i=1}^{m} e_i(t)$ is the aggregate, accumulated consumption process in the market, and where $\mathcal{F}_t$ is the information possibly available at time $t$, formally given by some filtration on a given probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual hypotheses, where $\mathcal{F} = \mathcal{F}_T$. The symbol $E$ stands for the expectation operator on this space, and we will sometimes use $E_t$ to signify conditional expectation given $\mathcal{F}_t$.

By integrating to an intermediate time point $s$, we use iterated expectation and the pricing principle in (2), in which case Eq. (2) can be written as

$$q_t = \frac{1}{u'(c_t, t)} E_t \left\{ \int_t^s (u'(c_v, v) \, dD_q(v) + d[D, u'(c)](v)) + u'(c_s, s)q_s |\mathcal{F}_t \right\}, \quad t \leq s \leq T.$$  

(3)

If there are no intermediate dividends before the final time point $T$, the pricing relation is

$$q_t = \frac{1}{u'(c_t, T)} E_t [u'(c_T, T)q_T], \quad t \leq T.$$  

(4)

In what follows, we shall assume that the intertemporal marginal utility function of the market is given by $u'(c, t) = e^{-\rho t} e^{-\eta c}$, where $\rho$ is the market’s impatience rate and $\eta$ the intertemporal coefficient of absolute risk aversion in the market. Thus, $(\rho, \eta)$ will be referred to as the preference parameters. This marginal utility function may result in equilibrium if all the agents have different utility functions of this class with individual preference parameters.
\((\rho_i, \eta_i), i = 1, 2, \ldots, m\), in which case \(\eta^{-1} = \sum_{i=1}^{m} \eta_i^{-1}\), i.e., the risk tolerance of the market is the sum of the agents’ individual risk tolerances.

The question whether the pricing rule (2) can be used also for assets other than the given primitive ones represented by \(S\), is related to the complete market issue. One way to complete such a market is to introduce trading of nonlinear contracts, i.e., contracts that cannot be written as stochastic integrals of self-financing portfolios with respect to the given primitive securities. Because of the above assumptions we have on preferences, we can take another path: In our model the aggregation property is satisfied, so exchange efficiency holds for our model. Here we appeal to the aggregation theorem of Borch (1962) (see also Wilson, 1968; Rubinstein, 1974). It relies on the observation that under certain conditions, even though the market may be incomplete, the equilibrium prices will be determined as if there were an otherwise similar complete (Arrow–Debreu) market.

We shall, however, interpret the model as a representative agent equilibrium in which all assets are priced by the relation (2). The introduction of a new asset may change the allocations and hence lead to a new equilibrium, where all assets, old and new satisfy Eq. (2). For other treatments where the consumption and dividend processes are semimartingales, see, e.g., Back (1991), Naik and Lee (1990), Aase (1993a,b).

2.2. Probability distributional assumptions on consumption

We now turn to the distributional assumptions, and by this we mean marginal probability distributions at each time \(t\), and we start with the aggregate, cumulative consumption process \(c(t)\), denoted \(c_t\), of all the agents at time \(t\).

**Assumption 1.** \(c_t\) has an inverse Gaussian probability density at each time \(t\).

This means that its density function is given by

\[
p_t(x) = \frac{x^{-3/2} \delta}{2 \sqrt{\pi}} e^{(-1/2x)(t\delta - x/\sqrt{\psi})^2}, \quad x > 0, \quad t > 0,
\]

(5)

and \(p_t(x) = 0\) for \(x \leq 0\) and all \(t\), where \(\delta > 0\) and \(\psi = (\alpha^2 - \beta^2) \geq 0\) are parameters, \(\delta\) being a scale parameter.

The inverse Gaussian process \(c = (c_t)_{0 \leq t \leq T}\) with parameters \(\delta, \alpha\) and \(\beta\) is now defined as the homogeneous Lévy process for which the probability density of \(c_t\) is given by Eq. (5). We remind our readers that an adapted process \(X = (X_t)_{0 \leq t \leq T}\) is a Lévy process if (i) \(X\) has independent increments, (ii) \(X\) has stationary increments, and (iii) \(X_t\) is continuous in probability.

Our assumption implies that the increment \((c_T - c_t)\) given \(\mathcal{F}_t\) has the same conditional distribution as \(c_{T-t}\) at time \((T-t)\) given \(\mathcal{F}_0\), assuming that \(c_0 = 0\). Thus, we know that \(c\) is a strong Markov process with Feller transition functions \(P_t(y, dx) = p_t(x-y) \, dx\), where \(P_t(x, A) = P(c_{s+t} \in A | c_s = y)\) for any Borel set \(A\) of reals. Let

\[
K_t(v) = \ln E(e^{-v c_t}).
\]

(6)

From (5), it then follows that

\[
K_t(v) = t\delta \sqrt{\psi} - \sqrt{\psi} v + 2v.
\]

(7)

From this expression, we can find the cumulants and consequently, e.g., the two first moments: \(E(c_t) = t\delta/\sqrt{\psi}\), and \(\text{var}(c_t) = t\delta/\psi^{3/2}\) for all \(t \geq 0\).

Note that the increments are here non-negative, and \(c_t\) is interpreted as cumulative consumption over the time interval \([0, t]\), in addition to also representing an aggregate quantity over all the investors. Because of the time homogeneity of \(c\), we may define the average consumption \(\overline{c}_t = c_t/t\) in the period \([0, t]\). The probability density \(p_{t\overline{c}}(x)\) of \(\overline{c}_t\) is then given by \(p_{t\overline{c}}(x) = p_t(tx)/t\), where \(p_t(x)\) is given in (5).

The process \(c\) is known to make an infinity of jumps in any finite time interval but the set of jump times has Lebesgue measure zero almost surely. The inverse Gaussian distribution has many applications (see, e.g., Seshadri,
1993). Bachelier (1900) discussed in his thesis “Théorie de la spéculation” a problem leading to the inverse Gaussian distribution. This distribution can accommodate a wide variety of shapes and, being a member of the exponential family, often lends a meaningful interpretation that provides satisfactory physical support to empirical fit. It is likely that this also may be the case in the present application to financial economics.

The distribution seems well suited to the modeling of cumulative consumption, which by assumption must be non-negative. Since the quantity in question is accumulated over time, an aggregate over agents, and also over goods and services in empirical applications, by the central limit theorem we could then expect something Gaussian looking. This can be accomplished by the present family of distributions, since the inverse Gaussian distribution tends towards normality as the parameter $\frac{t}{\sqrt{2}}$ increases. This can be inferred from the above cumulant generating function $K_t(v)$ in (7).

2.3. Probability distributional assumptions on asset returns

We start with the case $N = 1$, with one risky and one riskless security. We denote the accumulated return process of the risky asset by $\tilde{R}$, and assume it to be a homogeneous Lévy process.

Assumption 2. The conditional distribution of the accumulated return on the risky asset given the aggregate consumption $c_t$, we require to be normal with
1. mean $\mu c_t$ and
2. variance $\sigma^2 c_t$, where $\sigma$ is a positive constant.

This assumption is here crucial in order to obtain the desired marginal distributions on assets returns derived below. However, it is not as far-fetched as it may appear at first glance. Recall the classical asset pricing model by Black and Scholes (1973). It is possible to alternatively derive their results in a representative agent model with one consumption good much the same way as we do in our model (see, e.g., Bick, 1987 or Aase, 1998). This can be done using, e.g., a negative exponential utility function for the representative agent, if the return of the risky security is assumed to be conditionally normally distributed given $c_t$, where $c_t$ is normally distributed (see e.g., Aase, 1997, example 2).

The variance assumption in (2) is more peculiar to this model. It leads to a relationship with stochastic volatility models, since $c$ is a random process. We will concentrate on the case $\sigma = 1$ for matters of simplicity. Assumptions 1 and 2 lead directly to the marginal distribution for the accumulated return of the risky asset in the time period $[0, t]$, denoted by $\tilde{R}_t$, to be of the desired NiG type with density function

$$q_t(x) = a(\alpha, \beta, t\mu, t\delta)q \left( \frac{x - t\mu}{t\delta} \right)^{-1} K_1 \left( q \left( \frac{x - t\mu}{t\delta} \right) t\delta \alpha \right) e^{\beta x}, \quad t \geq 0, \quad x \in \mathbb{R},$$

where $a(\alpha, \beta, t\mu, t\delta) = (1/\pi) e^{\delta \sqrt{\alpha^2 - \beta^2}}$. $q(x) = \sqrt{1 + x^2}$, and $K_1(x)$ the modified Bessel function of the third-order and index 1. An integral representation of $K_1(x)$ is given in Eq. (39). Here $0 \leq |\beta| \leq \alpha$, where $\beta$ is a symmetry parameter, $\alpha$ a steepness parameter, $\delta$ a scale parameter and $\mu$ a location parameter.

Consider the accumulated return $(\tilde{R}_{t+s} - \tilde{R}_s)$ in the time interval $[s, s + t]$ of a risky asset. Assuming that the probability distribution of this quantity is given by (8) for all $s$ and $t \geq 0$. $(\tilde{R}_t)_{0 \leq t < T}$ is a strong Markov process with Feller transition probabilities $Q_t(x, dy) = q_t(y - x) dy$, and the moment generating function of the increments is

$$M(w; \alpha, \beta, t\mu, t\delta) = E e^{w(\tilde{R}_{t+s} - \tilde{R}_s)} = \exp \left\{ t\delta \left[ \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2} \right] + t\mu u \right\}.$$  

2 This is in the usual formal manner of convergence in distribution, and there will of course not be any probability mass on the negative real for any finite value of this parameter.

3 To our knowledge this is the first time the inverse Gaussian distribution has been suggested as a model for the aggregate consumption. An empirical investigation would eventually be needed to test the goodness of this hypothesis.
The NiG process is accordingly defined as the homogeneous Lévy process \( \tilde{R} = (\tilde{R}_t)_{0 \leq t \leq T} \) for which the moment generating function of \((\tilde{R}_{t+s} - \tilde{R}_s)\) is given by

\[
M_t(u; \alpha, \beta, \mu, \delta) = M(u; \alpha, \beta, \mu, \delta)^t,
\]

where \(M(u; \alpha, \beta, \mu, \delta)\) is given in (9) with \(t = 1\). Thus

\[
M_t(u; \alpha, \beta, \mu, \delta) = M(u; \alpha, \beta, t\mu, t\delta).
\]

From this, it follows that if \(\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n\) are independent NiG-distributed random variables with common parameters \(\alpha\) and \(\beta\) but having individual location and scale parameters \(\mu_i\) and \(\delta_i\), then \(\tilde{R} = \sum_{i=1}^n \tilde{R}_i\) is again distributed accordingly to the NiG law with parameters \((\alpha, \beta, \sum \mu_i, \sum \delta_i)\). Furthermore, if \((\tilde{R}_t - \tilde{R}_0)\) is distributed according to (8), then it has mean and variance

\[
E(\tilde{R}_t - \tilde{R}_0) = t\mu + t\delta \frac{\beta}{\alpha \sqrt{1 - (\beta/\alpha)^2}},
\]

\[
\text{var}(\tilde{R}_t - \tilde{R}_0) = \frac{t\delta^2}{\alpha(1 - (\beta/\alpha)^2)^{3/2}}.
\]

In particular \((\tilde{R} - bt)\) is a martingale, where the constant \(b\) is found from (12). A multivariate extension of the NiG distribution is possible to construct in view of the normal mixture representation (for details, see Barndorff-Nielsen, 1977, 1994). We give a brief discussion of several variables in Section 8.

The NiG process has the following properties: for \(\beta = 0, \alpha \to \infty\) and \(\delta/\alpha = \sigma^2\), the process tends to the Brownian motion with drift \(\mu\) and diffusion coefficient \(\sigma\) (sometimes called the Bachélier process). The process \((\tilde{R}_t - \tilde{R}_0)\) may be represented via a random time change of a Bachélier process \(B_t\) as \((\tilde{R}_t - \tilde{R}_0) = B_{\tau_t} + \mu t\), where \(B_t\) has drift \(\beta\) and diffusion coefficient 1, and where \(c_t\) is the inverse Gaussian process with parameters \(\delta\) and \(\sqrt{\Psi}\), \(c\) being independent of \(B\). Furthermore, it can be shown from the Lévy representation of \(\tilde{R}\) that the small jumps dominate the behavior of the process \(\tilde{R}\).

The distribution (8) can be used to closely approximate the hyperbolic distribution, which in its turn has displayed close fits to return data for common stocks (Eberlein and Keller, 1994; Küchler et al., 1994). However, the hyperbolic distribution does not have the convolution property. As mentioned above, the NiG distribution itself has proved to provide a close fit to stock return data. Both classes belong to the family of generalized hyperbolic distribution, and all of these are the so-called normal variance-mean mixtures. In the case of the latter class the mixing distribution is inverse Gaussian, which accounts for the terminology. Of all the properties of the processes \(\tilde{R}\) and \(c\) listed above, many of which are mentioned in Barndorff-Nielsen (1994), the conditional normality of \(\tilde{R}\) given \(c\) is of particular use to in Section 4.

2.4. A representation for the return process

Before we proceed, we consider a representation of the return process defined above. An expression for the characteristic function of the NiG process follows from conditioning on \(c_t\) : \(E e^{it\tilde{R}(t)} = \exp\{it\mu + K_t(1 + \tau^2 - i\tau\beta)\}\), where \(K_t(\cdot)\) is given in Eq. (7).\(^4\) It is now possible to derive the Lévy decomposition of the NiG process, which is

\[
\tilde{R} = (\theta + \mu)t + \int_{|y|<1} y[N(t; dy) - tv(dy)] + \int_{|y|\geq 1} yN(t; dy),
\]

where \(\theta = 2\pi^{-1}\delta\alpha J_1^1 \sinh(\beta x)K_1(\alpha x)\) dx and the Lévy measure \(v(dy)\) is

\[
v(dy) = \pi^{-1}\delta\alpha |y|^{-1} e^{\delta y} K_1(\alpha y)\) dy.
\]

\(^4\)Compare this expression to the moment generating function in (9).
Here $N(t; dy)$ is a random Poisson measure, meaning that $N_t^A = \int_A N(t; dy)$ is a Poisson process with parameter $\nu(A) = \int_A \nu(dy)$, and $N_t^A$ and $N_t^B$ are independent if $A$ and $B$ are disjoint sets of reals. Thus, $\tilde{R}$ is a process with paths of bounded variation. Any $(P, \mathcal{F}_t)$-martingale $M$ can be represented as

$$M(t) = M(0) + \int_0^t \int_{(-\infty, \infty)} \theta(s, y)\{N(ds; dy) - \nu(dy)\} ds.$$  

(16)

This representation property is obviously different from a representation in the form of linear spanning. Thus, our model is incomplete.

The price process of a risky asset having return process $Q_{Rt}$ we assume to be given by the following:

$$S(t) = S_0 e^{\tilde{R}_t}, \quad 0 \leq t < T.  \tag{17}$$

In other words, we interpret $\tilde{R}_t$ as the cumulative logarithmic returns by time $t$. In Section 8, we alternatively model $S$ by the semimartingale exponential, implying that we view the term $\tilde{R}_t$ as the cumulative simple returns for the period $[0, t]$.

### 3. Preference-free equilibrium pricing of derivative securities

We now turn to the pricing of derivative securities in the present model, where we first consider the case with one risky and one riskfree asset. Using the pricing rule (4), the equilibrium market price $\pi^\phi$ of a contingent claim having payoff $\phi(S(T))$ at the expiration time $T$ is given by

$$\pi^\phi(S, t) = \frac{1}{u'(c_t, t)} E_t[u'(c_T, T)\phi(S_T)]$$  \tag{18}

where $c$ is the aggregate consumption of all the consumers, accumulated from time 0 to time $t$, having probability density given in (5). Here the price of the underlying asset equals $S(T)$ at time $T$ and there are no intermediate dividends paid out to the holder of the contingent claim.

Below we shall consider the European call option as an illustration, where $\phi(x) = (x - k)^+$. In the case where the contingent claim also pays dividends, where $D^\phi = (D^\phi_t)_{0 \leq t \leq T}$ equals the accumulated dividend process, then the market price in (18) changes to

$$\pi^{\phi, D}(S, t) = \frac{1}{u'(c_t, t)} E_t\left[ u'(c_T, T)\phi(S_T) + \int_t^T u'(c_s, s) dD^\phi(s) + \int_t^T [D^\phi, u'(c)](s)|\mathcal{F}_s \right].$$  \tag{19}

We start by considering the real price $S(t)$ at time $t$. It satisfies

$$S(t) = \frac{1}{u'(c(t), t)} E_t[u'(c(T), T)S(T)] = \frac{S(t)}{u'(c(t), t)} E_t[u'(c(T), T)\exp(\tilde{R}_T - \tilde{R}_t)], \quad 0 \leq t \leq T,  \tag{20}$$

where we have used (17) and the fact that $S$ is an $\mathcal{F}_t$-adapted process. Conditional on $c_T - c_t$, the random variable $(R_T - R_t)$ is normally distributed having mean $(T - t)\mu + \beta c_T - c_t$ and variance $c_T - c_t$. In Eq. (20), we now condition on $(c_T - c_t)$, which gives

$$u'(c_t, t) = E_t[u'(c_T, T)\exp((\beta + \frac{1}{2})(c_T - c_t))] e^{\mu(T - t)}.$$  \tag{21}

Let us now make use of our assumption that the intertemporal marginal utility function of the market is given by $u'(c, t) = e^{-\rho t} e^{-\eta c}$, where $\rho$ is the market’s impatience rate and $\eta$ the intertemporal coefficient of absolute risk.

\[5\] The case with intermediate dividends of the Lévy type can be handled by formula (3).
aversion in the market. We want to demonstrate that the market values of contingent claims in the above model will not directly depend on the parameters \( (\rho, \eta) \). Thus, we obtain a similar kind of preference-free evaluations in the present model as in the equilibrium version of the model of Black and Scholes, a result we may have expected to be true only in a complete market (see, e.g., Aase, 1998).\(^6\)

To this end, observe that under the present conditions, expression (21) reduces to

\[
E_t \{ \exp \left( (\beta - \eta + \frac{1}{2})(c_T - c_t) \right) \} = e^{(T-t)(\rho-\mu)}. \tag{22}
\]

Using Eq. (7), we get

\[
e^{(T-t)(\rho-\mu)} = E_t \{ \exp \left( (\beta - \eta + \frac{1}{2})(c_T - c_t) \right) \} = e^{(T-t)\delta(\sqrt{\psi} - \sqrt{\psi + 2\eta - 2\beta - 1})}.
\]

Thus, we obtain the following relationship between the parameters:

\[
(\rho - \mu) = \delta(\sqrt{\psi} - \sqrt{\psi + 2\eta - 2\beta - 1}). \tag{23}
\]

The second equation between the parameters of the model is obtained as follows. The equilibrium interest rate \( r(t) \) satisfies the following equation:

\[
r(t) = -\frac{\mu_p(t)}{p(t)}, \quad p(t) = u'(c_t, t),
\]

where \( \mu_p(t) \) is the conditional expected rate of change in the process \( p \) (see, e.g., Duffie, 1988; Aase, 1993b), which can be written as follows:

\[
e^{-\rho t - \eta t} = E \left\{ \int_t^T e^{-\rho u - \eta u} r(u) \, du + E \left[ \int_T^T e^{-\rho u - \eta u} r(u) \, du \bigg| \mathcal{F}_s \right] \bigg| \mathcal{F}_t \right\}.
\]

Since \( E \left[ \int_T^T e^{-\rho u - \eta u} r(u) \, du \bigg| \mathcal{F}_s \right] = e^{-\rho s - \eta s} \), we obtain that

\[
1 = e^{\rho t + \eta t} E \left\{ \int_t^T e^{-\rho u - \eta u} r(u) \, du + e^{-\rho s - \eta s} \bigg| \mathcal{F}_t \right\}.
\]

We now use that the local characteristics of a Lévy process are deterministic and Eqs. (3.3)–(4) in Aase (1993b) to conclude that \( r \) is constant. Using relation (7), we get

\[
1 = re^{\rho t} \int_t^T e^{-\rho u - \eta u} \delta(\sqrt{\psi} - \sqrt{\psi + 2\eta}) \, du + e^{-\rho s} e^{-(s-t)\delta(\sqrt{\psi} - \sqrt{\psi + 2\eta})},
\]

which directly leads to the equilibrium interest rate \( r \) being given by

\[
r = \rho + \delta(\sqrt{\psi} - \sqrt{\psi + 2\eta}), \tag{24}
\]

where \( \psi = \alpha^2 - \beta^2 \), and \( \delta > 0 \) is the scale parameter.

Eq. (24) is the second equilibrium relation between the parameters.\(^7\) Note that since \( \eta \geq 0, r \leq \rho \), which we normally would expect to be the case. Also notice that if \( \eta = 0 \), so the market is risk neutral, then \( r = \rho \). Clearly Eqs. (23) and (24) can be solved and \((\eta, \rho)\) found as functions of the parameters \( \delta, \mu, \alpha, \beta \) and \( r \). We therefore propose the following concept.

**Definition 1.** An equilibrium model is preference-free if the expression for the market price of any risky security does not contain the preference parameters.

\(^6\) We may remark here that if all agents are identical with the above utility function, then the competitive equilibrium is fully Pareto optimal whatever the market structure, since there are no gains from trade (Hart, 1975).

\(^7\) Observe how probability distributions of asset prices are here being determined in equilibrium.
Because of jumps of random sizes at unpredictable time points in the pricing process of the primitive assets, our model is not complete according to the standard definition, but as already noted, the general pricing rule (2) still applies to contingent claims.

It may be useful to resort a notion of complete markets due to Borch (1962), who defined a market model to be complete if it assigns a unique value, its market price, to an arbitrary random process. This notion turns out to be useful in our setting.

Clearly a preference-free model is also complete in the meaning of Borch’s definition, but a Borch-complete model need not be preference-free. The word preference-free in the above definition should not be taken literally, since equilibrium prices will normally depend upon preferences, e.g., through the equilibrium riskfree interest rate which normally depends on the preference parameters. However, the prices on risky securities are not allowed to depend explicitly on these parameters according to the above definition. The importance of this concept seems rather obvious from a practical point of view, since the statistical estimation of the process parameters is a straightforward task compared to the statistical estimation of the preference parameters. We have essentially shown the following theorem.

**Theorem 1.** The model of this section, with constant, intertemporal absolute risk aversion $\eta$ and constant impatience rate $\rho$, is preference-free.

**Proof.** Any computation of the price of a contingent claim using formula (18) will depend upon the parameters $(\rho, \eta)$ as well as on the rest of the parameters of the model. However, $\rho$ and $\eta$ can be eliminated using Eqs. (23) and (24), and these equations are derived independent of any contingent claim possibly existing in the model. We obtain that the unique solution is

$$\eta = \frac{1}{8c^2}(\psi^2 + b^2 + a^4 - 2\psi b - 2\psi a^2 - 2ba^2), \quad (25)$$

$$\rho = r + \delta(\sqrt{\psi} + 2\eta - \sqrt{\psi}), \quad (26)$$

where

$$b = \psi - 2\beta - 1, \quad a = 2\sqrt{\psi} + \frac{1}{\delta}(\mu - r). \quad (27)$$

It is worth emphasizing again that all the return and consumption-related parameters $\delta, \mu, \alpha,$ and $\beta$ can readily be jointly estimated from return and consumption data, and $r$ is presumably observed in the market. Thus, the (estimated) values can be substituted for the unknown preference parameters $\eta$ and $\rho$, leading to numerical values for market price of risky assets even if the preference parameters are not known. Alternatively, one can view the above equations as relations from which the preference parameters can indirectly be estimated from market data.

4. **An option pricing formula**

We now want to demonstrate a connection between our equilibrium model and a certain class of option pricing models used in finance — the stochastic volatility models. To this end we start by computing the market price of a European call option with payoff $\phi(S(T)) = (S(T) - k)^+$ at expiration and no intermediate dividends. Using the

---

8 The random processes are assumed to belong to a certain set of processes, say the product space $L^2(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}[0, T], P \times \lambda)$, where $\mathcal{B}[0, T]$ is the Borel $\sigma$-field on $[0, T]$ and $\lambda$ the Lebesgue measure on the reals.

9 See, e.g., Küchler et al. (1994) for estimation procedures for NiG distributions.
pricing result in (18), we have to compute
\[ \pi(S, t) = \frac{1}{u'(c_t, t)} E[u'(c_T, T)(S(T) - k)^+ | \mathcal{F}_t] \]
in our model with \( N = 1 \). We simplify the expression by conditioning on \( c_{T-t} \), which gives
\[ \pi(S, t) = e^{-\rho(T-t)} E[e^{-\eta(c_T-c_t)} E[(S(t) e^{\tilde{R}_{T-t} - k)^+ | (c_T - c_t), \mathcal{F}_t)] | \mathcal{F}_t]. \]
The inner expectation can be computed using the independent increments of the Lévy process \( \tilde{R} \) and the assumption that \( (\tilde{R}_{T} - \tilde{R}_t) \) is conditionally Gaussian with mean \( \mu(T-t) + \beta(c_T - c_t) \). Thus
\[
E[(S(t) e^{\tilde{R}_{T-t} - k)^+ | (c_T - c_t), \mathcal{F}_t)] = \int_{-\infty}^{\infty} (S_t e^{\mu(T-t)+\beta(c_T-c_t)+y\sqrt{c_T-c_t} - k)^+ \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy.
\]
The computation of this latter integral is routine sailing, and the final result is
\[
\pi(S, t) = e^{-\rho(T-t)} S_t E \left\{ e^{(\beta+(1/2) - \eta)c_{T-t} \Phi \left( \frac{\ln(S_t / k) + \mu(T-t) + (\beta + 1)c_{T-t}}{\sqrt{c_T-c_t}} \right) \right\} \\
- e^{-\rho(T-t)} k E \left\{ e^{-\eta c_{T-t} \Phi \left( \frac{\ln(S_t / k) + \mu(T-t) + \beta c_{T-t}}{\sqrt{c_T-c_t}} \right) \right\}, \tag{28}
\]
where \( \Phi(\cdot) \) is the standard normal cumulative distribution function. Here we can eliminate the preference parameters \( (\rho, \eta) \) using \( \eta \) as derived in Eqs. (25) and (26), and \( \rho = r + \delta(\sqrt{\psi} + 2\eta - \sqrt{\psi}) \), so the market price does not directly depend upon these parameters. The expectation in (28) is taken with respect to the inverse Gaussian distribution of the random variable \( c_{T-t} \), having density \( p_{T-t}(x) \) given in Eq. (5). We have now shown the following theorem.

**Theorem 2.** The equilibrium market value at each time \( t \leq T \) of a European call option with strike price \( k \) and expiration time \( T \) is given by Eq. (28), which alternatively can be written as
\[
\pi(S, t) = e^{-\rho(T-t)} \int_{0}^{\infty} \left\{ S_t e^{\mu(T-t) e^{(\beta+(1/2) - \eta)x} \Phi \left( \frac{\ln(S_t / k) + \mu(T-t) + (\beta + 1)x}{\sqrt{x}} \right) \right\} \\
- k e^{-\eta x} \Phi \left( \frac{\ln(S_t / k) + \mu(T-t) + \beta x}{\sqrt{x}} \right) x^{-3/2} \frac{(T-t)\delta}{\sqrt{2\pi}} e^{-1/2x((T-t)\delta-x\sqrt{\psi})^2} dx. \tag{29}
\]
Formula (29) could be compared to the corresponding pricing result of stochastic volatility models in the arbitrage valuation theory (see, e.g., Hull and White, 1987, 1988; Naik, 1993; Amin and Ng, 1993; Andersen, 1994; Ball and Roma, 1994; Boudoukh, 1993; Heston, 1993; Hofman et al., 1992; Stein and Stein, 1991; Taylor, 1994; Wiggins, 1987 among others).

Referring to Hull and White (1987), e.g., the volatility process itself is allowed to follow a lognormal process independent of the risky asset, which is assumed to be lognormally distributed as well. The average volatility \( \bar{\nu} \) in their model should then be contrasted to our \( c \), resulting in some similarity between their equation (8) and our equation (29). The obvious advantage with our equilibrium approach is that the uncertainty enters into the corresponding volatility terms in an endogenous manner, whereas it is exogenously specified in the literature cited above. Furthermore, risk neutrality is assumed in general in the Hull and White model, except in the situation where “the volatility is uncorrelated with aggregate consumption”, where their formula is true also under risk aversion. In contrast, in our model the agents are risk averse and yet the pricing formula (29) is true regardless, as long as our assumptions hold. For practical purposes it is important to notice that formula (29) does not depend explicitly upon \( c_t \), which would, if the opposite were the case, have made it difficult to use in practice due to the relatively poor quality of aggregate consumption data.
5. A consumption-based capital asset pricing model

Any new pricing model ought to be tested against existing asset pricing puzzles. To this end, let us start with the process $\tilde{R}$, where we use the notation $E(d\tilde{R}_t|\mathcal{F}_t) = \mu_{\tilde{R}}(t)\, dt$, $\text{var}(d\tilde{R}_t|\mathcal{F}_t) = \sigma^2_{\tilde{R}}(t)\, dt$, $E(dc_t|\mathcal{F}_t) = \mu_c(t)\, dt$ and $\text{var}(dc_t|\mathcal{F}_t) = \sigma^2_c(t)\, dt$. First, we observe that $E(dc_t|\mathcal{F}_t) = \frac{p}{t}$, and using that $E(d\tilde{R}_t|\mathcal{F}_t) = \frac{\beta \delta}{\sqrt{\psi}}$, we get

$$\mu_{\tilde{R}}(t) - r = \left(\mu + \frac{\beta \delta}{\sqrt{\psi}} - r\right) \quad \text{for all } t > 0. \tag{30}$$

Alternatively we could have obtained this result directly from (12) setting $\tilde{R}_0 = 0$. Combining this with the equilibrium result for the riskless interest rate $r$ given in Eq. (24), we obtain

$$\mu_{\tilde{R}}(t) - r = (\mu - \rho \delta(\sqrt{\psi} + 2\eta - \sqrt{\psi})) + \frac{\beta \delta}{\sqrt{\psi}}, \tag{31}$$

or using Eq. (23)

$$\mu_{\tilde{R}}(t) - r = \delta(\sqrt{\psi} + 2\eta + \sqrt{\psi} + 2\eta - 2\beta - 1 - 2\sqrt{\psi}) + \frac{\beta \delta}{\sqrt{\psi}}. \tag{32}$$

Using the equilibrium interest rate $r$ given in (24), we get the following version of an intertemporal CCAPM

$$\mu_{\tilde{R}}(t) - r = (\mu - \rho) + \delta(\sqrt{\psi} + 2\eta - \sqrt{\psi}) + \frac{\beta \delta}{\sqrt{\psi}}. \tag{33}$$

Notice how the parameter $\beta$ can be given an interpretation close to the usual meaning of beta in the ordinary CCAPM, since the parameter $\delta$ is a scale parameter for the consumption process.

We know that the ordinary CCAPM performs poorly for many ordinary asset pricing models (see, e.g., Mehra and Prescott, 1985; Constantinides, 1990 among others). Below we calibrate the model (33) to the data reported by Mehra and Prescott (1985). They employed a variant of Lucas (1978) pure exchange economy and conducted a calibration experiment where they estimated the mean of the annual growth rate of per capita real consumption of non-durables and services in the years 1889–1978 to be 0.0183 with an estimated standard deviation of 0.0357; in the same period they estimated the annual real return on the Standard and Poor’s composite stock price index to have mean 0.0698 with an associated estimated standard deviation of 0.1654. From data of prime commercial papers, treasury bills and treasury certificates they estimated the risk-less rate to be 0.01 per year.

Postulating that the representative agent has a time- and state-separable utility, the puzzle is that they were unable to find a plausible pair of the subjective impatience rate and the coefficient of relative risk aversion (RRA) of the representative agent to match the sample mean of the annual real growth rate of consumption and of the equity premium over the same 90-year period. Taking into account the estimated standard deviation 0.1654 for the index and 0.0357 for the consumption rate reported by Mehra and Prescott, an estimate of the instantaneous covariance between the index and the growth rate of the consumption is 0.0059, so from the diffusion-based CCAPM, where the risk premium intensity equals the coefficient of inter-temporal RRA times the mentioned covariance, it follows that an estimate of the RRA coefficient equals 10.2. From independent studies, a reasonable value is considered to be in the range from 1 to 3, hence the equity premium puzzle.
Trying to calibrate the model in (33) to the above data, we obtain the following set of equations:

\[
\begin{align*}
\frac{\delta}{\sqrt{\psi}} &= 0.0183, \quad (I) \\
\frac{\delta}{\psi^{3/2}} &= 0.00127, \quad (II) \\
\mu + \frac{\delta \beta}{\sqrt{\psi}} &= 0.0698, \quad (III) \\
\frac{\delta}{\sqrt{\psi}} + \beta^2 \frac{\delta}{\psi^{3/2}} &= 0.02736, \quad (IV) \\
\rho &= \delta(\sqrt{\psi} + 2\eta - \sqrt{\psi}) + 0.01. \quad (V)
\end{align*}
\]

In addition, we have the equilibrium relation (23) between the parameters, and the CCAPM relation (33). Thus, we have seven equations to determine all the seven unknown parameters.

The first two equations come from (7) giving the first two conditional moments of \(dct\). Eq. (III) follows from (12) with the remark that \(R_0 = 0\). Eq. (IV) results from the fact that \(\text{var}(dR_t) = E(\text{var}(dR_t|dct)) + \text{var}(E(dR_t|dct))\) and from the independent increments of the processes \(R\) and \(c\). Thus, \(\sigma^2_R(t) dt = E(dct|F_t) + \text{var}(\mu dt + \beta dct|F_t) = (\delta/\sqrt{\psi}) + \beta^2 \delta/\psi^{3/2}) dt\). Eq. (V) is merely Eq. (24). The system of equations has two solutions. The first is \(\text{ARA} = \tilde{\eta} = 1.96, \text{RRA} = 3.23, \hat{\rho} = 0.044\), and the process/consumption related parameters are \(\tilde{\alpha} = 4.64, \hat{\beta} = 2.67, \tilde{\mu} = 0.0698, \hat{\mu} = 0.02, \delta = 0.0695\), and \(\psi = 14.41\). Here the definition \(\tilde{\alpha}^2 - \beta^2 = \psi\) constitutes the eighth equation.\(^{10}\)

Here we have the estimate of the RRA down to around 3 with an associated estimate of the impatience interest rate at approximately 4%, which must both be considered acceptable. The second solution has a negative beta, which we rule out on economic grounds as an equilibrium value.

Hence our model seems reasonably consistent with the observed price and consumption data for the New York Stock Exchange for the given 90-year period, and the values the preference parameters are anticipated to have, giving a possible resolution of the equity premium puzzle. Notice the accurate degree of parsimony in the above model. It contains just the right number of parameters so that these are determined by the natural number of estimates available. Thus, there are no free parameters, as we have seen in other models attempted to explain this puzzle.\(^{11}\)

We may further compare our results to the empirical findings of Siegel (1992), who demonstrated that the real interest rate in the period studied by Mehra and Prescott was not characteristic of the real rate in other periods, where it was found to be significantly higher. However, he found that the behavior of real return on equities showed no significant difference between the periods. This may explain an even lower RRA than our 3.2.

The equity premium puzzle has been discussed by many authors from several different perspectives. Rietz (1988) suggested that a small probability of a large, sudden fall in consumption can justify a large equity premium. Aase (1993a,b) included jumps in both the aggregate consumption process and the stock index, and constructed a CCAPM that alternatively may explain this puzzle. There is also a large literature on preferences that are not additive and separable (see, e.g., Constantinides, 1990; Detemple and Zapatero, 1991; Benartzi and Thaler, 1995).

\(^{10}\) Remember that \(c\) is accumulated consumption over the 90-year period, meaning that the estimate for RRA is RRA = \(\tilde{\eta} c = 1.96 \times 0.0183 \times 90 = 3.23\).

\(^{11}\) If there are enough parameters available, naturally anything can be explained, in which case, of course, not much has been learnt.
6. The relation between cumulative simple periodic returns and the pricing process

Here we first derive an analogue of the standard relation between the price process and cumulative simple periodic returns. To this end, define the cumulative return process \( R_t \) by

\[
R_t = \int_0^t \frac{1}{S(u-)} dS(u), \quad R(0) = 0
\]

(34)

with corresponding spot price process \( S_t \) given by

\[
dS(t) = S(t-) dR(t).
\]

(35)

The solution to (34) and (35) is given by the semimartingale exponential \( E^{\cdot R} \) (see e.g., Doléans-Dade, 1970), or

\[
S_t = S_0 \exp \left( \int_0^t \left( \Delta \tilde{R}_s \right) ds \right)
\]

(36)

where \( \Delta \tilde{R}_s = (\tilde{R}(t) - \tilde{R}(t-)) \) and the square brackets signify the quadratic variation process of \( \tilde{R} \). From representation (14), we know that \( \tilde{R} \) is of bounded variation, sometimes called a quadratic pure jump process, and thus the continuous part of the quadratic variation process \([\tilde{R}, \tilde{R}]_t = 0\) for all \( t \), implying that \([\tilde{R}, \tilde{R}]_t = \sum_{0<s \leq t} (\Delta \tilde{R}_s)^2\).

Accordingly

\[
S(t) = S(0) \exp \left( \tilde{R}_t - \sum_{0<s \leq t} (\Delta \tilde{R}_s) \right) \prod_{0<s \leq t} (1 + \Delta \tilde{R}_s).
\]

(37)

Since \( \tilde{R} \) is a pure jump process where \( \tilde{R}(0) = 0 \), clearly

\[
\tilde{R}(t) = \sum_{0<s \leq t} \Delta \tilde{R}_s.
\]

So from representation (37), we simply get

\[
S(t) = S(0) \prod_{0<s \leq t} (1 + \Delta \tilde{R}_s),
\]

(38)

which is indeed the present analogue of the standard relation between prices and simple periodic returns in finance.

Products are sometimes cumbersome to work with, so we may try to get rid of it. Below we demonstrate two ways of doing that. From the representation of the Lévy measure (15) and the integral representation of \( K_1(y) \) given by

\[
K_1(y) = \sqrt{2} \int_0^\infty e^{-\sqrt{1+2u^2}} du,
\]

(39)

it follows that \( K_1(y) \) behaves as \( y^{-1} \) as \( y \to 0 \), \( v(dy) \) behaves as \( \delta\pi^{-1}y^{-2} dy \) as \( y \to 0 \), implying that the small jumps are dominating the behavior of \( \tilde{R} = (\tilde{R}_t)_{0 \leq t \leq T} \). However, if \( \Delta \tilde{R}_t \leq -1 \) for some \( t \), then \( S_t \leq 0 \). At such a point in time the company is bankrupt and the stock is nil worth. Therefore, we technically kill the process \( \tilde{R} \) at the first time this happens. This we do as follows: let \( \tau = \inf \{ t > 0 : (\tilde{R}_t, \tilde{R}_t-) \notin C \} \), where \( C = \{(x, y) : x \leq y - 1, x \in (-\infty, \infty)\} \). Such a \( \tau \) is known to be an \( \mathcal{F}_t \)-stopping time. Now define

\[
R(t) = \begin{cases} 
\tilde{R}_t & \text{for } t < \tau, \\
\uparrow & \text{for } t \geq \tau,
\end{cases}
\]

(40)
where \( \dagger \) is the cemetery state, a point not in \((-\infty; \infty)\). Here we may choose \( \dagger = -\infty \). Our stopped process \( R = (R_t)_{0 \leq t \leq T} \) is a strong Markov process with transition probabilities \( P_t(x, A) = P(R_t \in A, \tau > t | R_0 = x) = P^x(R_t \in A, \tau > t) \) for \( x \in (-\infty, \infty) \), \( P_t(x, \dagger) = P^x(\tau < t) \), \( P_t(\dagger, \dagger) = 1 \).

From the representation (14) of \( \tilde{R} \) we notice that the last term is a compound Poisson process, and the term \( \int_{(-\infty,-1)} N(t; dy) \) is a Poisson process with parameter \( \lambda = \lambda(\alpha, \beta, \delta) = \nu(-\infty, -1) = \int_{(-\infty,-1)} \nu(dy) \), where \( \lambda \) can be numerically computed using (15). From this, it follows that

\[
P_t(x, \dagger) = P^x[\tau \leq t] = 1 - e^{-\lambda t}, \quad t \geq 0.
\] (41)

Returning to the version of \( \tilde{S} \) given in (37), the corresponding price process \( \tilde{S} = (\tilde{S}_t)_{0 \leq t \leq T} \) killed at \( \tau \) is

\[
\tilde{S}(t) = \begin{cases} \tilde{S}_0 \exp(\tilde{R}(t) - \int_{-1<\chi}N(t;dy) + \int_{-1<\chi} \ln(1+y)N(t;dy)); & 0 \leq t < \tau \land T, \\ 0, & \tau \leq t \leq T. \end{cases}
\]

Thus, if \( \tilde{R} \) hits \( \dagger \) before time \( T \), \( \tilde{S} \) is set to zero, or \( R_\tau = \dagger \Leftrightarrow \tilde{S}_\tau = 0 \) for \( \tau \leq T \). We simplify by noticing that since the small jumps dominate the behavior of the process \( R \), the term \( (\int \ln(1+y)N(t;dy) - \int yN(t;dy)) \) will be small for most values of \( t \), so we choose to ignore it in the remainder. This approximation seems reasonable when the process is estimated to fit stock market data, in which case it amounts to a common approximation for asset returns. Doing this, we get the following pricing process:

\[
S(t) = \begin{cases} S_0 e^{\tilde{R}}, & 0 \leq t < \tau \land T, \\ 0, & \tau \leq t \leq T. \end{cases}
\] (42)

which, in view of the definition of \( R = (R_t)_{0 \leq t \leq T} \) given in (40) can simply be written as

\[
S(t) = S(0) \exp[R(t)], \quad 0 \leq t \leq T.
\] (43)

The relationship (43) almost gives the usual connection between prices and accumulated logarithmic returns as in Eq. (17), but takes account of bankruptcy.

It may seem a little cryptic what we have done above, so let us present a second and perhaps more natural way to get rid of the product in (38). Consider the version of \( \tilde{S} \) given in Eq. (38), as soon as some \( \Delta \tilde{R}_t \leq -1 \) the associated price \( \tilde{S}(s) \) becomes zero, or negative, so we stop the process the first time \((= \tau)\) this happens. Taking natural logarithms in Eq. (38) and denoting the associated stopped price process by \( \hat{S} \), we then get

\[
\hat{S}(t) = \begin{cases} \hat{S}_0 \exp[\sum_{0<s\leq t} \ln(1+\Delta \tilde{R}_s)]; & 0 \leq t < \tau \land T, \\ 0, & \tau \leq t \leq T. \end{cases}
\] (44)

Define

\[
\hat{R}(t) = \begin{cases} \sum_{0<s\leq t} \ln(1+\Delta \tilde{R}_s); & 0 \leq t < \tau \land T, \\ -\infty, & \tau \leq t \leq T. \end{cases}
\] (45)

Then it follows that

\[
\hat{S}(t) = \hat{S}(0) e^{\hat{R}(t)}, \quad 0 \leq t \leq T.
\]

If the intermediate period returns \( \Delta \hat{R}(s) \) are small enough, typically the case for intra-day returns on common stocks, then \( \ln(1+\Delta \hat{R}(s)) \approx \Delta \hat{R}(s) \). Since the increments of the process \( \hat{R} \) are independent, and if the errors are not systematic to one side or the other, it is likely that the following approximation holds with reasonable accuracy:

\[
\sum_{0<s\leq t} \ln(1+\Delta \hat{R}(s)) \approx \hat{R}(t), \quad t < \tau,
\]
\[
\hat{R}(t) \approx \begin{cases} 
\hat{R}_t, & 0 \leq t < \tau \land T, \\
-\infty, & \tau \leq t \leq T,
\end{cases}
\] (46)

which means that \( \hat{R}(t) \approx R(t) \) as given in Eq. (40) in which case

\[
\hat{S}(t) \approx S(t) = S(0)e^{R(t)}, \quad 0 \leq t \leq T,
\]

where \( S(t) \) is the same as given in Eq. (43). We now make the following final approximation.

**Assumption 3.**

\[
P_t(x,A) = P^x(\hat{R}_t \in A, \tau > t) = Q_t(x,A) e^{-\lambda t}
\] (47)

for any \( t \geq 0 \), and Borel set \( A \) of real.

This is only correct if \( \tau \) is \( P^x \) independent of \( \hat{R} \) for all \( x \). In other words, we construct a new \( \bar{\tau} \), by possibly enlarging the probability space, having the same probability distribution as \( \tau \) and being \( P^x \) independent of \( \bar{\hat{R}} \) for all \( x \), and consider the process \( \bar{\hat{R}} \) killed at \( \bar{\tau} \). The Feller probability transition functions of this constructed process is given by the right-hand side of (47).

Assumption 3 may be reasonable in the present model if the default probability of the company is independent of the return level of its stock. The underlying process may not strictly satisfy this property, but we shall nevertheless work with the constructed process having transition probabilities given by the right-hand side of Eq. (47) in the sequel.

We can now alter the analysis of Sections 3 and 4 using the above approximations. Conditioning on the event \( \{ \omega : \tau(\omega) > T \} \) of no failure by time \( T \) in Eq. (20), then in Eq. (21) we get \( (\mu - \lambda) \) instead of \( \mu \) in the exponent. In Eq. (22), we get \( (\lambda - \mu) \) instead of \( -\mu \) in the exponent as well, and the equilibrium restriction between the parameters in Eq. (23) now becomes

\[
(\rho - \mu + \lambda) = \delta(\sqrt{\psi} - \sqrt{\psi} + 2\eta - 2\beta - 1).
\] (48)

However, Eq. (24) for the equilibrium interest rate \( r \) remains unchanged.

Eq. (27) gets a \( (\mu - \lambda) \) instead of \( \mu \) in the expression for \( a \), while the equations for the price of the European call option are changed slightly. Eq. (28) gets two corrections in both the exponents of the exponential multiplication factors, where the term \( \rho \) changes to \( (\rho + \lambda) \) both places, and finally, in Eq. (29) the exponent of the exponential in front of the integral sign is changed from \( \rho \) to \( (\rho + \lambda) \) as well. All these changes are of course now approximations.

### 7. The survival hypothesis

Survival rates for markets were used in the paper by Brown et al. (1995) to possibly explain the equity premium puzzle. They assumed that the real risk premium \((\mu_R - r)\) could be low, possibly as low as zero, and they argued that the observed risk premiums in the markets that actually survived ought to be larger. The estimates above are really constructed on the event \( \{ \tau(\omega) > 90 \} \), and in principle we should not be able to accurately estimate the failure rate from observing only one stretch of data that survived. However, since the parameters are related in equilibrium according to the constraint (48), we are able to estimate both the risk aversion, \( \lambda \) and \( \mu_R \). Below we demonstrate how this can be done.

We intend to use the stopped version of the previous section to give a possible test of the survival hypothesis for stock markets. Let us first derive a version of the CCAPM for the stopped version. To this end, consider the process \( R \) stopped at \( \tau = \inf \{ t ; \Delta R_t \leq -1 \} \), and let \( \mu_R(t) \) be the rate corresponding to \( R \). We now compute

\[
E(dR_t|\mathcal{F}_t) = E(d\tilde{R}_t|\tau > t+dt, \tau > t, \mathcal{F}_t)P(\tau > t+dt|\tau > t, \mathcal{F}_t) + E(d\tilde{R}_t|\tau \in (t, t+dt), \tau > t, \mathcal{F}_t)P(\tau \in (t, t+dt)|\tau > t, \mathcal{F}_t).
\]
Because of the limited liability in the stock market, the loss can be maximum 100%, so we must have

\[ E[dR_t | \mathcal{F}_t] = \mu_R(t) \int e^{-\lambda(t+dr)} e^{-\lambda t} \, dr - 1 \lambda e^{-\lambda t} \, dr, \]

or, letting \( E[dR_t | \mathcal{F}_t] = \mu_R(t) \, dr \), we obtain

\[ \mu_R(t) - r = \left( \mu - r - \lambda + \frac{\beta \delta}{\sqrt{\psi}} \right). \quad (49) \]

Eqs. (I)–(V) in Section 5 will be the same in the present version. In addition, we have Eq. (49) and the stopped version of Eq. (23), which is Eq. (48) above. The equilibrium ruin intensity \( \lambda = \int_{(-\infty, -1)} \psi(dy) \), where the Lévy measure \( \psi(dy) \) is given in (15). Some easy substitutions, using (39), imply that \( \lambda \) can be written as

\[ \lambda = \frac{\delta \alpha}{\pi} \int_1^\infty t (t^2 - 1)^{-1/2} \left( \int_{\alpha t + \beta}^\infty e^{-\frac{z}{\psi}} \, dz \right) \, dt. \quad (50) \]

This expression for \( \lambda \) constitutes the eighth equation, which, together with the remaining seven equations, determine the values of all the parameters, including \( \mu_R, \lambda, (\text{RRA}) = \eta c, \) and \( \rho \), calibrated to the above data.

This system of equations has also two solutions: The first is \( \lambda = 0.000005, \) \( \text{ARA} = 1.96, \) \( \text{RRA} = 3.23, \) \( \hat{\beta} = 0.044, \) and the process/consumption-related parameters are \( \hat{\alpha} = 4.64, \) \( \hat{\beta} = 2.67, \) \( \hat{\mu} = 0.0698, \) \( \hat{\rho} = 0.02, \) \( \delta = 0.0695, \) and \( \hat{\psi} = 14.41. \)

The changes from the previous solution are modest. Of course the most interesting new value is here the estimate for \( \lambda \). It yields an estimate for the survival probability for the 100-year period of the NYSE, which is 0.9995, giving a crash probability of 0.05% over the same period, a value somewhat smaller than conjectured by Brown et al. Thus, the data does not give much support for the crash hypothesis.

Increasing the crash probability from 0.05 to 24.42% (and thus ignoring Eq. (50)) lowers the estimate of RRA to 3.12 and the estimate for the impatience rate to 0.042, which are rather moderate changes. A crash probability during the last 100 years of 25% is more in line with the conjecture of Brown et al. In other words, such an increase in the market failure probability does not seem to affect the market parameters to any significant degree according to the model of Section 6.

8. Several primitive assets

Suppose we want to make the preferences more realistic, and thus more complicated, by including more linearly independent parameters in the intertemporal utility function of the market. We may then retain the conclusion of Theorem 1 by introducing more primitive assets in the model. For each new risky asset we obtain one new equation of the type (48) which may be used to eliminate each new preference parameter in the manner shown above. In this connection we consider the return vector \( \bar{R} \) of \( N \) risky assets. Conditional on the aggregate consumption \( c_t \), the distribution of \( \bar{R}(t) \) is assumed to be Gaussian with mean vector \( (\mu t + c_t \beta \Sigma) \) and covariance matrix \( c_t \Sigma \). Here \( \Sigma \) is an \( N \times N \) matrix, \( \mu \) and \( \beta \) are \( N \)-dimensional vectors. The covariance matrix \( \Lambda \) of the vector of returns \( \bar{R} \) is

\[ \Lambda = \delta (\alpha^2 - \beta' \Sigma \beta)^{-1/2} (\Sigma + (\alpha^2 - \beta' \Sigma \beta)^{-1} \Sigma \beta \beta' \Sigma). \]

Consequently, \( \Sigma \) relates to the covariance in a fairly complicated manner, involving all the other parameters as well. In the special case where \( \beta' \Sigma \beta \) is negligible compared to \( \alpha^2 \)

\[ \Lambda = \delta \left( \frac{\Sigma + 1}{\alpha^2} \Sigma \beta \beta' \Sigma \right) \approx \frac{\delta}{\alpha} \Sigma \]

because then the only omitted term is likely to be negligible as well. Then \( \Sigma \) diagonal corresponds to approximate uncorrelated returns (see Lillestøl, 2000).
If \( c \) is an inverse Gaussian process we obtain an \( N \)-dimensional generalization of the NiG process for \( \tilde{R} \). Using the pricing processes \( S_i^{(k)} = S_0^{(k)} \exp(R_i^{(k)}), \ k = 1, 2, \ldots, N \), where \( R_i^{(k)} \) is the stopped version of \( \tilde{R}_i^{(k)} \), we have a model for a stock market. Using the results of Barndorff-Nielsen (1994), we have the joint probability density function for \( \tilde{R} \) at time \( t = 1 \) is given by

\[
g_1(x; \alpha, \beta, \mu, \delta, \Sigma) = a(\alpha, \beta, \delta, \Sigma) b(x; \alpha, \beta, \mu, \delta, \Sigma) e^{\beta x'},
\]

where

\[
a(\alpha, \beta, \sigma, \Sigma) = ((2\delta)^{-N+1} \pi^{-N+1} |\Sigma|)^{-1/2} a(\Sigma)^{(N+1)/2} e^{\delta\sqrt{\alpha^2 - \beta \Sigma \beta'} - \beta \mu'},
\]

\[
b(x; \alpha, \beta, \mu, \delta, \Sigma) = q[d^{-2}(x - \mu) \Sigma^{-1} (x - \mu)' - (N-1)/2] K((N+1)/2) [d^{-2}(x - \mu) \Sigma^{-1} (x - \mu)'].
\]

For the parameters to be identifiable, it may be advantageous to choose the determinant of \( \Sigma \) equal to 1, i.e., \( |\Sigma| = 1 \). The prime symbol means the transpose of a vector.

Consider, e.g., the case \( N = 2 \). Then we have the return parameters \( \alpha, \delta, \mu = (\mu_1, \mu_2), \beta = (\beta_1, \beta_2) \), and

\[
\Sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} \\
\sigma_{12} & \sigma_{22}
\end{pmatrix}
\]

with the possible restriction \( \sigma_{11}\sigma_{22} - \sigma_{12}^2 = 1 \). Retaining our assumption about the utility function of the market, we now have that the equilibrium interest rate is given by

\[
r = \rho + \delta(\sqrt{\alpha^2 - \beta \Sigma \beta'} - \sqrt{\alpha^2 - \beta \Sigma \beta' + 2\eta})
\]

with the following two restrictions on the parameters of the model

\[
(\rho - \mu_1 + \lambda_1) = \delta(\sqrt{\alpha^2 - \beta \Sigma \beta'} - \sqrt{\alpha^2 - \beta \Sigma \beta' + 2\eta} - 2\beta \sigma_{11} - 2\beta_2 \sigma_{12} - 1),
\]

\[
(\rho - \mu_2 + \lambda_2) = \delta(\sqrt{\alpha^2 - \beta \Sigma \beta'} - \sqrt{\alpha^2 - \beta \Sigma \beta' + 2\eta} - 2\beta \sigma_{12} - 2\beta_2 \sigma_{22} - 1),
\]

where \( \lambda_i, i = 1, 2 \), are the corresponding ruin intensities of the two risky assets. Clearly, we can accept one more linearly independent parameter in the utility function, and still have the possibility of inverting the relevant transformation, retaining a preference-free model.

9. Summary

This paper presents a new asset pricing model based on NiG marginal distributions for asset returns. Due to discontinuous sample paths of the resulting price processes the model is incomplete. However, preference-free evaluations of contingent claims are still possible, a concept defined in the paper. NiG distributions have proved to fit stock returns remarkably well in empirical investigations, so there may be some hope that equilibrium models based on this assumption may perform well. This is investigated in Section 5, where a consumption-based capital asset pricing model is derived and calibrated to the data in the Mehra and Prescott (1985) study. The model gives a possible resolution of the equity premium puzzle.

One may wonder how much of this improvement is due to duration in the preferences of the representative agent, and how much can be attributed to the marginal return process of the risky stock index. Referring to Hindy and Huang (1993) in a paper discussing durability separately, they find that the effect is to lower the risk premium for a given risk aversion. Although these authors are considering a production economy, and thus another type of model, we are nevertheless directed towards the conclusion that the effect is due to the marginal distribution of returns alone, which must also be offsetting this reverse effect from durability.
Moreover the survival hypothesis of Brown et al. (1995) can be investigated based on the model of Section 6, in that a crash probability of the market as a whole can be estimated. Not much support was found for this hypothesis, when our model was used on the Standard and Poor stock index, although it may prove to be an interesting hypothesis for individual companies. The analysis in Section 6 is based on two approximations that seem somewhat hard to verify, so these results are of a more preliminary nature.

In Section 4, an option pricing formula is derived and compared to similar formulas in the literature of stochastic volatility models. The equilibrium theory has the advantage of providing an explanation for stochastic volatility, given Assumptions 1 and 2, without any exogenous specification of the uncertainty regarding the volatility. These assumptions are discussed in detail in the paper. Of the two main assumptions, Assumption 2(2) seems to be the most questionable, and it is precisely this one that gives the connection to stochastic volatility. It would undoubtedly be an advantage to derive this kind of assumption from other more fundamental economic primitives. The Lévy process structure fails to capture time series aspects of return data, such as second-order effects studied in ARCH and GARCH models (see, e.g., Duan, 1995). To include such effects could be one interesting direction for future research.

References


