Annuities under random rates of interest

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Abstract

We investigate the accumulated value of some annuities-certain over a period of years in which the rate of interest is a random variable under some restrictions. We aim at the expected value and the variance of the accumulated value, and we suggest two methods to derive these moments. In some cases both methods have similar difficulties, while in other cases one method is significantly preferable in terms of the simplicity of calculations. The novelty of our second approach lies in the fact that we find recursive relationships for the variance of the accumulated values, and solve these relationships, whilst previous investigators first obtained the second moment. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Many attempts have been made to investigate stochastic interest rate models and to evaluate their impact under various hypotheses: see Pollard (1971), Boyle (1976), Wilkie (1976), Westcott (1981) and McCutcheon and Scott (1986) to mention just some of the contributions. We investigate the accumulated value of some annuities-certain over a period of years in which the rate of interest is a random variable under some restrictions. We aim at the expected value and the variance of the accumulated value, and we suggest two methods to derive these moments. In some cases both methods have similar difficulties, while in other cases one method is significantly preferable in terms of the simplicity of calculations.

The novelty of our second approach lies in the fact that we find recursive relationships for the variance of the accumulated values, and solve these relationships, whilst previous investigators first obtained the second moment.

2. The case of a fixed interest rate

Suppose that the yearly rate of interest $j$ is fixed through the period of $n$ years. We recall that the yearly rate of discount $d$ is given by

$$(1 + j)d = j$$  \hspace{1cm} (2.1)
and the yearly present value \( v \) is given by
\[
(1 + j)v = 1, \tag{2.2}
\]
so that we have the equalities
\[
d = jv, \tag{2.3}
\]
\[
v + d = 1. \tag{2.4}
\]

We assume \( k \leq n \) throughout, unless otherwise specified. The accumulated value after \( k \) years (denoted \( \bar{s}_{kj} \)) of an annuity-due of \( k \) yearly payments of 1 is known to be given by the formula
\[
\bar{s}_{kj} = \frac{(1 + j)^k - 1}{d}, \tag{2.5}
\]
and we note that
\[
\bar{s}_{kj} = (1 + j)^k + \cdots + (1 + j) = (1 + j)(1 + \bar{s}_{k-1j}). \tag{2.6}
\]

The accumulated value after \( k \) years of an increasing annuity-due of \( k \) yearly payments of 1, 2, \ldots, \( k \), respectively, is denoted \( (I\bar{s})_{kj} \), and is known to be given by the formula
\[
(I\bar{s})_{kj} = \frac{\bar{s}_{kj} - j}{d}. \tag{2.7}
\]
We also note that
\[
(I\bar{s})_{kj} = (1 + j)^k + 2(1 + j)^{k-1} + \cdots + k(1 + j) = \frac{1 + j}{(I\bar{s})_{k-1j} + k}. \tag{2.8}
\]

The accumulated value after \( k \) years of an increasing annuity-due of \( k \) yearly payments of \( 1^2, 2^2, \ldots, k^2 \), respectively, is denoted by \( (I^2\bar{s})_{kj} \).

**Theorem 2.1.**
\[
(I^2\bar{s})_{kj} = \frac{2(I\bar{s})_{kj} - \bar{s}_{kj} - k^2}{d}. \tag{2.9}
\]

**Proof.** By definition, \( (I^2\bar{s})_{kj} = (1 + j)^k + 2^2(1 + j)^{k-1} + \cdots + k^2(1 + j) \). A straightforward calculation of \( (I^2\bar{s})_{kj} - v(I^2\bar{s})_{kj} \) leads to the following equation:
\[
d(I^2\bar{s})_{kj} = (1 + j)^k + \cdots + (2k - 1)(1 + j) - k^2,
\]
from which we obtain
\[
d(I^2\bar{s})_{kj} = 2[(1 + j)^k + \cdots + k(1 + j)] - [(1 + j)^k + \cdots + (1 + j)] - k^2
\]
and the result follows by using (2.6) and (2.8).

We observe that
\[
(I^2\bar{s})_{kj} = (1 + j)^k + 2^2(1 + j)^{k-1} + \cdots + k^2(1 + j) = (1 + j)((I^2\bar{s})_{k-1j} + k^2). \tag{2.10}
\]
Substituting the expression \( (I\bar{s})_{kj} \) from (2.7) in (2.9), we have the following corollary.
Corollary 2.1.

\[(I^2 \hat{s})_{k|j} = \frac{(1 + v)(\hat{s}_{k|j} + k^2) - 2k - 2k^2]{d}. \quad (2.11)\]

(Whenever the value of the rate of interest \(j\) is clear from the context, it is customary to omit it from the symbols and denote the corresponding values as \(\hat{s}_{k|j}\), \((I\hat{s})_{k|j}\) and so on.)

Similar arguments may be used to show that, for given \(m\), a recursive formula of type (2.9) can be derived for \((I^m \hat{s})_{k|j}\), which is defined as the accumulation of an annuity-due of \(k\) payments of \(1^m, 2^m, \ldots, k^m\).

3. Single payments and series of level payments

Let us suppose that the yearly rate of interest in the \(k\)th year is \(i_k\). We assume that, for each \(k\), we have \(E(i_k) = j\) and \(\text{Var}(i_k) = s^2\), and that \(i_1, \ldots, i_n\) are independent random variables. We write

\[E(1 + i_k) = 1 + j = \mu, \quad (3.1)\]
\[E[(1 + i_k)^2] = (1 + j)^2 + s^2 = 1 + f = m, \quad (3.2)\]

from which it follows that

\[f = 2j + j^2 + s^2, \quad (3.3)\]
\[\text{Var}(1 + i_k) = m - \mu^2. \quad (3.4)\]

We define \(r\) to be the solution of

\[1 + r = \frac{1 + f}{1 + j}. \quad (3.5)\]

That is,

\[r = \frac{f - j}{1 + j} \quad (3.6)\]

or, using (3.3), we have

\[r = j + \frac{s^2}{1 + j}. \quad (3.7)\]

For a \(k\)-year variable annuity-due with payments of \(c_1, \ldots, c_k\), respectively, we denote the accumulated value after \(k\) years by \(C_k\). We assume that \(c_1 = 1\) and we write

\[E(C_k) = \mu_k, \quad (3.8)\]
\[E(C_k^2) = m_k. \quad (3.9)\]

It follows that \(\mu_1 = \mu, m_1 = m\), and

\[\text{Var}(C_k) = m_k - \mu_k^2. \quad (3.10)\]

Example 3.1. Consider the case when \(c_1 = 1, c_2 = c_3 = \cdots = c_k = 0\). This is, in fact, the case of a single investment at the beginning of the first year. In this case we have

\[C_k = (1 + i_1) \cdots (1 + i_k) = C_{k-1}(1 + i_k) \quad \text{for } k = 2, \ldots, n. \quad (3.11)\]
From this we easily find that (see, e.g., Kellsion (1975, Ch. 11)) \( \mu_k = \mu_{k-1}\mu \) and hence

\[ \mu_k = \mu^k. \]  
(3.12)

We also have \( m_k = m_{k-1}m \), and hence

\[ m_k = m^k. \]  
(3.13)

Therefore

\[ \text{Var}(C_k) = m^k - \mu^{2k} = ((1 + j)^2 + s^2)^k - (1 + j)^{2k}. \]  
(3.14)

In particular, after \( n \) years we have

\[ E(C_n) = (1 + j)^n \quad \text{and} \quad \text{Var}(C_n) = ((1 + j)^2 + s^2)^n - (1 + j)^{2n}. \]  
(3.15)

**Example 3.2.** In this example we assume \( c_1 = \cdots = c_n = 1 \); i.e., \( C_k \) is the accumulation after \( k \) years of an annuity-due of \( k \) yearly payments of 1. In this case we have

\[ C_k = (1 + i_k)(1 + C_{k-1}) \quad \text{for} \quad k = 2, \ldots, n. \]  
(3.16)

From this we find by straightforward reasoning, that

\[ \mu_k = \mu(1 + \mu_{k-1}), \]  
(3.17)

\[ m_k = m(1 + 2\mu_{k-1} + m_{k-1}). \]  
(3.18)

Applying (2.6) to (3.17) we obtain the following result.

**Theorem 3.1.** If \( C_k \) denotes the future value after \( k \) years of an annuity-due of \( k \) yearly payments of 1 and if the yearly rate of interest during the \( k \)th year is a random variable \( i_k \) such that \( E(1 + i_k) = 1 + j \) and \( \text{Var}(1 + i_k) = s^2 \), and \( i_1, \ldots, i_n \) are independent variables, then \( \mu_k = E(C_k) = \bar{s}_{ij}; \) in particular, \( \mu_n = E(C_n) = \bar{s}_{1j}. \)

The expected value of \( C_k \) is given thus in Theorem 3.1, so that using (2.5), we have a formula to compute \( E(C_k) \).

To derive the value of \( \text{Var}(C_k) \) we can use (3.10) for which we need the value of \( E(C_k^2) = m_k \). The value of \( m_k \) is given recursively in (3.18), but we wish to find a closed form. We shall use induction, here and in the other examples. We note that one could use difference equations, as in Kellsion (1975).

**Lemma 3.1.** Under the assumptions of Theorem 3.1 we have

\[ m_k = (m + \cdots + m^k) + 2(m\bar{s}_{k-1|j} + m^2\bar{s}_{k-2|j} + \cdots + m^{k-1}\bar{s}_{1|j}). \]

**Proof.** We proceed by induction. Let

\[ M_{1k} = m + \cdots + m^k, \]  
(3.19)

\[ M_{2k} = m\bar{s}_{k-1|j} + \cdots + m^{k-1}\bar{s}_{1|j}, \]  
(3.20)

so we must show that \( m_k = M_{1k} + 2M_{2k} \). When \( k = 2 \), this follows by (3.18), since \( \mu_1 = \bar{s}_{1|j} \) and \( m_1 = m \).

Assuming our result is true for a given \( k(2 \leq k \leq n - 1) \), it follows by formula (3.18) that it is also true for \( k + 1 \).

This completes the proof by induction. \( \square \)

Since, by (3.2), \( m = 1 + f \), we apply (3.11) to find that

\[ M_{1k} = \bar{s}_{k|f}. \]  
(3.21)
We recall the value of \( r \) as given in (3.5), and that of \( m \) as given in (3.2), and apply (3.11) to derive
\[
M_{2k} = (1 + f) \frac{(1 + j)^{k-1} - 1}{d} + \cdots + (1 + f) \frac{(1 + j)^{k-1} - 1}{d}.
\] (3.22)

By applying the value of \( d \) from (2.1), we easily derive
\[
M_{2k} = \frac{(1 + j)^{k+1} s_{k|f} - (1 + j) s_{k|f}}{j}.
\] (3.23)

Since \( m_k = M_{1k} + M_{2k} \) then we can summarize (3.21) and (3.23) to obtain the following lemma.

**Lemma 3.2.** Under the assumptions of Theorem 3.1, we have
\[
m_k = \frac{2(1 + j)^{k+1} s_{k|f} - (2 + j) s_{k|f}}{j}.
\] (3.24)

**Proof.** Since we have
\[
m_k = \frac{2(1 + j)^{k+1} s_{k|f} - (2 + j) s_{k|f}}{j},
\]
the equality follows as stated. \( \square \)

We have thus reached a formula for \( E(C_k^2) \).

To conclude, we need a formula for \( (E(C_k))^2 \), i.e., for \( \mu_k^2 = (s_{k|j})^2 \). To this extent we have the following lemma.

**Lemma 3.3.**
\[
(s_{k|j})^2 = \frac{\tilde{s}_{k|j} - 2 \tilde{s}_{k|j}}{d}.
\]

**Proof.** By (2.5), we have
\[
(s_{k|j})^2 = \frac{[(1 + j)^{k} - 1]^2}{d^2} = \frac{(1 + j)^{2k} - 2(1 + j)^{k} + 1}{d^2} = \frac{[(1 + j)^{2k} - 1] - 2[(1 + j)^{k} - 1]}{d^2} = \frac{(\tilde{s}_{k|j} - 2 \tilde{s}_{k|j})}{d}.
\]

We have thus reached the following theorem.

**Theorem 3.2.** Under the assumption of Theorem 3.1, \( E(C_k) = \tilde{s}_{k|j} \) and
\[
\text{Var}(C_k) = \frac{2(1 + j)^{k+1} s_{k|f} - (2 + j) s_{k|f} - (1 + j) \tilde{s}_{k|j} + 2(1 + j) \tilde{s}_{k|j}}{j}.
\] (3.25)

This establishes a formula for \( \text{Var}(C_k) \) in terms of future values of annuities due for periods of \( k \) and \( 2k \) and in terms of rates of interest \( j, f \) and \( r \). In this example, we studied fixed annuities due.
4. Varying series of payments

A natural extension is the case in which \( c_i = i \) for \( i = 1, \ldots, n \) which corresponds to the increasing annuity-due of \( n \) payments of \( 1, 2, \ldots, n \). In this example, we have

\[
C_k = (1 + i_k)(C_{k-1} + k).
\]

From this we derive, by straightforward reasoning,

\[
\mu_k = \mu(\mu_{k-1} + k) \quad \text{for} \quad k = 2, \ldots, n,
\]

\[
m_k = m(m_{k-1} + 2k\mu_{k-1} + k^2) \quad \text{for} \quad k = 2, \ldots, n.
\]

Since \( \mu = 1 + j = (I\bar{s})_{\bar{i}|j} \) applying (2.8) to (4.2) we obtain the following theorem.

**Theorem 4.1.** If \( C_k \) denotes the future value after \( k \) years of an increasing annuity-due of \( k \) yearly payments of \( 1, \ldots, k \), and if the yearly rate of interest during the \( k \)th year is a random variable, \( i_k \), so that \( E(1 + i_k) = 1 + j \) and \( \text{Var}(1 + i_k) = s^2 \), and so that \( i_1, \ldots, i_n \) are independent variables, then

\[
\mu_k = E(C_k) = (I\bar{s})_{\bar{i}|j} \quad \text{for} \quad k = 1, \ldots, n
\]

and, in particular, \( \mu_n = E(C_n) = (I\bar{s})_{n|j} \).

To overcome the inconvenience of the recursive equation for \( m_k \) \( (k = 2, \ldots, n) \), we proceed as in Example 3.2.

**Lemma 4.1.** Under the assumption of Theorem 4.1, we have

\[
m_k = (m^k + 2^2m^{k-1} + \cdots + k^2m) + 2(2m^{k-1}(I\bar{s})_{\bar{i}|j} + \cdots + km(I\bar{s})_{k-1|j}).
\]

**Proof.** Using (4.3), the proof follows by induction, using straightforward computations (like those in Lemma 3.1).

We write

\[
M_{1k} = m^k + 2^2m^{k-1} + \cdots + k^2m,
\]

\[
M_{2k} = 2m^{k-1}(I\bar{s})_{\bar{i}|j} + \cdots + km(I\bar{s})_{k-1|j}.
\]

So we have \( m_k = M_{1k} + 2M_{2k} \).

Since \( m = 1 + f \), we apply (2.10) to obtain

\[
M_{1k} = (I\bar{s})_{\bar{i}|f}
\]

and, following (2.7) and (3.5), we may deduce that
Under the assumptions of Theorem 4.1, we have the following theorem.

Lemma 4.2. Under the assumptions of Theorem 4.1, we have

\[ M_{2k} = \frac{2(1 + f)^{k-1}[\bar{x}_{ij} - 1] + \cdots + k(1 + f)(\bar{x}_{ij} - (k - 1))}{d} \]

\[ = \frac{2(1 + f)^{k-1}[(1 + j) - 1]/d + \cdots + k(1 + f)((1 + j)^{k-1} - 1)/d}{d} \]

\[ = 2(1 + f)^{k-1} + \cdots + k(k - 1)(1 + f) \]

\[ = \frac{(1 + j)^{k}[2(1 + r)^{k-1} + \cdots + k(1 + r)]}{d^2} - \frac{2(1 + f)^{k-1} + \cdots + k(1 + f)}{d^2} \]

\[ = 2^2(1 + f)^{k-1} + \cdots + k^2(1 + f) + \frac{2(1 + f)^{k-1} + \cdots + k(1 + f)}{d^2} \]

\[ = \frac{(1 + j)^k[(IS)_{k|f} - (1 + r)^k]}{d^2} - \frac{(IS)_{k|f} - (1 + f)^k}{d^2} - \frac{(I^2s)_{k|f} - (1 + f)^k}{d^2} + \frac{(I\bar{s})_{k|f} - (1 + f)^k}{d^2} \]

\[ = \frac{1 + j)^k(I\bar{s})_{k|f} - (1 + j)(I\bar{s})_{k|f} - j(1 + j)(I^2\bar{s})_{k|f}}{j^2} \]

Hence we have the following lemma.

Lemma 4.2. Under the assumptions of Theorem 4.1, we have

\[ M_{2k} = \frac{(1 + j)^{k+2}(IS)_{k|f} - (1 + j)(I\bar{s})_{k|f} - j(1 + j)(I^2\bar{s})_{k|f}}{j^2}. \] (4.7)

Hence we have the following theorem.

Theorem 4.2. Under the assumptions of Theorem 4.1, we have

\[ m_k = \frac{2(1 + j)^{k+2}(IS)_{k|f} - (2 + 2k - j^2)(IS)_{k|f} - (1 + j)(I^2\bar{s})_{k|f}}{j^2}. \] (4.8)

To evaluate \( \text{Var}(C_k) \), we need \( E(C_k)^2 = \mu_k^2 = (I\bar{s})_{k|f}^2 \), and to this extent we establish the following lemma.

Lemma 4.3.

\[ (I\bar{s})_{k|f}^2 = \frac{(I\bar{s})_{k|f} - 2(1 + k\bar{d})(IS)_{k|f} - k^2}{d^2}. \]

Proof. By (2.7) and Lemma 3.3, we have

\[ (I\bar{s})_{k|f}^2 = \frac{(\bar{s}_{k|f} - k)^2}{d^2} = \frac{\bar{s}_{k|f}^2 - 2k\bar{s}_{k|f} + k^2}{d^2} = \frac{(\bar{s}_{k|f} - 2\bar{s}_{k|f})/d - 2\bar{s}_{k|f} + k^2}{d^2} \]

and since \( \bar{s}_{k|f} - 2\bar{s}_{k|f} = (\bar{s}_{k|f} - 2k) - 2(\bar{s}_{k|f} - k) \), we have

\[ (I\bar{s})_{k|f}^2 = \frac{(I\bar{s})_{k|f} - 2(I\bar{s})_{k|f} - 2k(\bar{s}_{k|f} - k) - k^2}{d^2}. \]
Theorem 4.4. Hence Theorem 4.4 follows.

Summarizing, we obtain the following theorem.

Theorem 4.3. Under the assumption of Theorem 4.1, \(E(C_k) = (I\bar{s})_{\bar{k}ij}\) and

\[
\text{Var}(C_k) = \frac{2(1 + j)^{k+2}(I\bar{s})_{\bar{k}ij} - (2 + 2j - j^2)(I\bar{s})_{\bar{k}ij} - (j + j^2)(I^2\bar{s})_{\bar{k}ij}}{j^2} - \frac{(I\bar{s})_{\bar{k}ij} - 2(1 + kd)(I\bar{s})_{\bar{k}ij} - k^2}{d^2}.
\]

(4.9)

We have thus established formulae to evaluate \(E(C_k)\) and \(\text{Var}(C_k)\) in terms of future values of increasing annuities due for periods of \(k\) and \(2k\), and in terms of rates of interest \(j\), \(f\) and \(r\).

The next example is the decreasing case, i.e., the case in which \(c_i = n - i + 1\) for \(i = 1, \ldots, n\). In this example, we have

\[C_k = (1 + i_k)(C_{k-1} + n - k + 1).
\]

(4.10)

From this we derive, by straightforward reasoning,

\[\mu_k = \mu(\mu_{k-1} + n - k + 1), \quad \mu = 1 + j.
\]

(4.11)

\[m_k = m(m_{k-1} + 2(n - k + 1)\mu_{k-1} + (n - k + 1)^2), \quad m = 1 + f.
\]

(4.12)

Observe that \(\mu_1 = n(1 + j), m_1 = n^2(1 + f)\) and \(\text{Var}(C_1) = n^2s^2\), and recall that \((I\bar{s})_{\bar{k}ij} + \mu_k = (n + 1)\bar{s}_{\bar{k}ij}\).

Hence Theorem 4.4 follows.

Theorem 4.4. If \(C_k\) denotes the future value after \(k\) years of the decreasing annuity-due of \(k\) yearly payments of \(n, n-1, \ldots, n-k+1\), and if the yearly rate of interest during the \(k\)th year is a random variable \(i_k\), so that \(E(1 + i_k) = j\) and \(\text{Var}(1 + i_k) = s^2\) and so that \(i_1, \ldots, i_n\) are independent variables, then

\[\mu_k = E(C_k) = (n + 1)\bar{s}_{\bar{k}ij} - (I\bar{s})_{\bar{k}ij} \quad \text{for } k = 1, \ldots, n
\]

(4.13)

or, equivalently,

\[\mu_k = \left(n - \frac{1}{j}\right)\bar{s}_{\bar{k}ij} + \frac{k}{d}.
\]

(4.14)

Instead of finding \(m_k\), we shall find a recursive formula for \(\text{Var}(C_k)\) and hence an explicit expression for \(\text{Var}(C_k)\).

\[
\text{Var}(C_k) = m_k - \mu_k^2 = m(m_{k-1} + 2(n - k + 1)\mu_{k-1} + (n - k + 1)^2) - \mu_k^2
\]

and we derive

\[
\text{Var}(C_k) = m(m_{k-1} - \mu_k^2) + m(\mu_{k-1} + n - k + 1)^2 - \mu_k^2
\]

\[
= m \text{Var}(C_{k-1}) + \frac{1 + f}{(1 + j)^2}\mu_k^2 - \mu_k^2
\]

\[
= m \text{Var}(C_{k-1}) + \left(\frac{s}{1 + j}\right)^2\mu_k^2, \quad \text{setting } C_0 \equiv 0.
\]

This recursive equation for \(\text{Var}(C_k)\) yields by straightforward verification the following lemma.
Lemma 4.4. Under the hypothesis of Theorem 4.4, we have

\[
\text{Var}(C_k) = \left( \frac{s}{1 + j} \right)^2 \left( m^k - 1 \right) \mu_1^2 + m^k - 2 \mu_2^2 + \cdots + \mu_k^2. \tag{4.15}
\]

By substituting the values of \( m = 1 + f \) and \( \mu_k \) as given in (4.14), and using Lemma 3.3, we can derive

\[
\text{Var}(C_k) = \left( \frac{s}{1 + j} \right)^2 \left\{ (1 + f)^k - 1 \right\} \left[ (n - 1) \frac{s_{ij}^2 + 1/d}{d} + \cdots + \left( n - 1 \right) \frac{s_{kj}^2 + 1/d}{d} \right]
\]

\[
+ 2 \left( n - 1 \right) \left( \frac{s}{1 + j} \right)^2 \left( 1 + f \right)^k \frac{s_{ij}^2 + \cdots + k s_{kj}^2}{d} + \left( \frac{s}{1 + j} \right)^2 \left( 1 + f \right)^k \frac{s_{ij}^2 + \cdots + k^2}{d^2}
\]

It follows that

\[
\left[ \frac{d(1 + j)}{s} \right]^2 \text{Var}(C_k) = \left( n - 1 \right)^2 U + 2 \left( n - 1 \right) V + W,
\]

where \( U, V, W \) are given as

\[
U = [(1 + f)^k - 1] (1 + j)^2 + \cdots + (1 + j)^{2k} - 2 [(1 + f)^k - 1] (1 + j) + \cdots + (1 + j)^k,
\]

\[
V = [(1 + f)^k - 1] (1 + j) + \cdots + k (1 + j)^k - [(1 + f)^k - 1] + \cdots + k,
\]

\[
W = (1 + f)^k - 1 + \cdots + k^2.
\]

By setting \( 1 + f = (1 + j)(1 + r) \) and \( 1 + f = (1 + j)^2 (1 + \ell) \), we have \( \ell = (s/(1 + j))^2 \), and we may summarize the results in the below theorem.

Theorem 4.5. Under the assumption of Theorem 4.4, we have

\[
E(C_k) = (n + 1) s_{ij} - (1 + j) s_{ij},
\]

\[
\text{Var}(C_k) = \left( \frac{\ell}{d} \right)^2 \left[ \frac{(n - 1/j)^2 (1 + j)^2 s_{ij}^2}{1 + \ell} - \frac{(n - 1/j) (1 + j)^k s_{ij}^2}{1 + r} + \frac{(n - 1/j)^2 s_{ij}^2}{1 + f} + \frac{2 (n - 1/j) (1 + j)^k s_{ij}^2}{1 + r} + \frac{2 (n - 1/j) s_{ij} s_{ijf}}{1 + f} + \frac{(I^2 s_{ij})^2}{1 + f} \right].
\]

Another interesting example arises in the case of the annuity-due of \( n \) payments of \( 1, (1 + u), \ldots, (1 + u)^{n-1} \), where \( u \) denotes a fixed rate of increase of the payments. We set \( 1 + u = (1 + j)(1 + r) \) to derive

\[
C_k = (1 + j)(1 + u)(1 + i) \cdots (1 + i_{k-1}) \cdots (1 + u_k) \cdots (1 + i)
\]

or

\[
C_k = (1 + i_k)(C_{k-1} + (1 + u)^{k-1})
\]

and for \( \mu_k = E(C_k) \) and \( m_k = E(C_k^2) \), we obtain

\[
\mu_k = \mu (\mu_k - 1) + (1 + u)^{k-1}, \quad m_k = m (m_k - 1) + 2 (1 + u)^{k-1} \mu_k - 1 + (1 + u)^{2(k-1)}
\]

and, in particular, \( \mu_1 = 1 + j \) and \( m_1 = 1 + f \).
As in the previous example, we can proceed directly to evaluate the variance of $C_k$, noting that $\mu_k = (1 + j)^k \hat{s}_{i|t}/(1 + t)$, so that

$$\text{Var}(C_k) = m_k - \mu_k^2 = m(m_{k-1} - \mu_{k-1}^2) + m(\mu_{k-1} + (1 + u)^{k-1})^2 - \mu_k^2$$

$$= m \text{Var}(C_{k-1}) + \left( \frac{m}{(1 + j)^2} - 1 \right) \mu_k^2.$$ 

One can easily verify that this equation holds in the case of the fixed annuity as well as in the case of the increasing one

$$\text{Var}(C_k) = m \text{Var}(C_{k-1}) + \left( \frac{s}{1 + j} \right)^2 \mu_k^2, \quad \text{setting } C_0 = 0.$$ 

It follows that (compare with the previous example)

$$\text{Var}(C_k) = \left( \frac{s}{1 + j} \right)^2 [m^{k-1} \mu_1^2 + \cdots + \mu_k^2]$$

and, using Lemma 3.3, we obtain

$$\text{Var}(C_k) = \left( \frac{s}{1 + j} \right)^2 \left\{ [1 + f]^{k-1} \left[ (1 + j)^2 \hat{s}_{i|t} + (1 + f) s_{i|t} \right] + \cdots + (1 + j)^2 \hat{s}_{i|t} \right\}$$

$$= \left( \frac{s}{1 + j} \right)^2 \left[ \frac{(1 + j)^2k-2(1 + r)^2k-2}{1 + f} \hat{s}_{i|t} - \frac{(1 + j)^{k-2} - 2(1 + u)}{1 + f} \hat{s}_{i|t} + 2 \frac{(1 + j)^{k-1}}{1 + f} \hat{s}_{i|t} \right].$$

By setting $1 + f = (1 + j)(1 + r)$ and $1 + f = (1 + j)^2(1 + \ell)$ we have

$$\ell = \left( \frac{s}{1 + j} \right)^2,$$

and the following equalities hold:

$$1 + u = (1 + j)(1 + t)$$

$$1 + f = (1 + u)(1 + m) = (1 + j)^2(1 + t)^2(1 + h).$$

So, using the above notations, we can summarize as follows.

**Theorem 4.6.** If $C_k$ denotes the future value after $k$ years, of an annuity-due of $k$ yearly payments of $1, 1 + u, \ldots, (1 + u)^{k-1}$, and if the yearly rate of interest during the $k$th year is a random variable $i_k$, so that $E(1 + i_k) = 1 + j$ and $\text{Var}(1 + i_k) = s^2$, and so that $i_1, \ldots, i_n$ are independent variables, then

$$\mu_k = E(C_k) = \left( \frac{1 + j}{1 + t} \right)^k \hat{s}_{i|t},$$

$$\text{Var}(C_k) = \ell \hat{s}_{i|t} \left[ (1 + u) s_{i|t} \right] - \ell \hat{s}_{i|t} \left[ (1 + j) s_{i|t} \right] + 2 \ell (1 + u) \hat{s}_{i|t} + 2 \ell (1 + j) \hat{s}_{i|t}.$$
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