Comparison of individual risk models

Claude Lefèvre a,*, Sergey Utev b

a Institut de Statistique et de Recherche Opérationnelle, Université Libre de Bruxelles,
Boulevard du Triomphe, CP 210, B-1050 Bruxelles, Belgium
b NCEPH, The Australian National University, Canberra, ACT 0200, Australia

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Abstract

This paper is concerned with the stochastic comparison of two individual risk models for homogeneous portfolios with different claim size distributions. It is shown that a Lorenz order between the claim sizes, or a hamr-order if the claim sizes are NBUE, are transferred to the corresponding individual risk models. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

An important problem in actuarial sciences is the evaluation of the distribution of aggregate claims on a portfolio of business over a certain period of time. Let us assume, as traditionally made, that the portfolio is homogeneous (after splitting), so that claim sizes for the different risks are viewed as a sequence of non-negative independent identically distributed random variables \(X_k, k = 1, 2, \ldots\), distributed as \(X\) say. With each risk \(k\), we associate a Bernoulli random variable \(I_k\) to indicate whether the claim actually occurs (\(I_k = 1\) if it does). Furthermore, we also introduce, for each \(k\), a scaling parameter represented by a positive discrete random variable \(A_k\), in order to take account of an interest factor which may fluctuate stochastically with time. Finally, let \(N\) denote the total number of possible claims in the portfolio (random or not). Thus, the total claim sizes for this portfolio is defined as

\[
S_X = \sum_{k=1}^{N} I_k A_k X_k.
\] (1.1)

The claim sizes \(X_k\) are assumed to be independent of the events triplets \((N, I_k, A_k)\). Notice, however, that \((N, I_k, A_k)\) are allowed to be dependent.

The model (1.1) is usually named an individual risk model. It has received considerable attention in the literature. The reader is referred, e.g., to the books by Bowers et al. (1986) and De Vylder (1996) for the classical case without random interest factors \(A_k\). The insertion of factors of this type has been proposed and studied by, e.g., Dufresne
(1990) and Willmot (1989). It is worth mentioning that models such as (1.1) arise too in a number of other application fields (see Vervaat (1979) and the references therein).

Now, let us consider two homogeneous portfolios that differ only through the claim size distributions. More precisely, we construct two similar versions of the individual risk model (1.1) associated with the claim severities $X$ and $Y$, respectively. Let us suppose that $X$ and $Y$ satisfy a stochastic order relation

$$ X \preceq Y, $$  

which means that $X$ is less severe than $Y$ in some probabilistic sense. The question we address is whether comparison (1.2) is transferred to the comparison between the total claims $S_X$ and $S_Y$, i.e., (A) under what conditions a comparison $X \preceq Y$ does imply

$$ S_X \preceq S_Y. $$  

Our purpose in the present paper is to discuss that problem for various stochastic orders of convex-type. In Section 2, we will easily see that the implication holds for integral convex-type orders, such as the $s$-convex or $s$-increasing convex orders. Then, in Sections 3–5, we will investigate in some details two other stochastic orders, classical in actuarial theory, that are not of integral form, namely the Lorenz order and the harmonic average mean remaining life (hamr) order.

The random variables under interest throughout the work are within the class $G$ of non-negative random variables with strictly positive and finite mean. We recall that $X$ is smaller than $Y$ in the hamr-order (denoted by $X \preceq_{\text{hamr}} Y$) when

$$ \frac{E(X - u)^+}{EX} \leq \frac{E(Y - u)^+}{EY} \quad \text{for all } u \geq 0. $$  

$X$ is said to be smaller than $Y$ in the Lorenz order (denoted by $X \preceq_{\text{l}} Y$) when

$$ \frac{X}{EX} \preceq_{\text{ct}} \frac{Y}{EY}, \text{ i.e. when } Ef\left(\frac{X}{EX}\right) \leq Ef\left(\frac{Y}{EY}\right) \quad \text{for all convex functions } f, $$

provided that the expectations exist. Standard references on the theory of stochastic orders are the books by Shaked and Shanthikumar (1994) and Stoyan (1983); see also, e.g., Arnold (1987) and Ross (1983). Actuarial applications and related topics can be found in Goovaerts et al. (1990).

2. Preliminaries

First, it is clear that due to the assumptions, $S_X$ and $S_Y$ can be represented as a mixture of r.v.’s

$$ X_B := \sum_{i \in B} a_i X_i \quad \text{and} \quad Y_B := \sum_{i \in B} a_i Y_i, $$  

where $B$ is any subset in $\{1, \ldots, n\}$, $n = 1, 2, \ldots$, and the mixing densities are given by

$$ f(n, B, a) := P\left( N = n, \prod_{i \in B} (U_i = 1) \cap (A_i = a_i) \right). $$  

We also note that $X_B$ and $Y_B$ are sums of independent random variables. Thus question (A) has a positive answer if the order $\preceq$ is preserved (1) under mixture, (2) under convolution and (3) under scaling.
Specifically, assume that we have the comparison $F_k \preceq G_k$, $k = 1, \ldots, m$. The order $\preceq$ satisfies a *mixture property* if for any non-negative $p_1, \ldots, p_m$ with $p_1 + \cdots + p_m = 1$,
\begin{equation}
\sum_{k=1}^{m} F_k p_k \preceq \sum_{k=1}^{m} G_k p_k,
\end{equation}
$\preceq$ satisfies a *convolution property* if
\begin{equation}
F_1 \ast \cdots \ast F_m \preceq G_1 \ast \cdots \ast G_m,
\end{equation}
and $\preceq$ satisfies a *scaling property* if $X \preceq Y$ implies that
\begin{equation}
aX \preceq aY \quad \text{for any } a \geq 0.
\end{equation}

The validity of these properties is easily verified for integral stochastic orders, which are defined through some class of functions $F$ by
\begin{equation}
X \preceq_Y Y \quad \text{if } \quad Ef(X) \leq Ef(Y) \quad \text{for all } f \in F,
\end{equation}
provided that the expectations exist (see, e.g., Müller, 1997). Indeed, the mixture property is then immediate. Under that property, the convolution property will hold if for any $f \in F$ we have $f(a + x) =: f_a \in F$. For the scaling property, the condition is that $f(ax) =: g_a \in F$ whenever $f \in F$.

In particular, it is directly seen that the three properties hold true for functions that are of the $s$-convex form, and thus for the stochastic $s$-convex orders and its modifications (see, e.g., Denuit et al. 1998).

3. Preservation under convolution

It is well known that a stochastic order $\preceq$ satisfies the convolution property (2.4) if and only if for any random variable $X$ and independent random variables $Y_1$ and $Y_2$ with $Y_1 \preceq Y_2$,
\begin{equation}
X + Y_1 \preceq X + Y_2.
\end{equation}

In this part, we are going to show that both orders $\preceq_{\text{hamr}}$ and $\preceq_{\text{ld}}$ do not satisfy the convolution property (3.1) and thus (2.4). Furthermore, we will point out, for each order, an additional condition that allows us to obtain the convolution property (2.4).

3.1. Convolution and hamr-order

We write that $X \preceq_{\text{hamr}} Y$ when $X \preceq Y$ and $Y \preceq X$. Let $\preceq_d$ denote the usual order in distribution. For any $u \geq 0$, we put $H_X(u) := E(X - u)^+ / EX$.

**Property 3.1.** Let $X, Y \in \mathcal{G}$ with $EX \leq EY$. Then, $X \preceq_{\text{hamr}} Y$ if and only if there exists a Bernoulli distributed r.v. $v$ independent of $Y$ and such that $X \preceq_d vY$.

**Proof.** Let us begin with the sufficiency. Denoting $P(v = 1) = p$, we get
\begin{align*}
H_X(u) = H_{vY}(u) &= \frac{E(vY - u)^+}{E(vY)} = \frac{E(Y - u)^+ P(v = 1)}{E(Y) P(v = 1)} = H_Y(u) \quad \text{for all } u \geq 0,
\end{align*}
inence the result. Now, for the necessity, suppose that $X \preceq_{\text{hamr}} Y$. By (1.4) this means that
\begin{align*}
\frac{\int_{u}^{\infty} P(X > v) \, dv}{EX} &\geq \frac{\int_{u}^{\infty} P(Y > v) \, dv}{EY} \quad \text{for all } u \geq 0,
\end{align*}
which implies that
\[ \int_{u}^{\infty} \left( \frac{P(X > v)}{EX} - \frac{P(Y > v)}{EY} \right) dv = 0 \quad \text{for all} \ u \geq 0. \]

Therefore, we get
\[ P(X > v) = \left( \frac{EX}{EY} \right) P(Y > v) \quad \text{a.s.,} \tag{3.2} \]
and the probability distribution being a càdlàg function, the equality in (3.2) holds true for all \( v \geq 0 \). Since \( EX \leq EY \), we deduce that the law of \( X \) can be expressed as announced.

**Remark 1.** From Property 3.1, we can write that there exist r.v.’s \( X, Y \in \mathcal{G} \) such that \( Y \preceq_{\text{hamr}} X \preceq_{\text{hamr}} Y \) and \( EX = 2EY > 0 \) say. Therefore, we see that
\[ X \preceq_{\text{hamr}} Y \quad \text{does not imply} \quad EX \leq EY. \tag{3.3} \]
We indicate that Heilmann and Schröter (1991) have proved, however, that
\[ X \preceq_{\text{hamr}} Y \quad \text{implies} \quad E(X|X > 0) \leq E(Y|Y > 0). \tag{3.4} \]

**Remark 2.** A question often examined in the literature is (E) whether some given stochastic order \( X \preceq Y \) does imply the inequality \( EX \leq EY \) (which is sometimes called the engineering order)?

We notice that the property (E) can be seen, in a sense, as a necessary condition for the convolution property (2.4). Indeed, assume that the order \( \preceq \) satisfies the scaling property (2.5) and is preserved under weak convergence for uniformly integrable r.v.’s. Then, property (2.4) together with the law of large numbers imply (E). Thus, the relatively simple property (E) may clarify whether the convolution property (2.4) holds true. This observation is strongly related to Proposition 1.1.1 in Stoyan (1983).

For the hamr-order, that can be supported by the following elementary argument. Assume that \( X \preceq_{\text{hamr}} Y \) and the following simplified version of the convolution property:
\[ X + a \preceq_{\text{hamr}} Y + a \quad \text{for any} \ a > 0 \]
holds. Then, we see that when \( a > u \),
\[ \frac{E(X + a - u)^+}{EX + a} \leq \frac{E(Y + a - u)^+}{EY + a} \]
if and only if \( EX \leq EY \).

We recall that a random variable \( X \) is NBUE (new better than used in expectation) if \( E(X - u|X \geq u) \leq EX \) for all \( u \geq 0 \).

**Property 3.2.** Let \( \{X_k : k = 1, \ldots, m\} \) and \( \{Y_k : k = 1, \ldots, m\} \) be two sequences of independent random variables in \( \mathcal{G} \) such that
\[ X_k \preceq_{\text{hamr}} Y_k, \quad \text{and} \tag{3.5} \]
\[ EX_k \leq EY_k, \quad k = 1, \ldots, m, \tag{3.6} \]
\[ X_k \text{ and } Y_k \text{ are all NBUE except possibly for one } X_{k_1} \text{ and one } Y_{k_2}, \ k_1 \neq k_2. \tag{3.7} \]
Then,
\[
\sum_{k=1}^{m} X_k \leq_{\text{hamr}} \sum_{k=1}^{m} Y_k. \tag{3.8}
\]

This property can be established exactly as in Pellerey (1995, 1996), provided that condition (3.6) is added. The necessity of this condition is illustrated below.

**Counterexample 1.** We show that condition (3.6) has to be added to (3.5) and (3.7) for obtaining (3.8).

For that, we start by proving that without (3.6), the hamr-order does not satisfy the property (3.1) for any random variable \(X\). Let \(I, X, X_1, X_2\) be non-negative independent random variables. Assume that \(I\) has a Bernoulli distribution with parameter \(0 < p < \frac{1}{2}\) and that \(X, X_1, X_2\) are NBUE, identically distributed and non-degenerated at 0, for example with negative exponential distribution or being degenerated at some positive point. Take \(Y_1 = X_1\) and \(Y_2 = IX_2\). Note that \(X, Y_1, Y_2\) are mutually independent. By Property 3.1, \(Y_1 \leq_{\text{hamr}} Y_2\).

Now, since \(Y_2 \leq_{\text{hamr}} Y_1\) and \(EY_2 < EY_1\), we obtain by Property 3.2 that
\[
X + Y_2 \leq_{\text{hamr}} X + Y_1. \tag{3.9}
\]

Let us suppose that the opposite inequality (i.e., (3.1)) holds true, yielding \(X + Y_2 =_{\text{hamr}} X + Y_1\). Since \(EY_2 < EY_1\), Property 3.1 and construction imply that there exists an independent Bernoulli distributed r.v. \(v\) such that
\[
X_1 + IX_2 =_d X + Y_2 =_d v(X + Y_1) =_d v(X_1 + X_2). \tag{3.10}
\]

Let \(P(v = 1) = u\) and \(f(z) = Ez^X\). Equating the moment generating functions for the l.h.s of (3.10) \((Ez^{X_1 + IX_2} = f(z)[1 - p + pf(z)])\) and the r.h.s \((Ez^{v(X_1 + X_2)} = [1 - u + uf^2(z)])\), we find that
\[
f^2(z)(u - p) - f(z)(1 - p) + (1 - u) = 0 \tag{3.11}
\]
for all \(z \in [0, 1]\). Since \(X\) is non-degenerated at 0, the continuous function \(f(z)\) takes an arbitrary value in an open interval \((f(0), f(1)) = (A, 1)\). Thus, the identity (3.11) holds true for all \(z \in [0, 1]\) if and only if \(x^2(u - p) - x(1 - p) + (1 - u) = 0\) for all \(x \in (A, 1)\), resulting in \(p = u = 1\) which is in contradiction with \(0 < p < \frac{1}{2}\).

Let us turn to Property 3.2. Let \(X_1, X_2, Y_1, Y_2\) be independent of random variables, \(X_1 =_d X_2 =_d Y_1 =_d X\) with NBUE distributions and \(Y_2 =_d vX\), where \(P(v = 1) = 1 - P(v = 0) = p\) and \(v\) is independent of \(X\). Then,
\[
X_1 \leq_{\text{hamr}} Y_1 \quad \text{and} \quad X_2 \leq_{\text{hamr}} Y_2,
\]
but the above argument shows that there exists \(0 < p < \frac{1}{2}\) such that \(X_1 + X_2\) is not smaller than \(Y_1 + Y_2\) in the hamr-order.

**Remark 3.** The starting point of this work was our observation that the convolution property (3.8) derived by Pellerey (1995, 1996) is valid only when the r.v.’s satisfy condition (3.6). By (3.3) and (3.4), this condition is satisfied for strictly positive r.v.'s, but not, as claimed by Pellerey, for non-negative r.v.’s. We also mention that a non-negative NBUE random variable being necessarily strictly positive, condition (3.6) is verified if in condition (3.7), all the \(X_k\) and \(Y_k\)’s are taken as NBUE.

The next proposition shows that condition (3.7) in Property 3.2 is actually necessary (and not only sufficient) to have the convolution property (3.8) — under the additional condition \(EY_1 \leq EY_2\). For that, we will show that the convolution of two r.v.’s is larger, in the hamr-order, than one of these r.v.’s if and only if that random variable is NBUE.
Property 3.3. Let $X \in \mathcal{G}$. The following assertions are equivalent:

1. $X \leq_{\text{shamr}} X + Y$ for any r.v. $Y \in \mathcal{G}$ independent of $X$,
2. $X$ is NBUE,
3. $X + Y_1 \leq_{\text{shamr}} X + Y_2$ for any r.v. $Y_1, Y_2 \in \mathcal{G}$ independent of $X$ and such that $EY_1 \leq EY_2$.

Proof. Firstly, we establish the equivalence of statements (1) and (2). Given any non-negative $a$, let

$$Z_u(a) = E[(X + a - u)^+ - (X - u)^+] = \int_{u-a}^u P(X \geq v) \, dv.$$ 

By definition, $X \leq_{\text{shamr}} X + Y$ means that

$$EY[(X - u)^+] \leq EXZ_u(Y) \quad \text{for all } u \geq 0,$$

which holds for any pair of independent r.v.’s $X, Y \in \mathcal{G}$ if and only if

$$aE(X - u)^+ \leq EXZ_u(a) \quad \text{for all } u \text{ and } a \geq 0. \quad (3.12)$$

On the other hand, $E(X - u|X \geq u) \leq EX$ is equivalent to

$$EX(X - u)^+ \leq EXP(X \geq u) \quad \text{for all } u \geq 0. \quad (3.13)$$

Thus, we see that (3.13) is implied by (3.12). Reciprocally, dividing (3.12) by $a$ and passing to the limit with $a \downarrow 0$ yields (3.13).

Now, the implication (2) $\Rightarrow$ (3) is derived in Pellerey (1996).

Finally, let us prove the implication (3) $\Rightarrow$ (1). Observe that

$$H_u(a) = \left(\frac{1-u}{a}\right)^+ \uparrow \text{ as } a \uparrow,$$

i.e., $\leq_{\text{shamr}} a$ when $0 < \varepsilon < a$. Take $Y_1 = \varepsilon$ and $Y_2 = a$. The comparison $X + Y_1 \leq_{\text{shamr}} X + Y_2$ means that

$$(EX + a)E(X + \varepsilon - u)^+ \leq (EX + \varepsilon)E(X + a - u)^+ \quad \text{for all } u \geq 0. \quad (3.14)$$

Passing to the limit with $\varepsilon \downarrow 0$, inequality (3.14) yields (3.12), which was shown to be equivalent to (2) in the proof of the equivalence between (1) and (2). \hfill \Box

3.2. Convolution and Lorenz order

Property 3.4. Let $\{X_k : k = 1, \ldots, m\}$ and $\{Y_k : k = 1, \ldots, m\}$ be two sequences of independent random variables in $\mathcal{G}$ such that

$$X_k \preceq_{\text{l}} Y_k, \quad \text{and} \quad \frac{EX_k}{EY_k} = c, \quad k = 1, \ldots, m \quad (3.15)$$

for some constant $c$. Then,

$$\sum_{k=1}^m X_k \preceq_{\text{l}} \sum_{k=1}^m Y_k. \quad (3.17)$$
Proof. We begin by observing that by (3.16),
\[
    r_k := \frac{E X_k}{\sum_{j=1}^{m} E X_j} = \frac{E Y_k}{\sum_{j=1}^{m} E Y_j}, \quad k = 1, \ldots, m.
\]  
(3.18)

Now, by (3.15) and the scaling property (2.5) of the convex order,
\[
    r_k(X_k/EX_k) \preceq_{cx} r_k(Y_k/EY_k), \quad k = 1, \ldots, m.
\]
Using the convolution property (2.4) of the convex order, we then get
\[
    \sum_{k=1}^{m} r_k(X_k/EX_k) \preceq_{cx} \sum_{k=1}^{m} r_k(Y_k/EY_k), \quad k = 1, \ldots, m,
\]
which can be rewritten, by (3.18), as
\[
    \sum_{k=1}^{m} \frac{X_k}{EX_k} \preceq_{cx} \sum_{k=1}^{m} \frac{Y_k}{EY_k},
\]
i.e., (3.17).

Counterexample 2. We show that without (3.16), the order \( \preceq_j \) does not verify property (3.1) for any random variable \( X \). Thus, the additional condition (3.16) is necessary for obtaining (3.17). Indeed, assume that (3.1) is satisfied by \( X \) with \( EX = 1 \). Fix \( Y \) with a non-degenerated distribution and \( EY = 1 \). We see that for \( Z, U \in \mathcal{G} \),
\[
    Z \sim \sim U \quad \text{if and only if} \quad Z/EZ \sim \sim U/EU,
\]  
(3.19)
and moreover,
\[
    aZ \sim \sim bZ \quad \text{for any} \quad a, b > 0.
\]  
(3.20)
Thus, (3.1) with \( Y_1 =_d aY \) and \( Y_2 =_d bY \) would imply
\[
    X + aY \preceq_j X + bY \quad \text{for any} \quad a, b > 0.
\]  
(3.21)
Note that (3.16) is not satisfied. Since \( a \) and \( b \) can be permuted, the sign \( \preceq_j \) in (3.21) is in fact a sign \( \sim \), so that by (3.19) and taking \( a = 1 \), we would have
\[
    X + Y =_d (X + bY) \left( \frac{2}{1+b} \right) \quad \text{for any} \quad b > 0.
\]
But this equality holds if and only if either \( X = Y = 1 \) a.s., or \( X \) and \( Y \) have a positive Cauchy distribution, which has no finite mean, hence the contradiction.

4. Preservation under mixture

In this part, we are going to show that both orders \( \preceq_{hame} \) and \( \preceq_j \) do not satisfy the mixture property (2.3) in general. Furthermore, we will prove that, surprisingly, the property holds true when condition (3.16) is added to the hypotheses.
4.1. Mixture and hamr-order

Property 4.1. The mixture property is satisfied for $\preceq_{\text{hamr}}$ under the additional condition (3.16).

Proof. By definition, (2.3) for $\preceq_{\text{hamr}}$ means that

$$\sum_{k,j=1}^{m} p_i p_k E(X_k - u)^+ EY_i \leq \sum_{k,j=1}^{m} p_i p_k E(Y_k - u)^+ EX_j. \tag{4.1}$$

Now, by (3.16) and since $X_k \preceq_{\text{hamr}} Y_k$, we find that

$$E(X_k - u)^+ EY_i = \frac{E(X_k - u)^+ EX_k EY_i}{EX_k} \leq \frac{E(Y_k - u)^+ EX_k EY_i}{EY_k}$$

$$= E(Y_k - u)^+ EX_i \left( \frac{EX_k}{EY_k} / \frac{EX_i}{EY_i} \right) = E(Y_k - u)^+ EX_i,$$

which implies (4.1).

Counterexample 3. We show that the order $\preceq_{\text{hamr}}$ does not satisfy the mixture property (2.3) in general, and thus that the additional condition (3.16) is necessary. Fix positive $a$ and $u$ such that $1 < 1 + a < u < \frac{3}{2}$. In the context of property (2.3), we take

$$m = 2, \quad p_1 = p_2 = 0.5 \quad \text{and} \quad X_1 = 1, \quad X_2 = 4, \quad Y_1 = 1 + a, \quad Y_2 = 4 + a.$$

First, we note that

$$X_1 \preceq_{\text{hamr}} Y_1 \quad \text{and} \quad X_2 \preceq_{\text{hamr}} Y_2,$$

and (3.16) is not satisfied. Now, let $L$ and $R$ be the l.h.s. and r.h.s. of (4.1) calculated for these r.v.’s and at point $u$. Since $1 + a < u < 4$, we get

$$L = \frac{1}{4} E(X_2 - u)^+ (EX_1 + EY_2) = \frac{1}{4}(4-u)(5+2a),$$

and

$$R = \frac{1}{4} E(Y_2 - u)^+ (EX_1 + EX_2) = \frac{1}{4}(4+a-u)5.$$

Thus $L > R$ when $u < \frac{3}{2}$, i.e., the mixture property (2.3) does not hold.

4.2. Mixture and Lorenz order

Property 4.2. The mixture property is satisfied for $\preceq_l$ under the additional condition (3.16).

Proof. We proceed as for Property 3.4. By (3.16), we have that

$$s_k := \frac{EX_k}{\sum_{j=1}^{m} EX_j p_j} = \frac{EY_k}{\sum_{j=1}^{m} EY_j p_j}, \quad k = 1, \ldots, m. \tag{4.2}$$

Now, by (2.5) and (3.15), we get

$$s_k(X_k/EX_k) \preceq_{\text{ce}} s_k(Y_k/EY_k), \quad k = 1, \ldots, m.$$
Thus, for any convex function $f$,

$$
\sum_{k=1}^{m} p_k E[f(s_k(X_k/EX_k))] \leq \sum_{k=1}^{m} p_k E[f(s_k(Y_k/EY_k))],
$$

i.e.,

$$
\sum_{k=1}^{m} p_k E\left[\frac{X_k}{\sum_{j=1}^{m} EX_j p_j}\right] \leq \sum_{k=1}^{m} p_k E\left[\frac{Y_k}{\sum_{j=1}^{m} EY_j p_j}\right],
$$

which is equivalent to (2.3).

**Counterexample 4.** We show that the order $\preceq_l$ does not satisfy the mixture property (2.3) in general, meaning that the extra condition (3.16) is necessary. Let $X, Z$ be two independent r.v.'s with a negative exponential distribution of parameter 1. Take

$$
X_1 \overset{d}{=} X, X_2 \overset{d}{=} X, Y_1 \overset{d}{=} X, Y_2 \overset{d}{=} Xa.
$$

Then, $X_1 \preceq_l Y_1$ and $X_2 \preceq_l Y_2$.

Now, let us suppose that the mixture property is satisfied with $m = 2$ and $0 < p_1 = 1 - p_2 < 1$. Arguing as before (in Counterexample 2), we get

$$
F_X p_1 + F_X p_2 = F_X b p_1 + F_X a p_2 \quad \text{for any } a, b > 0,
$$

which yields

$$
e^{-x} = e^{-xb} p_1 + e^{-xb/a} p_2 \quad \text{for any } x \geq 0.
$$

But this is possible only when $b = 1 = a$, i.e., for the mixture of identically distributed r.v.'s.

### 5. Main result

Let us come back to the original question (A). By combining the results of previous sections, we obtain the following answer.

**Property 5.1.** Assume that comparison (1.2) between two claim severities $X$ and $Y$ holds for an $s$-convex order, or for the Lorenz order, or for the hamr-order provided that these severities are NBUE. Then, comparison (1.3) between the associated individual risk models holds for the same order.

**Remark 4.** We observe that the result can be directly extended to the case where the two individual risk models are constructed for claim sizes $\{X_k\}$ and $\{Y_k\}$ that form two sequences of non-negative independent r.v.'s with fixed means $\mu_X$ and $\mu_Y$ (and not necessarily equidistributed), respectively.

**Remark 5.** Let us look at possible links between the Lorenz and hamr-orders. First, we establish the implication (5.1), which seems to be little known:

$$
\text{if } X \preceq_l Y \text{ and } EX \leq EY, \quad \text{then } X \preceq_{\text{hamr}} Y.
$$

(5.1)
Indeed, under these conditions, we see that
\[
\frac{E(X - u)^+}{EX} \leq \frac{E\left(\frac{Y}{EY} - \frac{u}{EX}\right)^+}{EY} = \frac{E\left(\frac{Y}{EY} - \frac{u}{EX}\right)^+}{EY} \leq \frac{E(Y - u)^+}{EY},
\]
since \(E(Y - z)^+ \downarrow\) as \(0 < z \uparrow\).

Now, it is worth noting that the inverse is not true, i.e.,
\[
X \preceq_{\text{harr}} Y \quad \text{and} \quad EX \leq EY \quad \text{do not imply} \quad X \preceq_{\text{f}} Y. \tag{5.2}
\]

Indeed, referring to Property 3.1, take \(X \sim_{\text{d}} vY\), with \(0 < P(v = 1) < 1\) and \(EY > 0\). Then, \(X \preceq_{\text{harr}} Y\), but it is easily checked that \(Y \preceq_{\text{f}} X\) (strictly).

Using (5.1), we deduce the following unexpected comparison for our problem.

**Corollary 5.1.** If \(X \preceq_{\text{f}} Y\) and \(EX \leq EY\), then
\[
S_X \preceq_{\text{f}} S_Y \quad \text{and} \quad S_X \preceq_{\text{harr}} S_Y. \tag{5.3}
\]

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**References**